On quantum curves

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The general problem: how can we reconstruct **geometry** from a **quantum-mechanical** framework?

Quantum Mechanics → Space

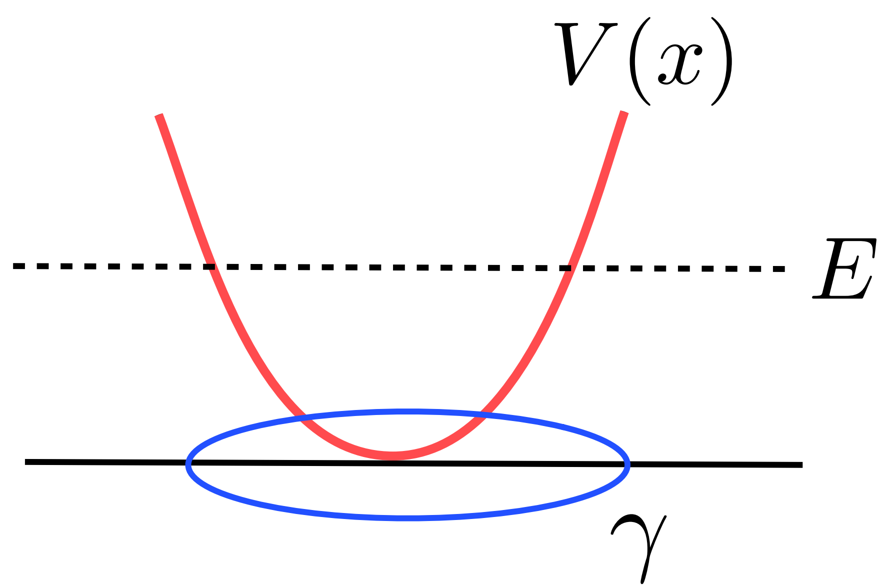
Quantum Mechanics	Space
discrete	smooth
reality/positivity conditions	complex/analytic
spectral theory	geometry and topology

This is not obvious!

In my view, the theory of quantum curves should be a toy model of how to bridge this gap

A simple example

Let us consider the action in classical mechanics



$$H(x,p) = \frac{p^2}{2} + V(x) = E$$

$$J(E) = \oint_\gamma p(x,E)\mathrm{d}x$$

This is just the leading order approximation to a "quantum" action

$$J(E;\hbar) = \sum_{n\geq 0} \hbar^{2n} \oint_\gamma p_{2n}(x,E)$$

To determine the quantum action one can take the following perspective:

First, we "quantize" the classical curve:

$$x \to \mathsf{x} \qquad\qquad p \to \mathsf{p} \to -\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}$$

$$H\left(x, -\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi(x) = E\psi(x)$$

Second, we use a WKB ansatz for the wavefunction:

$$\psi(x) = \frac{1}{\sqrt{P(x,\hbar)}}\exp\left(\frac{\mathrm{i}}{\hbar}\int^{x} P(x',\hbar)\mathrm{d}x'\right)$$

Third, we solve in terms of a formal power series expansion

$$P(x, \hbar) = \sum_{n \geq 0} p_{2n}(x, E) \hbar^{2n}$$

Many works on quantum curves follow morally this pattern. Often the formal power series obtained in this way contain interesting geometric information ("quantum" invariants) associated to the curve we started with.

However, this approach is rather insufficient, from the point of view of QM:

1) It is based on formal series expansions. Therefore, it is **purely perturbative!**

2) Although the curve is quantized, **there is no Hilbert space!**

Why Hilbert spaces are useful

Let us now consider the actual spectral problem in $L^2(\mathbb{R})$

Under favorable conditions, the Hamiltonian operator will have a discrete spectrum $\{E_n\}_{n\geq 0}$

There is an all-orders, perturbative WKB quantization condition

$$J(E;\hbar) = \sum_{n\geq 0} \oint_\gamma p_{2n}(x,E)\mathrm{d}x = 2\pi\hbar\left(n+\frac{1}{2}\right)$$

We can now ask the following question [Balian-Parisi-Voros]:

Is there a **well-defined** function $J_{\rm exact}(E;\hbar)$ such that

$$J_{\rm exact}(E;\hbar) = 2\pi\hbar\left(n+\frac{1}{2}\right)$$

gives the **exact** spectrum?

