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An introduction to topological string theory

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Abstract: Notes for a mini-course on topological string theory.

Contents

1 Introduction

1.1 Topological strings and their applications

Topological string theory is a topological sector of type II string theory described by a special class of CFTs (which are two-dimensional sigma models) coupled to gravity. The target of the sigma model is usually a Calabi–Yau manifold, which in the context of type II string theory is simply the compactification manifold. It is often possible to compute the amplitudes of these theories in an exact form, for both the open and the closed string sector, and they probably provide most of the exactly computable quantities in string theory. The amplitudes that we typically compute in topological string theory are functions of the moduli, and we denote them in the closed and open sectors as

$$F_g(t), \quad F_{g,h}(t,z), \tag{1.1}$$

respectively. Here g, h are the genus and the number of holes of the Riemann surface, and t, z are the closed and open moduli. We will now give a list of the problems in string theory and mathematical physics where these amplitudes are relevant.

1. The $F_g(t)$ and the $F_{g,h}(t,z)$ contain information about the enumerative geometry of the Calabi–Yau target. They can be interpreted in terms of Gromov–Witten theory, one of the cornerstones of modern algebraic geometry.

2. The $F_g(t)$ are couplings in the $\mathcal{N} = 2$ 4d supergravity theory which appears when one compactifies type II string theory in a Calabi–Yau manifold. Appropriate choices of the target make possible to derive *field theory results* from the amplitudes $F_g(t)$. For example, the counting of instantons in $\mathcal{N} = 2$ gauge theory can be done by using topological string theory. This is called the *geometric engineering* of $\mathcal{N} = 2$ theories.

3. The open and closed amplitudes count BPS states of the type II compactification. One can use them to count microstates of a certain class of black holes in type II/M theory.

4. On certain Calabi–Yau manifolds, the open and closed amplitudes are related by large N dualities to the 't Hooft resummation of amplitudes in Chern–Simons theory and matrix models, very much in the spirit of the AdS/CFT correspondence. This makes possible to test large N dualities to all orders in $1/N$.

1.2 The structure of topological string theory

String theories involve typically a conformal field theory (CFT) on a general Riemann surface Σ_g which is then coupled to 2d gravity. If we write the CFT action as

$$S[\phi, g_{\mu\nu}], \tag{1.2}$$

where $g_{\mu\nu}$ is the two–dimensional metric and ϕ are the "matter" fields, then the basic object we want to compute is the total free energy

$$F = \sum_{g=0}^{\infty} g_s^{2g-2} F_g, \qquad F_g = \int \mathcal{D}h \, \mathcal{D}\phi \, \mathrm{e}^{-S[\phi,h]} \tag{1.3}$$

where g_s is the string coupling constant, and the path integral in F_g is over field configurations on the Riemann surface Σ_g. We can also perturb the CFT with various operators $\{\mathcal{O}_a\}$ leading to a general action which we write

$$S[\phi, h, t] = S[\phi, h] + \sum_a t^a \mathcal{O}_a. \tag{1.4}$$

In this case, the free energies at genus g will depend as well on the couplings t^a, and we write $F_g(t)$ to indicate this explicit dependence.

The computation of the free energies in (1.3) is a phenomenal problem, and many approaches have been developed in order to solve it. In the continuum approach, one fixes diffeomorphism invariance and finds that there is a critical central charge $c = 26$ in the CFT for which the 2d metric *decouples* and one is left with an integration over a *finite* set of coordinates τ parametrizing the moduli space $\mathcal{M}_g$

$$F_g(t) \sim \int_{\mathcal{M}_g} \mathrm{d}\tau \int \mathcal{D}\phi \, \mathrm{e}^{-S[\phi,\tau,t]} \tag{1.5}$$

This is the so-called *critical string theory.* For $c \neq 26$ (noncritical string theories), the metric has, on top of the finite set of moduli τ, a dynamical degree of freedom–the Liouville field. Critical strings are manageable but still very hard to solve, since integrating over the moduli space remains a difficult problem. Noncritical strings for generic values of c are so far intractable.

There are however two classes of examples where we have tools to compute $F_g(t)$. The first class are noncritical strings for $c < 1$. Here one can not only solve for the Liouville dynamics, but also perform the resulting integrals over the moduli and obtain explicit results for $F_g(t)$ at all genera. This is achieved by using matrix models to *discretize* the worldsheet of the string. The second class of examples are topological string theories, which are the topic of this course.

2 Calabi–Yau manifolds

Good introductions to complex and Calabi–Yau manifolds can be found in [1] and in the second volume of [2]. We will also use some standard results in algebraic geometry. The reference for these is [3].

2.1 Definition and properties

The first thing to know about Calabi–Yau manifolds is that they are *complex*, i.e. they have complex coordinates (in local patches) that we will denote by

$$x^I, \quad x^{\overline{I}}, \qquad I = 1, \cdots, d, \tag{2.1}$$

where d is the complex dimension. These manifolds are endowed with a Riemannian metric that is *Hermitian*, i.e. it only mixes holomorphic with antiholomorphic coordinates, and in a local patch it looks like

$$G_{I\overline{J}}. \tag{2.2}$$

The Hermitian character implies that, if v^I is a vector of complex components, then

$$G_{I\overline{J}} v^I (v^J)^* \geq 0 \tag{2.3}$$

and it vanishes only when $v^I = 0$.

To build a Calabi–Yau manifold we need another condition, i.e. that the manifold is *Kähler*. This means that the *Kähler form*

$$\omega = \mathrm{i} G_{I\overline{J}} \mathrm{d}x^I \wedge \mathrm{d}x^{\overline{J}} \tag{2.4}$$

is *closed*:

$$\mathrm{d}\omega = 0 \tag{2.5}$$

Notice that, since

$$G^*_{I\overline{J}} = G_{J\overline{I}} \tag{2.6}$$

the Kähler form is a real two-form. In components (2.5) leads to

$$\partial_K G_{I\overline{J}} = \partial_I G_{K\overline{J}}, \qquad \partial_{\overline{K}} G_{I\overline{J}} = \partial_{\overline{J}} G_{I\overline{K}}. \tag{2.7}$$

Locally, one can then write the Hermitian metric as

$$G_{I\overline{J}} = \frac{\partial^2 K}{\partial x^I \partial x^{\overline{J}}}, \tag{2.8}$$

where $K(x^I, x^{\overline{I}})$ is called the *Kähler potential*. It is easy to check that for a Kähler manifold the Christoffel symbols do not have mixed indices, i.e. their only nonvanishing components are

$$\Gamma^I_{JK}, \qquad \Gamma^{\overline{I}}_{\overline{JK}}. \tag{2.9}$$

The Calabi–Yau condition requires on top of all this that the metric $G_{I\overline{J}}$ is Ricci–flat:

$$R_{IJ} = 0. \tag{2.10}$$

CYs first appeared in string theory as compactification manifolds, since the CY condition appears a a requirement to have a supersymmetric Minkowski vacuum. From the point of view of the sigma models, (2.10) guarantees conformal invariance of the underlying sigma model. It was conjectured by Calabi and then proved by Yau that Ricci flatness is equivalent to a topological condition, namely that the first Chern class of the manifold vanishes

$$c_1(X) = 0. \tag{2.11}$$

One of the most important properties of Calabi-Yau manifolds (which can actually be taken as their defining feature) is that they have a holomorphic, nonvanishing section Ω of the canonical bundle $K_X = \Omega^{d,0}(X)$. In local coordinates, this is written as

$$\Omega = \Omega_{I_1 \cdots I_d} \mathrm{d}x^{I_1} \wedge \cdots \wedge \mathrm{d}x^{I_d}. \tag{2.12}$$

Since the section is nowhere vanishing, the canonical line bundle is trivial and $c_1(K_X) = 0$.