This function must have the asymptotic expansion

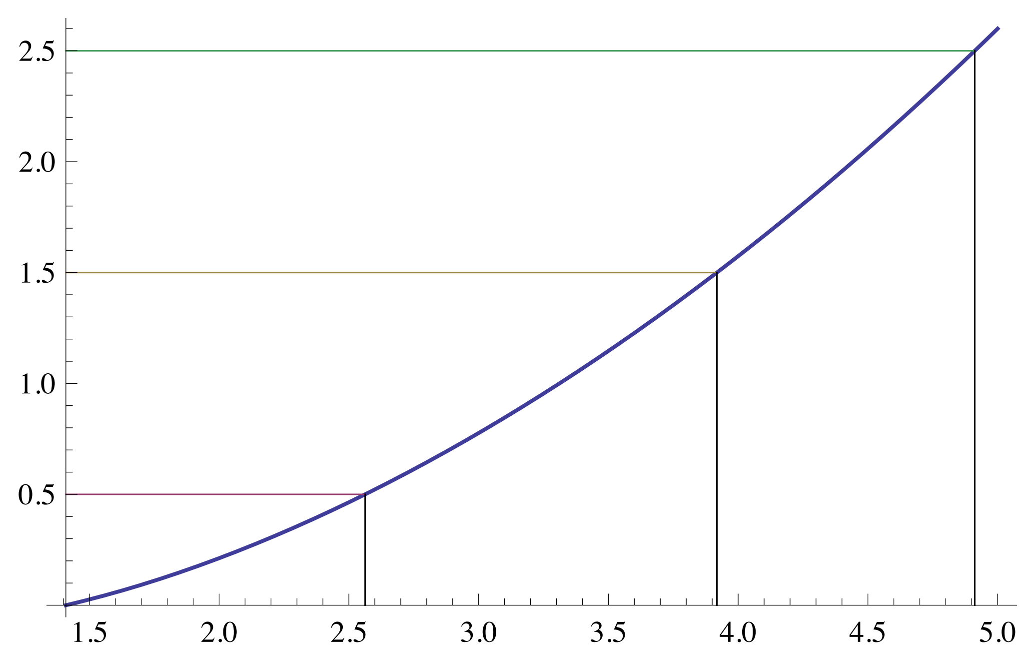
$$J_{\rm exact}(E;\hbar) \sim \sum_{n\geq 0} \hbar^{2n} \oint_\gamma p_{2n}(x,E){\rm d}x$$

However, in many cases it will contain additional information from non-perturbative sectors

Moral:

Go to full-fledged QM (with a Hilbert space) and look for well-defined functions, defined by spectral problems. Recover your favorite formal series as the **asymptotic expansion** of such a function.

However, as already emphasized by [Balian-Parisi-Voros] the spectral problem gives just a discrete set of points. Finding a full-fledged function which "interpolates" between them is maybe asking for too much

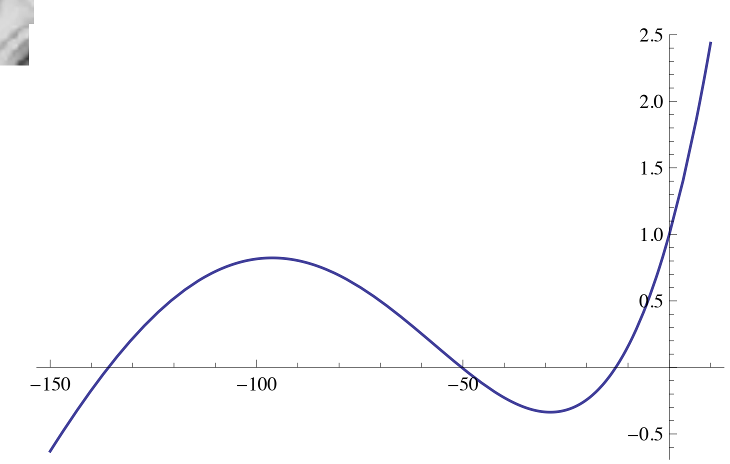


Let us suppose that we have a trace class operator ρ with spectrum $\left\{ \mathrm{e}^{-E_n} \right\}_{n \in \mathbb{Z}_{\geq 0}}$

Then, there is a wonderful machine called the **Fredholm determinant**

spectrum →  → **entire function!**

$$\Xi_\rho(\kappa) = \prod_{n=0}^{\infty} \left(1 + \kappa\, \mathrm{e}^{-E_n}\right)$$



Classical mirror curves

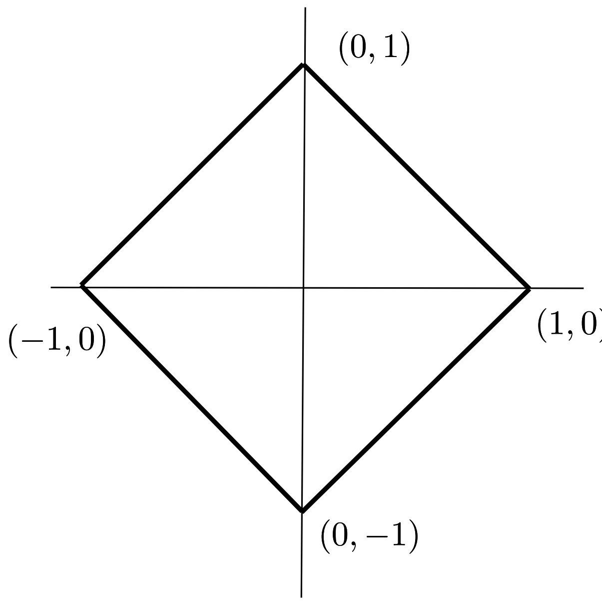
The simplest yet non-trivial Calabi-Yau (CY) threefolds are **toric CYs**, which are noncompact. They can be described by Newton polygons. Their mirror manifolds reduce to algebraic curves

$$W_X(\mathrm{e}^x, \mathrm{e}^y) = 0$$

given by the Newton polynomial of the polygon.

Example: the canonical bundle over $\mathbb{F}_0$

"local $\mathbb{F}_0$ "



inner point of the polygon

$$W_X(\mathrm{e}^x, \mathrm{e}^y) = \mathrm{e}^x + \mathrm{e}^{-x} + \mathrm{e}^y + \mathrm{e}^{-y} + \kappa = 0$$

The topological string free energies at genus *g* encode the Gromov-Witten invariants of these threefolds:

$$F_g(t) = \sum_{d\geq 1} N_{g,d}\, \mathrm{e}^{-dt}$$

The total topological string free energy is the formal (and asymptotically divergent) series

$$F(t, g_s) = \sum_{g=0}^{\infty} F_g(t) g_s^{2g-2}$$

$$F_g(t) \sim (2g)! \qquad\qquad g \gg 1$$

Quantum mirror curves

Weyl quantization of the Newton polynomial produces a self-adjoint operator on $L^2(\mathbb{R})$

$$W_X(e^x, e^y) \to \mathsf{O}_X$$

e.g. local $\mathbb{F}_0$

$$\mathsf{O} = e^{\mathsf{x}} + e^{-\mathsf{x}} + e^{\mathsf{y}} + e^{-\mathsf{y}}$$

Theorem

[Grassi-Hatsuda-M.M., Kashaev-M.M., Laptev-Schimmer-Takhtajan]

The operator $\rho_X = \mathsf{O}_X^{-1}$ on $L^2(\mathbb{R})$ is positive definite and of trace class

The integral kernel of ρ_X can be explicitly computed in some cases by using Faddeev's quantum dilogarithm [Kashaev-M.M.]

Open problem 1: compute the integral kernel of ρ_X for **any** toric CY. Do we need generalizations of the quantum dilog?

First asymptotic conjecture

The Fredholm determinant of ρ_X is therefore well-defined and analytic. In the double-scaling limit

$$\kappa = \mathrm{e}^{\mu} \qquad \mu \to \infty, \quad \hbar \to \infty, \qquad \frac{\mu}{\hbar} = t$$

we have

$$\log \Xi_X(\kappa) \sim \sum_{g \geq 0} F_g(t) \hbar^{2-2g}$$

We have then a well-defined (even entire) function, obtained from the spectrum of the "quantum curve", which gives the enumerative invariants by asymptotic expansion

There is an **exact** version of this conjecture which gives explicit expressions for the Fredholm determinant in terms of BPS invariants of the CY

This leads to a precise correspondence between spectral theory and topological strings, or