Using Hodge theory, Poincaré duality and the CY condition, one can see that the Hodge diamond of a CY threefold

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{0,1} & & h^{1,0} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array} \tag{2.13}$$

has the structure

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{2.14}$$

and therefore it only depends on two integers $h^{1,1}(X)$, $h^{2,1}(X)$. It also follows that the Euler characteristic of a CY is

$$\chi(X) = 2(h^{1,1}(X) - h^{2,1}(X)). \tag{2.15}$$

2.2 Examples

Example 2.1. *Tori.* The simplest examples of CY manifolds are tori, $X = \mathbb{T}^{2d}$. In one complex dimension, these are elliptic curves,

$$y^2 = 4x^3 - g_2 x - g_3. \tag{2.16}$$

Notice that as we change g_2, g_3, we change the shape of the elliptic curve. These are in fact simple instances of *complex deformation parameters*.

Example 2.2. *Non-compact CYs.* Let us first consider non-compact Calabi–Yau manifolds whose building block is a one-dimensional compact manifold. These manifolds will be given by a Riemann surface together with an appropriate bundle over it, and geometrically they can be regarded as the local geometry of an embedded Riemann surface in a general Calabi–Yau space. Indeed, consider a Riemann surface Σ_g holomorphically embedded inside a Calabi–Yau threefold X, and let us look at the holomorphic tangent bundle of X restricted to Σ_g. We have

$$TX|_{\Sigma_g} = T\Sigma_g \oplus \mathcal{N}_{\Sigma_g}, \tag{2.17}$$

where $\mathcal{N}_{\Sigma_g}$ is a holomorphic rank-two complex vector bundle over Σ_g, called the normal bundle of Σ_g, and the Calabi–Yau condition $c_1(X) = 0$ gives

$$c_1(\mathcal{N}_{\Sigma_g}) = 2g - 2. \tag{2.18}$$

The Calabi–Yau X 'near Σ_g' then looks like the total space of the bundle

$$\mathcal{N} \to \Sigma_g, \tag{2.19}$$

where $\mathcal{N}$ is regarded here as a rank-two bundle over Σ_g satisfying (2.18). The non-compact space (2.19) is an example of a *local* Calabi–Yau threefold.

When $g = 0$ and $\Sigma_g = \mathbb{P}^1$ it is possible to be more precise about the bundle $\mathcal{N}$. A theorem due to Grothendieck says that any holomorphic bundle over $\mathbb{P}^1$ splits into a direct sum of line bundles (for a proof, see for example [3], pp. 516–7). Line bundles over $\mathbb{P}^1$ are all of the form $\mathcal{O}(n)$, where $n \in \mathbf{Z}$. In Maxwell theory, the bundle $\mathcal{O}(n)$ is associated to a monopole of charge n. It can be easily described in terms of two charts on $\mathbb{P}^1$: the north-pole chart, with co-ordinates z, Φ for the base and the fibre, respectively, and the south-pole chart, with co-ordinates z', Φ'. The change of co-ordinates is given by

$$z' = 1/z, \quad \Phi' = z^{-n}\Phi. \tag{2.20}$$

We also have that $c_1(\mathcal{O}(n)) = n$. We then find that local Calabi–Yau manifolds that are made out of a two-sphere together with a bundle over it are all of the form

$$\mathcal{O}(-a) \oplus \mathcal{O}(a-2) \to \mathbb{P}^1, \tag{2.21}$$

since the degrees of the bundles have to sum up to -2 due to (2.18). An important case occurs when $a = 1$. The resulting non-compact manifold,

$$\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1, \tag{2.22}$$

is called the *resolved conifold*.

We can also consider non-compact Calabi–Yau threefolds based on compact complex surfaces. Consider a complex surface S embedded in a Calabi–Yau manifold X. As before, we can split the tangent bundle as

$$TX|_S = TS \oplus \mathcal{N}_S, \tag{2.23}$$

where the normal bundle $\mathcal{N}_S$ is now of rank one. The Calabi–Yau condition leads to

$$c_1(\mathcal{N}_S) = c_1(K_S), \tag{2.24}$$

where K_S is the canonical line bundle over S, and we used that $c_1(TS) = -c_1(K_S)$. Therefore, we have $\mathcal{N}_S = K_S$. The Calabi–Yau X 'near S' looks like the total space of the bundle

$$K_S \to S. \tag{2.25}$$

This construction gives a whole family of non-compact Calabi–Yau manifolds that are also referred to as local Calabi–Yau manifolds. A well-known example is $S = \mathbb{P}^2$, the two-dimensional projective space, which leads to the Calabi–Yau manifold

$$\mathcal{O}(-3) \to \mathbb{P}^2, \tag{2.26}$$

also known as *local* $\mathbb{P}^2$. Another important example is $S = \mathbb{P}^1 \times \mathbb{P}^1$, which leads to *local* $\mathbb{P}^1 \times \mathbb{P}^1$.

Example 2.3. *K3 surface.* The K3 surface is the unique, simply-connected complex surface which has $c_1(X) = 0$ and is therefore CY. Let us find a concrete realization. Since we want a surface, we can look for a projective hypersurface X in $\mathbb{P}^3$ of the form

$$x_0^d + x_1^d + x_2^d + x_3^d = 0. \tag{2.27}$$

Let us show that, for an appropriate choice of d, this is indeed a K3 surface. By the Lefshetz hyperplane theorem, this hypersurface is simply-connected. The only thing to impose is then that $c_1(X) = 0$. We first notice that the tangent bundle of $\mathbb{P}^3$ along X satisfies

$$T\mathbb{P}^3\big|_X = T_X \oplus N_X \tag{2.28}$$

where T_X and N_X are respectively the tangent and normal bundle to X. The total Chern class satisfies, by Whitney's formula,

$$c(\mathbb{P}^3) = c(T_X) \cdot c(N_X). \tag{2.29}$$

We now use some elementary facts from algebraic geometry. The total Chern class of $\mathbb{P}^k$ is given by

$$c(\mathbb{P}^k) = (1 + x)^{k+1}, \tag{2.30}$$

where $x = [H]$ is the generator of $H^2(\mathbb{P}^k, \mathbb{Z})$ and coincides with the hyperplane class (i.e. the Poincaré dual to a hyperplane $H \simeq \mathbb{P}^{k-1}$ in $\mathbb{P}^k$). On the other hand, a basic consequence of the *adjunction formula* says that the normal bundle N_X to a hypersurface of degree d in $\mathbb{P}^K$ (which is a line bundle) has first Chern class given by

$$c_1(N_X) = [X] \tag{2.31}$$

where $[X]$ is the Poincaré dual to X, understood as a cycle of complex codimension one in $\mathbb{P}^k$. One also has that

$$[X] = dx \tag{2.32}$$

Using these facts, we obtain for the hypersurface (2.27),

$$c(T_X) = \frac{(1+x)^4}{1+dx} = 1 + (4-d)x + \cdots . \tag{2.33}$$

Therefore, $c_1(X) = 0$ if and only if $d = 4$ in (2.27).