TS/ST correspondence

Large *N* limits

Another mechanism for obtaining a "smooth" structure from a discrete one is to consider the **limit of large quantum numbers**

It turns out that the relevant large *N* limit in our theory of quantum curves involves the "fermionic" spectral traces

$$\Xi_\rho(\kappa) = 1 + \sum_{N \geq 1} Z(N) \kappa^N$$

The fermionic spectral trace has a very nice physical interpretation. It can be regarded as the canonical partition function of a non-interacting gas of *N* fermions with one-particle density matrix ρ

It can be also written as an *N*-dimensional integral [Fredholm,1903]

$$Z(N) = \frac{1}{N!} \int_{\mathbb{R}^N} \det_{i,j} \rho(x_i, x_j) \mathrm{d}^N x$$

When written down explicitly for the operators ρ_X, this trace is a **matrix model** with 't Hooft parameter [M.M.-Kashaev-Zakany]

$$\lambda = \frac{N}{\hbar}$$

In some examples one finds a generalized *O(2)* model:

$$Z_X(N,\hbar) = \int_{\mathbb{R}^N} \frac{\mathrm{d}^N u}{(2\pi)^N} \prod_{i=1}^{N} \mathrm{e}^{-\hbar V(u_i,\hbar)} \frac{\prod_{i<j} 4\sinh^2\left(\frac{u_i-u_j}{2}\right)}{\prod_{i,j} 2\cosh\left(\frac{u_i-u_j}{2}+C\right)}$$

Open problem 2: obtain explicit expressions for these matrix models for **any** toric CY. Note that these are well-defined and convergent matrix models

Second asymptotic conjecture

In the double-scaling limit

$$N \to \infty, \quad \hbar \to \infty, \qquad \frac{N}{\hbar} = \lambda$$

we have

$$\log Z_X(N, \hbar) \sim \sum_{g \geq 0} F_g^D(\lambda) \hbar^{2-2g}$$

dual topological string free energies

For the experts: these are the topological string free energies in the conifold frame. They are related by a symplectic transformation to the usual Gromov-Witten generating functionals

Open problem 3: show that the matrix model for $Z_X(N, \hbar)$ is described, at large *N*, by a spectral curve which agrees with the mirror curve (and is algebraic). Show in addition that it satisfies the topological recursion [recent progress by Zakany]

or, alternatively,

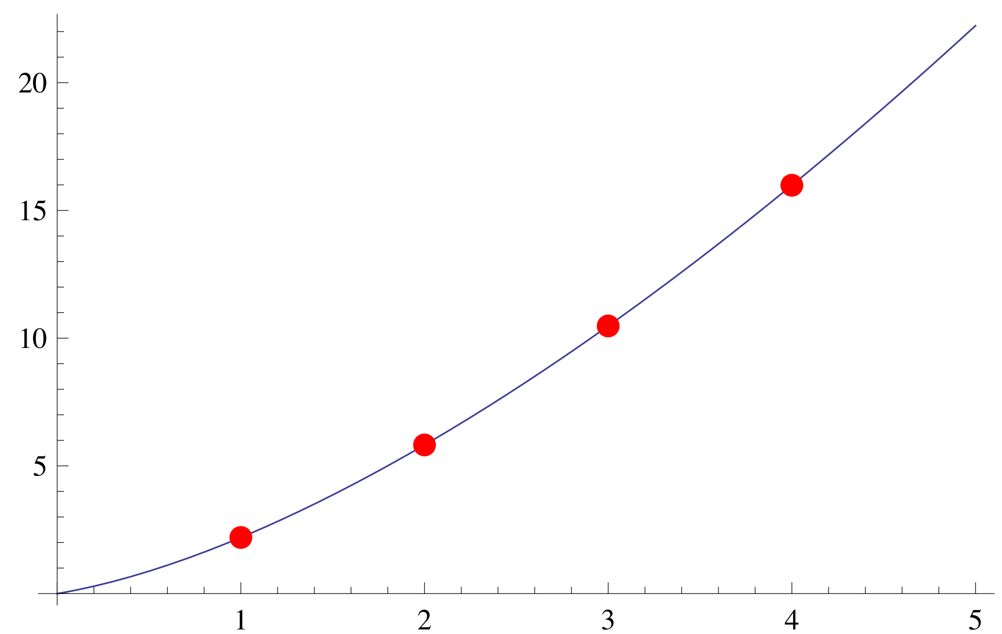
Open problem 4: prove the second asymptotic conjecture (this also follows from **3** and BKMP)

Open problem 5: prove the first asymptotic conjecture

Open problem 6: prove the exact version of the conjecture

It turns out that topological string theory provides an **exact** expression for the fermionic spectral trace which promotes it to an **analytic** function of *N*

$$\frac{1}{N!} \to \frac{1}{\Gamma(N+1)}$$



In this case interpolation is again possible

Can we see quantum curves?