Example 2.4. *The quintic CY.* This can be understood as the three-dimensional analogue of the hypersurface (2.27). It is defined by an algebraic equation in $\mathbb{P}^4$

$$x_0^d + x_1^d + x_2^d + x_3^d + x_4^d = 0. \tag{2.34}$$

An argument similar to the one for the K3 surface shows that the CY condition implies now

$$d = 5 \tag{2.35}$$

hence the name quintic. The Lefshetz hyperplane theorem says that

$$h^{1,1}(X) = h^{1,1}(\mathbb{P}^4) = 1, \tag{2.36}$$

and its generator is $\omega = i^*(x)$, where x is the hyperplane class of $\mathbb{P}^4$ and $i : X \hookrightarrow \mathbb{P}^4$ is the inclusion map. The triple intersection number

$$\kappa = \int_X \omega^3 = \int_{\mathbb{P}^4} \omega^3 \wedge [X] = 5 \int_{\mathbb{P}^4} x^4 = 5. \tag{2.37}$$

We can also compute the Euler characteristic

$$\chi(X) = \int_X c_3(X) = \int_{\mathbb{P}^4} c_3(X) \wedge [X] = -\int_{\mathbb{P}^4} 40x^3 \cdot 5x = -200, \tag{2.38}$$

and from here we deduce $h^{2,1}(X) = 101$.

Example 2.5. *The Enriques CY.* A very simple example of a CY is, with our definition,

$$\mathrm{K3} \times \mathbb{T}^2. \tag{2.39}$$

This is a very special case, however. It is easy to see that a generic CY has $SU(3)$ holonomy, while (2.39) has $SU(2)$ holonomy. Correspondingly, when we compactify type II string theory on (2.39) we have $\mathcal{N} = 4$ supersymmetry, and not $\mathcal{N} = 2$. However, one can find a less nontrivial example closely related to this, which is called the *Enriques Calabi–Yau*. The Enriques Calabi-Yau X can be viewed as the first non-trivial generalization of the product space $\mathbb{T}^2 \times \mathrm{K3}$. It is defined as the orbifold $(\mathbb{T}^2 \times \mathrm{K3})/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts as a free involution. This involution inverts the coordinates of the torus and acts as the Enriques involution on the K3 surface. It can be shown that

$$h^{2,1}(X) = h^{1,1}(X) = 11, \tag{2.40}$$

therefore

$$\chi(X) = 0. \tag{2.41}$$

From the 11 $(1,1)$ classes, one –that we will denote by ω_S– comes from $\mathbb{T}^2$, while the remaining 10 come from the quotient $\mathrm{K3}/\mathbb{Z}_2$, which is called the Enriques surface E, and we denote them by ω_a, $a = 1, \cdots, 10$. If we define the intersection matrix of the two classes in the Enriques surface

$$C_{ab} = \int_E \omega_a \wedge \omega_b, \tag{2.42}$$

then the only nontrivial triple intersection number of $(1,1)$ classes in the Enriques CY is

$$\kappa_{abS} = \int_X \omega_a \wedge \omega_b \wedge \omega_S = C_{ab}. \tag{2.43}$$

3 Topological sigma models

Topological strings are, as all string theories, CFTs coupled to 2d gravity. The underlyng CFTs are *2d nonlinear sigma models* with a *target space* X, and they were introduced in [4–6]. We will take X to be a CY manifold. This will guarantee among other things the conformal invariance of the model.

There are two different sigma models that can be used to construct topological strings, and they are referred to as type A and type B topological sigma models. Let us first explain what is common to both of them.

3.1 General properties of topological sigma models

First of all, both models are based on a scalar, commuting field which is a map $x : \Sigma_g \to X$. On top of that they both have Grassmann-valued fields and possess a Grassmann, scalar symmetry $\mathcal{Q}$ which acts as a derivation and has the following properties:

1. It is nilpotent $\mathcal{Q}^2 = 0$.

2. The sigma model action is $\mathcal{Q}$-exact:

$$S = \{\mathcal{Q}, V\} \tag{3.1}$$

   The quantity V is sometimes called the *gauge fermion*.

3. The energy-momentum tensor of this theory is also $\mathcal{Q}$-exact,

$$T_{\mu\nu} = \{\mathcal{Q}, G_{\mu\nu}\}, \tag{3.2}$$

   where $G_{\mu\nu} = \delta V / \delta g^{\mu\nu}$.

The last property is what makes these theories topological. Indeed, one can show that the partition function of a theory with this property does not depend on the background two-dimensional metric. This is easily proved, at least at a formal level. The partition function is given by

$$Z = \int \mathcal{D}\phi \, \mathrm{e}^{-S}, \tag{3.3}$$

where ϕ denotes the set of fields of the theory, and we compute it in the background of a two-dimensional metric $h_{\alpha\beta}$ on the Riemann surface. Since $T_{\alpha\beta} = \delta S / \delta g^{\alpha\beta}$, we find that

$$\frac{\delta Z}{\delta g^{\mu\nu}} = -\langle \{\mathcal{Q}, G_{\mu\nu}\} \rangle, \tag{3.4}$$

where the bracket denotes an unnormalized vacuum expectation value. Since $\mathcal{Q}$ is a symmetry of the theory, the above vacuum expectation value vanishes, and we find that Z is metric-independent. Notice that $\mathcal{Q}$ is formally identical to a BRST operator, and this suggests that the right operators to look at in the model are the $\mathcal{Q}$ cohomology classes, i.e. operators $\mathcal{O}$ which satisfy

$$\{\mathcal{Q}, \mathcal{O}\} = 0, \qquad \mathcal{O} \neq \{\mathcal{Q}, \Psi\}. \tag{3.5}$$

It is easy to see that the correlation functions of these operators are also metric independent. This is because

$$\begin{aligned}\frac{\delta}{\delta g^{\mu\nu}}\langle \mathcal{O}_{i_1}\mathcal{O}_{i_2}\cdots\mathcal{O}_{i_n}\rangle &= \langle \mathcal{O}_{i_1}\mathcal{O}_{i_2}\cdots\mathcal{O}_{i_n}T_{\mu\nu}\rangle \\ &= \langle \mathcal{O}_{i_1}\mathcal{O}_{i_2}\cdots\mathcal{O}_{i_n}\{\mathcal{Q},G_{\mu\nu}\}\rangle = \pm\langle\{\mathcal{Q},\mathcal{O}_{i_1}\mathcal{O}_{i_2}\cdots\mathcal{O}_{i_n}G_{\mu\nu}\}\rangle = 0.\end{aligned} \tag{3.6}$$

In (3.4) and (3.6) we have 'integrated by parts' in field space, therefore we have assumed that there are no contributions coming from boundary terms. In some situations this assumptions does not hold, and we have contributions coming from the boundary of field space that lead to anomalies in the $\mathcal{Q}$ symmetry, This is indeed the origin of the holomorphic anomaly equations in topological string theory, which we will address later.

The metric-independence of the correlation functions in a topological field theory is very surprising, since correlators of a generic QFT on a curved background will depend on the metric –after all, this is what QFT in curved space is about!

The second surprising implication of $\mathcal{Q}$-exactness of the action is that *the semiclassical approximation is exact.* To see this we explicitly introduce a coupling constant $\hbar$,

$$Z = \int \mathcal{D}\phi\, \mathrm{e}^{-\frac{1}{\hbar}S}, \tag{3.7}$$

and

$$\frac{\delta Z}{\delta \hbar^{-1}} = -\langle\{\mathcal{Q},V\}\rangle = 0, \tag{3.8}$$

In particular, we can evaluate the partition function (and the correlation functions of $\mathcal{Q}$ invariant operators) at $\hbar \to 0$, which is the semiclassical limit.

The structure of topological quantum field theories, as we have reviewed them here, leads immediately to a procedure for constructing non-local observables starting from local ones. Let us suppose that we have found an operator $\phi^{(0)}$ which is in the cohomology of $\mathcal{Q}$, as well as operators $\phi^{(n)}$, $n = 1, 2$, that are differential forms of degree n on Σ such that,

$$\mathrm{d}\phi^{(n)} = \mathcal{Q}\phi^{(n+1)}, \quad n \geq 0. \tag{3.9}$$

In this equation, d denotes the exterior derivative on Σ. The operators $\phi^{(n)}$ are called the *topological descendants* of $\phi^{(0)}$. It is easy to see that the operator

$$W^{(\gamma_n)}_{\phi^{(0)}} = \int_{\gamma_n} \phi^{(n)}, \tag{3.10}$$

where $\gamma_n \in H_n(\Sigma)$, is a topological observable:

$$\mathcal{Q}W^{(\gamma_n)}_{\phi^{(0)}} = \int_{\gamma_n} \mathcal{Q}\phi^{(n)} = \int_{\gamma_n} \mathrm{d}\phi^{(n-1)} = \int_{\partial\gamma_n} \phi^{(n-1)} = 0, \tag{3.11}$$

since $\partial\gamma_n = 0$. Similarly, it is easy to show if γ_n is trivial in homology (i.e. if it is ∂-exact), then $W^{(\gamma_n)}_{\phi^{(0)}} = 0$ is $\mathcal{Q}$-exact. Equations (3.9) are called *descent equations.* The conclusion of this analysis is that, given a (scalar) topological observable $\phi^{(0)}$ and a solution to the descent equations (3.9), one can construct a family of topological observables:

$$W^{(\gamma_{i_n})}_{\phi^{(0)}}, \quad i_n = 1, \cdots, b_n, \quad n = 1, 2, \tag{3.12}$$

in one-to-one correspondence with the homology classes of the Riemann surface Σ.

It is easy to see that in any theory where (3.2) is satisfied there is a simple procedure to construct a solution to (3.9) given a scalar observable $\phi^{(0)}$. If (3.2) holds, then one has:

$$P_\mu = T_{0\mu} = \{\mathcal{Q}, G_\mu\}, \tag{3.13}$$

where

$$G_\mu \equiv G_{0\mu}. \tag{3.14}$$

Since $\mathcal{Q}$ is a Grassmannian symmetry, G_μ is an anti-commuting operator and a one-form on Σ. If we are given a $\mathcal{Q}$-invariant operator $\phi^{(0)}(x)$, we can use (3.14) to construct

$$\phi^{(n)}_{\mu_1\mu_2\cdots\mu_n}(x) = G_{\mu_1}G_{\mu_2}\cdots G_{\mu_n}\phi^{(0)}(x), \tag{3.15}$$

where $n \leq 1, 2$. On the other hand, since the G_{μ_i} anti-commute,

$$\phi^{(n)} = \frac{1}{n!}\phi^{(n)}_{\mu_1\mu_2\cdots\mu_n}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_n}, \tag{3.16}$$

is an n-form on Σ. By using (3.13), the $\mathcal{Q}$-invariance of $\phi^{(0)}$, as well as $P_\mu = \partial_\mu$, one can easily check that these forms satisfy the descent equations (3.9). This solution to (3.9) is usually called the *canonical solution* to the descent equations.

A simple consequence of the descent procedure is that the *perturbed* action with the operators (3.10) for $n = 2$

$$S(t^a) = S + \sum_a t^a W^{(\Sigma)}_{\phi^{(0)}_a} \tag{3.17}$$

is also $\mathcal{Q}$-closed. Therefore, given an observable $\phi^{(0)}$, we can associate to it a deformation of the theory which preserves $\mathcal{Q}$-invariance and also $\mathcal{Q}$-exactness of the energy-momentum tensor (this is due to the fact that the perturbation involves integrating a two-form, so it is independent of the metric on Σ). These are the topological analogues of *marginal deformations* in conventional quantum field theory.

3.2 The type A model

We will denote real indices in the tangent space of the target manifold X by i, with $i = 1, \cdots, 2d$. We will also use complex coordinates on the worldsheet $z = x^1 + \mathrm{i}x^2$, $\bar{z} = x^1 - \mathrm{i}x^2$. Locally, we can always find a flat Euclidean metric whose components are given in these complex coordinates by

$$g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}, \qquad g_{zz} = g_{\bar{z}\bar{z}} = 0. \tag{3.18}$$

The components of the epsilon tensor $\epsilon^{\mu\nu}$ in complex coordinates are

$$\epsilon^{\bar{z}z} = -\epsilon^{z\bar{z}} = 2\mathrm{i}. \tag{3.19}$$

Finally, our choice of measure is $\mathrm{d}^2 z = -\mathrm{i}\mathrm{d}z\wedge\mathrm{d}\bar{z}$.

The *field content* of the type-A topological sigma model is the following. First, we have a map $x : \Sigma_g \to X$ from a Riemann surface of genus g to a Kähler manifold X of complex dimension d. We also have Grassmann fields $\chi \in x^*(TX)$, which are scalars on Σ_g, and a Grassmannian one-form ρ_μ with values in $x^*(TX)$. This last field satisfies a self-duality condition which implies that its only non-zero components are

$$\rho^I_{\bar{z}} \in x^*(T^{(1,0)}X), \qquad \rho^{\bar{I}}_z \in x^*(T^{(0,1)}X). \tag{3.20}$$

We also have an auxiliary field F^i_μ which is also self-dual. The $\mathcal{Q}$ symmetry acts on the fields as follows:

$$\begin{aligned}
&[\mathcal{Q}, x^i] = \chi^i, \\
&\{\mathcal{Q}, \chi^i\} = 0, \\
&\{\mathcal{Q}, \rho^I_{\bar{z}}\} = 2\partial_{\bar{z}} x^I + F^I_{\bar{z}} - \Gamma^I_{JK}\chi^J \rho^K_{\bar{z}}, \\
&\{\mathcal{Q}, \rho^{\bar{I}}_z\} = 2\partial_z x^{\bar{I}} + F^{\bar{I}}_z - \Gamma^{\bar{I}}_{\bar{J}\bar{K}}\chi^{\bar{J}} \rho^{\bar{K}}_z, \\
&[\mathcal{Q}, F^I_{\bar{z}}] = -2D_{\bar{z}}\chi^I - \Gamma^I_{JK}\chi^J F^K_{\bar{z}} - R^I{}_{K\bar{J}L}\chi^K\chi^{\bar{J}}\rho^L_{\bar{z}}, \\
&[\mathcal{Q}, F^{\bar{I}}_z] = -2D_z\chi^{\bar{I}} - \Gamma^{\bar{I}}_{\bar{J}\bar{K}}\chi^{\bar{J}} F^{\bar{K}}_z + R^{\bar{I}}{}_{\bar{J}L\bar{K}}\chi^L\chi^{\bar{J}}\rho^{\bar{K}}_z.
\end{aligned} \tag{3.21}$$

Notice that, by construction, one has $\mathcal{Q}^2 = 0$. Geometrically, if we interpret χ^i as the basis $\mathrm{d}x^i$ of differential forms on X, we see that $\mathcal{Q}$ acts on x^i, χ^i like the de Rham differential operator on the target space X.