Quantum curves should be "fuzzy" curves.
How do we get something fuzzy but smooth, out of a discontinuous quantum system?

One possibility is to consider **Wigner distributions** in phase space:

$$f_\psi(x,p) = \frac{1}{2\pi\hbar}\int_{\mathbb{R}} \psi^*\left(x+\frac{y}{2}\right)\psi\left(x-\frac{y}{2}\right)\mathrm{e}^{\mathrm{i}py/\hbar}\mathrm{d}y$$

In the standard classical limit of Quantum Mechanics
(in one dimension)

$$n \to \infty, \quad \hbar \to 0, \qquad n\hbar = \xi$$

(here *n* is the quantum number) the limiting Wigner distribution is supported on the classical energy curve

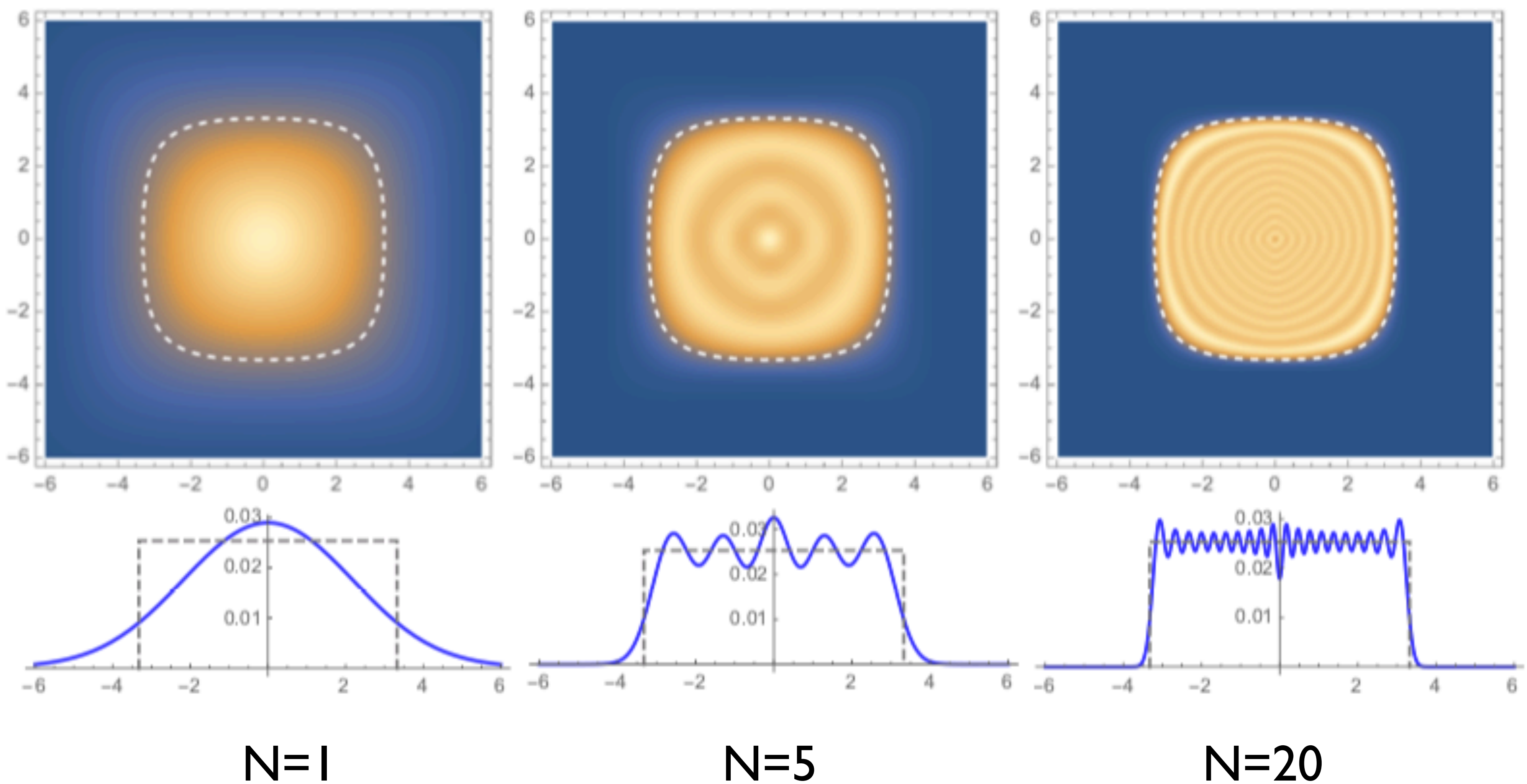
$$H(x,p) = E(\xi)$$

where the energy is determined by ξ

Something similar can be done for quantum mirror curves. One considers in this case the so-called "reduced density matrix" of the underlying Fermi gas of *N* particles, and its Wigner transform, which is a function on phase space *(x,p)* [M.M.-Zakany]

The classical limit is here the ’t Hooft limit considered before, and the ’t Hooft parameter sets the modulus of the curve.

$$e^{x} + e^{-x} + e^{p} + e^{-p} = \kappa(\lambda)$$



$$\lambda = \frac{N}{\hbar} = \frac{1}{2\pi}$$

Quantization(s)

Note that, in our theory, the standard topological string emerges in the **strong coupling limit** of the spectral problem

$$\hbar \to \infty$$

This is in clear tension with a statement found often in the literature on quantum curves

topological recursion= WKB

It is almost obvious to me that this statement cannot be true literally and/or in general (although a version of this is true for curves of genus zero [Bouchard-Eynard])

The correct statement seems to be rather the following:

Given a classical curve, there are (at least) **two different quantizations**: one is obtained from WKB, and the other is obtained from topological recursion.

In particular, they lead to different free energies

$$F_g^{\rm NS} \neq F_g \qquad g \geq 1$$

However, in the case of mirror curves, there is a non-trivial relation involving "quantum theta functions" [Grassi-Gu, Huang et al.], which are in turn related to the "non-perturbative" version of the topological recursion [Grassi-Hatsuda-M.M.,Grassi]

Limiting cases

Although toric CYs lead to exponentiated operators, it is possible to find more conventional operators by taking special cases and appropriate limits

If the CY is an A_N fibration over a two-sphere, there is a "geometric engineering" limit in which the mirror curve becomes the curve of Seiberg-Witten theory

This leads to an operator which is a "mild" deformation of standard Quantum Mechanics [Grassi-M.M.] and has a very interesting spectral theory:

$$\mathsf{H}_N = \Lambda^N \left(\mathrm{e}^{\mathsf{p}} + \mathrm{e}^{-\mathsf{p}} \right) + \mathsf{x}^N + \sum_{k=2}^{N-1} \mathsf{x}^{N-k} h_k$$

Complexifications

Note that, for odd *N*, this operator does not have a trace class inverse (unbounded potential). This also happens for mirror curves, for some choices of the parameters

e.g. $\mathrm{e}^{\mathsf{x}} + m_{\mathbb{F}_0}\mathrm{e}^{-\mathsf{x}} + \mathrm{e}^{\mathsf{p}} + \mathrm{e}^{-\mathsf{p}}$ with $m_{\mathbb{F}_0} \leq 0$

Interestingly, the resulting spectral theory can be extended by analytic continuation if we consider **resonances**. Doing this in spectral theory is more delicate than in the topological string (which lives in a complexified realm from the very beginning)

However, it can be seen that the correspondence between both theories extends to the complex case.

To conclude

I believe that the theory of quantum curves gains a lot from really being quantum, with a Hilbert space and correspondingly well-defined spectral problems.

In the case of mirror curves, the correspondence between ST and TS is extremely rich and leads to a wealth of new and unexpected results in both fields. I have pointed out some important open problems but there are much more, specially on the foundational side.

I believe that this correspondence is saying something deep about quantization which should be unveiled…

Thank you for your attention and…