The action for the theory is,

$$\begin{aligned}
S =& \frac{1}{2}\int_{\Sigma_g} \mathrm{d}^2 z \Big[G_{I\bar{J}}\big(4\partial_{\bar{z}}x^I\partial_z x^{\bar{J}} - 2\rho^I_{\bar{z}} D_z\chi^{\bar{J}} - 2\rho^{\bar{J}}_z D_{\bar{z}}\chi^I - F^I_{\bar{z}} F^{\bar{J}}_z\big) \\
&+ R_{\bar{I}J\bar{K}L}\rho^{\bar{I}}_{\bar{z}}\rho^J_z\chi^{\bar{K}}\chi^L\Big].
\end{aligned} \tag{3.22}$$

One can easily check that

$$S_A = \{\mathcal{Q}, V\}, \tag{3.23}$$

where

$$V = \frac{1}{4}\int_{\Sigma_g} \mathrm{d}^2 z\sqrt{g} g^{\mu\nu} G_{I\bar{J}}\Big[\rho^I_\mu\partial_\nu x^{\bar{J}} + \rho^{\bar{J}}_\mu\partial_\nu x^I - \frac{1}{2}\rho^I_\mu F^{\bar{J}}_\nu - \frac{1}{2}\rho^{\bar{J}}_\mu F^I_\nu\Big]. \tag{3.24}$$

In other words, the action is $\mathcal{Q}$-exact. Since the action of $\mathcal{Q}$ does not depend on the two-dimensional metric on Σ_g, it immediately follows that the energy-momentum tensor is also $\mathcal{Q}$-exact. Therefore, the A model is a topological field theory of the cohomological type. In this theory we also have a $U(1)$ ghost number symmetry. The ghost numbers of the fields x, χ and ρ, F are 0, 1, -1 and 0, respectively. Notice that the Grassmannian charge $\mathcal{Q}$ then has ghost number 1. The 'antighost' $G_{\mu\nu}$ appearing in (3.2) has ghost number -1.

The bosonic term in (3.22) can be written as

$$\frac{1}{2}\int_{\Sigma_g} \mathrm{d}^2 z\sqrt{g} G_{I\bar{J}}(g^{\mu\nu} - \mathrm{i}\epsilon^{\mu\nu})\partial_\mu x^I\partial_\nu x^{\bar{J}} = \frac{1}{2}\int_{\Sigma_g} \mathrm{d}^2 z\sqrt{g}\, G_{I\bar{J}} g^{\mu\nu}\partial_\mu x^I\partial_\nu x^{\bar{J}} + \int_{\Sigma_g} x^*(\omega), \tag{3.25}$$

where ω is given in (2.4). This is because, in our conventions,

$$x^*(A) = \mathrm{i}A_{I\bar{J}}\Big(\partial_z x^I\partial_{\bar{z}} x^{\bar{J}} - \partial_{\bar{z}} x^I\partial_z x^{\bar{J}}\Big)\mathrm{d}^2 z = -\frac{1}{2}A_{I\bar{J}}\epsilon^{\mu\nu}\partial_\mu x^I\partial_\nu x^{\bar{J}}\sqrt{g}\mathrm{d}^2 z \tag{3.26}$$

for any $(1,1)$ form $A_{I\bar{J}}$.

On the other hand, the bosonic part of the standard nonlinear sigma model is given by

$$S_\sigma = \frac{1}{2}\int_{\Sigma_g} \mathrm{d}^2 z\sqrt{g}\,\Big(G_{I\bar{J}} g^{\mu\nu} + \mathrm{i}B_{I\bar{J}}\epsilon^{\mu\nu}\Big)\partial_\mu x^I\partial_\nu x^{\bar{J}}, \tag{3.27}$$

where $B_{I\overline{J}}$ are the components of a form

$$B = B_{I\overline{J}} \mathrm{d}x^I \wedge \mathrm{d}x^{\overline{J}} \in H^{1,1}(X), \tag{3.28}$$

called the B *field*. This is a real two-form, and its components satisfy

$$B^*_{I\overline{J}} = -B_{J\overline{I}}. \tag{3.29}$$

In view of the above, we can write

$$\begin{aligned} S_\sigma &= \int_{\Sigma_g} \mathrm{d}^2 z\, \mathcal{G}_{I\overline{J}} \partial_z x^I \partial_{\bar{z}} x^{\overline{J}} + \int_{\Sigma_g} \mathrm{d}^2 z\, \overline{\mathcal{G}}_{I\overline{J}} \partial_{\bar{z}} x^I \partial_z x^{\overline{J}} \\ &= \int_{\Sigma_g} \mathrm{d}^2 z\, G_{I\overline{J}} 2 \partial_{\bar{z}} x^I \partial_z x^{\overline{J}} - \int_{\Sigma_g} x^*(\mathcal{J}) \end{aligned} \tag{3.30}$$

where

$$\mathcal{G}_{I\overline{J}} = G_{I\overline{J}} + B_{I\overline{J}}, \qquad \overline{\mathcal{G}}_{I\overline{J}} = G_{I\overline{J}} - B_{I\overline{J}} \tag{3.31}$$

and

$$\mathcal{J} = \omega + \mathrm{i} B \tag{3.32}$$

is called the *complexified Kähler form*.

Remark 3.1. Although we are focusing on CY manifolds, the A model can be formulated on any Kähler (in fact, almost-Hermitian) manifold, see [4].

Since the A model is a cohomological field theory, the relevant operators, as we discussed in the previous section, are the observables, i.e. the operators that belong to the $\mathcal{Q}$-cohomology. One can easily check that the $\mathcal{Q}$-cohomology is given by operators of the form

$$\mathcal{O}_\phi = \phi_{i_1 \cdots i_p} \chi^{i_1} \cdots \chi^{i_p}, \tag{3.33}$$

where $\phi = \phi_{i_1 \cdots i_p} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}$ is a closed p-form representing a non-trivial class in $H^p(X)$. Therefore, in this case the $\mathcal{Q}$-cohomology is in one-to-one correspondence with the de Rham cohomology of the target manifold X. This is in agreement with the fact that $\mathcal{Q}$ can be interpreted as the de Rham differential on X. Notice that the degree of the differential form corresponds to the ghost number of the operator. Moreover, one can derive a selection rule for correlation functions of such operators: the vacuum expectation value $\langle \mathcal{O}_{\phi_1} \cdots \mathcal{O}_{\phi_\ell} \rangle$ vanishes unless

$$\sum_{k=1}^{\ell} \deg(\mathcal{O}_{\phi_k}) = 2d(1-g) + 2 \int_{\Sigma_g} x^*(c_1(X)), \tag{3.34}$$

where $\deg(\mathcal{O}_{\phi_k}) = \deg(\phi_k)$ and $c_1(X)$ is the first Chern class of the Kähler manifold X. This selection rule arises as follows: the twisted theory has a $U(1)$ ghost current, a global $U(1)$ symmetry that rotates the twisted fermions. Since χ and ρ have opposite ghost numbers, this symmetry is anomalous, and the anomaly is given by the r.h.s. of (3.34), which calculates the number of zero modes of the twisted Dirac operator (in other words, the r.h.s. is minus the ghost number of the vacuum). As usual in quantum field theory, the operators with non-trivial vacuum expectation values have to soak up the zero modes associated to the anomaly. It is interesting to note that, for Calabi–Yau threefolds, i.e. Kähler manifolds of complex dimension 3, and such that $c_1(X) = 0$, the last term in (3.34) vanishes and the ghost number anomaly is $6 - 6g$, as in the usual bosonic string. This also indicates that, for $g > 1$, the theory is trivial since the correlation functions of topological observables vanish.

As we have seen, in such a cohomological field theory one can compute correlation functions by just doing a semi-classical computation. From the argument in (3.8), any field configuration which gives a nonzero contribution to $\{\mathcal{Q}, V\}$ will not contribute to the path integral, since it will be suppressed in the $\hbar \to 0$ limit. But it is clear by looking at (3.22) that any *holomorphic* map $x : \Sigma_g \to X$, i.e. a map satisfying

$$\partial_{\bar{z}} x^I = 0 \tag{3.35}$$

leads to a vanishing bosonic action. The holomorphic maps are the instanton configurations of the nonlinear sigma model on a Kähler target. Therefore, the semiclassical evaluation of the path integral in the A model will involve a sum over these instanton sectors. These sectors are classified topologically by the homology class

$$\beta = x_*[(\Sigma_g)] \in H_2(X, \mathbf{Z}). \tag{3.36}$$

Sometimes it is useful to introduce a basis $[S_a]$ of $H_2(X, \mathbf{Z})$, where $a = 1, \cdots, h^{1,1}(X)$, in such a way that we can expand β as

$$\beta = \sum_a n_a [S_a]. \tag{3.37}$$

The instanton sectors are then labelled by $h^{1,1}(X)$ integers n_a. These instantons are also called *worldsheet instantons*. If we introduce *the complexified Kähler parameters* with respect to this basis,

$$t^a = \int_{S_a} \mathcal{J}, \quad i = 1, \cdots, h^{1,1}(X), \tag{3.38}$$

where $\mathcal{J}$ is the complexified Kähler form of X, we can write

$$\int_{\Sigma_g} x^*(\mathcal{J}) = \int_\beta \mathcal{J} = \sum_a n_a t^a. \tag{3.39}$$

A simple analysis (see, for example, [4, 6]) shows that the contribution of an instanton sector to the path integral reduces to an integration over the moduli space of instantons in that sector.

We will now focus on operators of the form

$$\phi_A^{(0)} = A_{I\bar{J}} \chi^I \chi^{\bar{J}}, \tag{3.40}$$

where

$$A = A_{I\bar{J}} \mathrm{d}x^I \wedge \mathrm{d}x^{\bar{J}} \tag{3.41}$$

is a closed $(1,1)$-form. It can be seen that the 2-form operator

$$- A_{I\bar{J}} \Big(\partial_z x^I \partial_{\bar{z}} x^{\bar{J}} - \partial_{\bar{z}} x^I \partial_z x^{\bar{J}} \Big), \tag{3.42}$$

satisfies the descent equations (3.9) (see below, (3.121), for a derivation). Therefore, due to our general discussion, we can use to perturb the action. The corresponding integrated operator can be written as

$$\int_{\Sigma_g} x^*(\mathrm{i}A), \tag{3.43}$$

where A is the differential form (3.41). This perturbation has an easy interpretation: it corresponds to a shift

$$\mathcal{J} \to \mathcal{J} - \mathrm{i}A \tag{3.44}$$

in the second term of (3.30), involving the complexified Kähler form. We can interpret it as a deformation that changes the Kähler form chosen to construct the model. This means that the observables of the form (3.40), of which there are $h^{1,1}(X)$, correspond to *deformations of the Kähler structure*.

Let us now present some results for the correlation functions of the A model on a Calabi–Yau threefold X involving three operators $\phi_A^{(0)}$ (3.40). The selection rule (3.34) says that, in genus $g = 0$ (i.e. when the Riemann surface is a sphere) and on a Calabi–Yau threefold, these correlation functions are generically non-vanishing. The computation of a given correlation function involves summing over the different topological sectors of worldsheet instantons. In the trivial sector, i.e. when $\beta = 0$, the image of the sphere is a point in the target. The moduli space of instantons is just the target space X, and the correlation function is just

$$\kappa_{abc} = \int_X A_a \wedge A_b \wedge A_c, \tag{3.45}$$

which is the classical intersection number of the divisors associated to these forms. The non-trivial instanton sectors give an infinite series. The final answer is

$$C_{abc} = -\langle \phi_{A_a}^{(0)} \phi_{A_b}^{(0)} \phi_{A_c}^{(0)} \rangle = -\kappa_{abc} - \sum_\beta I_{0,3,\beta}(A_a, A_b, A_c) Q^\beta. \tag{3.46}$$

Here we have used the notation

$$Q_a = \mathrm{e}^{-t^a}, \qquad Q^\beta = \prod_a Q_a^{n_a}. \tag{3.47}$$

This correlation function is called the *Yukawa coupling* for historical reasons. The coefficient $I_{0,3,\beta}(A_a, A_b, A_c)$ 'counts' in an appropriate sense the number of holomorphic maps from the sphere to the Calabi–Yau that send the point of insertion of $\phi_{A_i}^{(0)}$ to the divisor D_i which is Poincaré dual to A_i. It can be shown that the coefficients $I_{0,3,\beta}(A_a, A_b, A_c)$ can be written as

$$I_{0,3,\beta}(A_a, A_b, A_c) = N_{0,\beta} \int_\beta A_a \int_\beta A_b \int_\beta A_c, \tag{3.48}$$

in terms of invariants $N_{0,\beta}$ that encode all the information about the three-point functions (3.46) of the topological sigma model. The invariants $N_{0,\beta}$ are our first example of *Gromov–Witten invariants*. It is convenient to put all these invariants together in a generating functional called the *prepotential*:

$$F_0(t) = -\frac{1}{3!} \kappa_{abc} t^a t^b t^c + \sum_\beta N_{0,\beta}\, Q^\beta. \tag{3.49}$$

This prepotential depends on the $h^{1,1}(X)$ complexified Kähler parameters of the Calabi–Yau X.

Exercise 3.2. Show that

$$\frac{\partial^3 F_0}{\partial t_a \partial t_b \partial t_c} = C_{abc} = -\langle A_a A_b A_c \rangle. \tag{3.50}$$

3.3 The type B model

The field content of the type B model is the following: a map $x : \Sigma_g \to X$, which is a scalar, commuting field, two sets of Grassmann fields $\eta^{\overline{I}}, \theta^{\overline{I}} \in x^*(\overline{TX})$, which are scalars on Σ_g, and a Grassmannian one-form on Σ_g, ρ^I_α, with values in $x^*(TX)$. We also have commuting auxiliary fields $F^I, F^{\overline{I}}$. We follow here the formulation presented in [7]. The $\mathcal{Q}$-transformations read

$$\begin{aligned} [\mathcal{Q}, x^I] &= 0, & \{\mathcal{Q}, \eta^{\overline{I}}\} &= 0, \\ [\mathcal{Q}, x^{\overline{I}}] &= \eta^{\overline{I}}, & \{\mathcal{Q}, \theta_I\} &= G_{I\overline{J}} F^{\overline{J}}, \\ \{\mathcal{Q}, \rho^I_z\} &= \partial_z x^I & [\mathcal{Q}, F^I] &= D_z \rho^I_{\bar{z}} - D_{\bar{z}} \rho^I_z + R^I{}_{J\overline{L}K} \eta^{\overline{L}} \rho^J_z \rho^K_{\bar{z}}, \\ \{\mathcal{Q}, \rho^I_{\bar{z}}\} &= \partial_{\bar{z}} x^I, & [\mathcal{Q}, F^{\overline{I}}] &= -\Gamma^{\overline{I}}_{\overline{JK}} \eta^{\overline{J}} F^{\overline{K}}, \end{aligned} \tag{3.51}$$

and satisfy $\mathcal{Q}^2 = 0$. Notice that $\mathcal{Q}$ acts differently on holomorphic and anti-holomorphic co-ordinates. In contrast to what happens in the type-A model, it depends explicitly on the splitting between holomorphic and anti-holomorphic co-ordinates on X, in other words, *it depends explicitly on the choice of complex structure on* X. If we interpret $\eta^{\overline{I}}$ as a basis for anti-holomorphic differential forms on X, the action of $\mathcal{Q}$ on x^I, $x^{\overline{I}}$ may be interpreted as the Dolbeault anti-holomorphic differential $\overline{\partial}$. The action for the theory is

$$\begin{aligned} S_B = \int_{\Sigma_g} \mathrm{d}^2 z \Big[& G_{I\overline{J}} \big(\partial_z x^I \partial_{\bar{z}} x^{\overline{J}} + \partial_{\bar{z}} x^I \partial_z x^{\overline{J}} \big) - \rho^I_z \big(G_{I\overline{J}} D_{\bar{z}} \eta^{\overline{J}} + D_{\bar{z}} \theta_I \big) \\ & - \rho^I_{\bar{z}} \big(G_{I\overline{J}} D_z \eta^{\overline{J}} - D_z \theta_I \big) - R^I{}_{J\overline{L}K} \eta^{\overline{L}} \rho^J_z \rho^K_{\bar{z}} \theta_I - G_{I\overline{J}} F^I F^{\overline{J}} \Big]. \end{aligned} \tag{3.52}$$

We can explicitly introduce the metric on Σ_g in this action and verify that it is $\mathcal{Q}$-exact:

$$S_B = \{\mathcal{Q}, V\}, \tag{3.53}$$

where V is now given by

$$V = \int_{\Sigma_g} \mathrm{d}^2 z \sqrt{g} \left[G_{I\bar{J}} g^{\mu\nu} \rho^I_\mu \partial_\nu x^{\bar{J}} - F^I \theta_I \right]. \tag{3.54}$$

Finally, we also have a $U(1)$ ghost number symmetry, in which x, η, θ and ρ have ghost numbers 0, 1, 1, and -1, respectively.

Since the action is $\mathcal{Q}$-exact, the theory is topological and the semi-classical approximation is exact. In contrast to the type-A model, only *constant* maps $x : \Sigma_g \to X$ contribute to the path integral in the B model. This is because the bosonic part of the action in (3.52) does *not* vanish unless x is constant (notice that it contains the bosonic part of the A model Lagrangian, plus its complex conjugate). Any nonconstant configuration will lead to a nonzero action which is suppressed as $\hbar \to 0$ in (3.8). It follows that path integrals in the type-B model reduce to integrals over X, as found by Witten [6].

What are the observables in this theory? It is easy to see that the operators in the $\mathcal{Q}$-cohomology are of the form

$$\mathcal{O}_\phi = \phi^{J_1 \cdots J_q}_{\overline{I}_1 \cdots \overline{I}_p} \eta^{\overline{I}_1} \cdots \eta^{\overline{I}_p} \theta_{J_1} \cdots \theta_{J_q}, \tag{3.55}$$

where

$$\phi = \phi^{J_1 \cdots J_q}_{\overline{I}_1 \cdots \overline{I}_p} \mathrm{d}x^{\overline{I}_1} \wedge \cdots \wedge \mathrm{d}x^{\overline{I}_p} \frac{\partial}{\partial x^{J_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{J_q}} \tag{3.56}$$

is an element of $H^p_{\overline{\partial}}(X, \wedge^q TX)$ (more precisely, these operators are closed on-shell, by using the equations of motion which set $F^I = 0$). Therefore, the $\mathcal{Q}$-cohomology is in one-to-one correspondence with the twisted Dolbeault cohomology of the target manifold X. We can then consider correlation functions of the form

$$\langle \prod_a \mathcal{O}_{\phi_a} \rangle. \tag{3.57}$$

This correlation function vanishes unless the following selection rule is satisfied:

$$\sum_a p_a = \sum_a q_a = d(1-g), \tag{3.58}$$

where g is the genus of the Riemann surface. This selection rule comes from a $U(1)_L \times U(1)_R$ anomalous, global symmetry of the model which refines the ghost number, as we will se below. Due to the arguments presented above, this correlation function can be computed in the semi-classical limit, where the path integral reduces to an integration over the target X. The product of operators in (3.57) corresponds to a form in $H^d_{\overline{\partial}}(X, \wedge^d TX)$. To integrate such a form over X we crucially need the Calabi–Yau condition. This arises as follows. One of the most important properties of Calabi–Yau manifolds (which can actually be taken as their defining feature) is that they have a holomorphic, nowhere-vanishing section Ω of the canonical bundle $K_X = \Omega^{d,0}(X)$. Since the section is nowhere-vanishing, the canonical line bundle is trivial and we recover the condition $c_1(K_X) = c_1(X) = 0$. This means in particular that we have an invertible map

$$' : \Omega^{0,p}(\wedge^q TX) \longrightarrow \Omega^{d-q,p}(X) \tag{3.59}$$

which sends ϕ in (3.56) to

$$\phi' \equiv \phi \wedge \Omega = \Omega_{I_1 \cdots I_q I_{q+1} \cdots I_d} \phi^{I_1 \cdots I_q}_{\overline{J}_1 \cdots \overline{J}_p} \mathrm{d}x^{I_{q+1}} \cdots \wedge \mathrm{d}x^{I_d} \wedge \mathrm{d}x^{\overline{J}_1} \cdots \wedge \mathrm{d}x^{\overline{J}_p} \tag{3.60}$$

where the $(d,0)$-form Ω is used to contract the indices. Since Ω is holomorphic, this descends to the $\overline{\partial}$-cohomology. It then follows that an element in $H^d_{\overline{\partial}}(X, \wedge^d TX)$ maps to an element in $H^{0,d}_{\overline{\partial}}(X)$. After further multiplication by Ω, one can then integrate a (d,d)-form over X. This is the prescription to compute correlation functions like (3.57).

An important example of this procedure is the case of a Calabi–Yau threefold, $d = 3$, and operators of the form

$$\mathcal{O}_\phi = \phi^J_{\overline{I}} \eta^{\overline{I}} \theta_J \tag{3.61}$$

associated to forms in $H^1_{\overline{\partial}}(X, TX)$, or by using (3.60), to forms in $H^{2,1}_{\overline{\partial}}(X)$. These operators are important since, as we will see, they correspond to infinitesimal deformations of the complex structure of X. The selection rule (3.58) says that we have to integrate three of these operators, and the correlation function reads in this case

$$C_{abc} = \langle \mathcal{O}_{\phi_a} \mathcal{O}_{\phi_b} \mathcal{O}_{\phi_c} \rangle = \int_X \Omega \wedge (\phi_a)^{I_1}_{\overline{J}_1} (\phi_b)^{I_2}_{\overline{J}_2} (\phi_c)^{I_3}_{\overline{J}_3} \, \Omega_{I_1 I_2 I_3} \mathrm{d}z^{\overline{J}_1} \wedge \mathrm{d}z^{\overline{J}_2} \wedge \mathrm{d}z^{\overline{J}_3}. \tag{3.62}$$

This correlation function is the B-model version of the Yukawa coupling. It turns out that, in contrast to the A-model correlation functions, the Yukawa couplings of the B-model can be computed more or less straightforwardly. In order to do this, we have to relate them to a more general geometric problem which is the *variation of complex structures* of the CY manifold X, and which is addressed in the next subsection.

Some remarks are in order:

1. The computation of the Yukawa coupling depends on the normalization of Ω. Since this is a nowhere vanishing holomorphic form, it can be redefined by multiplying it by a nonvanishing constant on X. As we will see when we consider the problem in the context of variation of complex structures, this means that Ω is actually a *section* of a line bundle $\mathcal{L}$ over the moduli space of complex structures $\mathcal{M}$, therefore C_{abc} is a section of $\mathcal{L}^2$.

2. Notice that the map (3.60) is invertible. In the three-fold case, the only one we will consider in certain detail, the inverse map is simply

$$\phi^I_{\overline{J}} = \frac{1}{2|\Omega|^2}\overline{\Omega}^{IKL}\phi'_{KL\overline{J}} \tag{3.63}$$

where

$$|\Omega|^2 = \frac{1}{3!}\Omega_{IJK}\overline{\Omega}^{IJK} \tag{3.64}$$

3. One can use the operators (3.61) and the descent procedure to construct topological perturbations of the B-model. Here, these perturbations are more subtle, as pointed out in [6], due to the fact that the observables are only closed on-shell. A detailed treatment of these perturbations and their connection to Kodaira–Spencer theory can be found in [7].

3.4 $\mathcal{N} = 2$ supersymmetry

Let us first summarize the main results on the topological sigma models.

	A model	B model
field configurations	holomorphic maps	constant maps
moduli/observables	Kähler parameters	complex parameters
prepotential F_0	Gromov–Witten invariants	period integrals

In this and the next section, we derive these models from what is called *twisted* $\mathcal{N} = 2$ *supersymmetry*. First we give a brief summary of the $\mathcal{N} = 2$ supersymmetric sigma model. We will follow very closely the presentation of [5], but adapt the conventions to those in [8]. This amounts to exchange $z \leftrightarrow \bar{z}$ in [5].

Let us first state our notation and conventions for $\mathcal{N} = 2$ supersymmetry. Our choice of Euclidean Dirac matrices γ^μ is

$$(\gamma^1)_\alpha{}^\beta = \sigma^1, \qquad (\gamma^2)_\alpha{}^\beta = -\sigma^2,$$

where σ^1, σ^2 are Pauli matrices. We will denote the spinor indices by $\alpha = +, -$, and they are lowered and raised by the matrix $C_{\alpha\beta} = \sigma^1$, so that $(\gamma^1)_{\alpha\beta} = \mathbf{1}_{\alpha\beta}$, $(\gamma^2)_{\alpha\beta} = \mathrm{i}(\sigma^3)_{\alpha\beta}$.

The generators of $\mathcal{N} = 2$ supersymmetry are denoted by $Q_{\alpha a}$, where $\alpha = +, -$ are Lorentz indices, and $a = +, -$ are R-charge indices. The $\mathcal{N} = 2$ supersymmetry algebra contains the following relations:

$$\begin{aligned} \{Q_{\alpha+}, Q_{\beta-}\} &= \gamma^\mu_{\alpha\beta}P_\mu, & [F_R, Q_{+\pm}] &= \pm\frac{1}{2}Q_{+\pm}, \\ \{Q_{\alpha\pm}, Q_{\beta\pm}\} &= 0, & [F_R, Q_{-\pm}] &= 0, \\ [J, Q_{\pm a}] &= \pm\frac{1}{2}Q_{\pm a}, & [F_L, Q_{+\pm}] &= 0, \\ & & [F_L, Q_{-\pm}] &= \pm\frac{1}{2}Q_{-\pm}. \end{aligned} \tag{3.65}$$

We have assumed that there are no central charges in the algebra. J is the generator of Lorentz $SO(2)$ transformations and $F_{L,R}$ are the left and right internal $U(1)$ currents, respectively. They combine into vectorial and axial currents F_V, F_A as follows:

$$F_V = F_R + F_L, \qquad F_A = F_R - F_L. \tag{3.66}$$

Notice that

$$\{Q_{-,+}, Q_{-,-}\} = 2\partial_z, \qquad \{Q_{+,+}, Q_{+,-}\} = 2\partial_{\bar{z}}. \tag{3.67}$$

We will consider here a theory for an $\mathcal{N}=2$ chiral multiplet. A convenient description of this multiplet is by using $\mathcal{N}=2$ superspace in two dimensions, which is essentially identical to the usual $\mathcal{N}=1$ superspace in four dimensions. In $\mathcal{N}=2$ superspace we have superspace covariant derivatives $D_{\alpha a}$ satisfying the following algebra:

$$\{D_{-,+}\ , D_{-,-}\} = 2\partial_z, \quad \{D_{+,+}\ , D_{+,-}\} = 2\partial_{\bar{z}}, \tag{3.68}$$

while all other anti-commutators among the $D_{\alpha a}$ vanish. The two basic $\mathcal{N}=2$ multiplets are described by a scalar $\mathcal{N}=2$ superfield Φ satisfying the following relations:

$$\begin{aligned} &D_{+,-}\Phi = D_{-,-}\Phi = 0, \qquad \text{chiral}, \\ &D_{+,-}\Phi = D_{-,+}\Phi = 0, \qquad \text{twisted chiral}. \end{aligned} \tag{3.69}$$

There exist also the antichiral and the twisted antichiral versions of these multiplets,

$$\begin{aligned} &D_{+,+}\overline{\Phi} = D_{-,+}\overline{\Phi} = 0, \qquad \text{antichiral}, \\ &D_{+,+}\overline{\Phi} = D_{-,-}\overline{\Phi} = 0, \qquad \text{twisted antichiral}. \end{aligned} \tag{3.70}$$

Let us now consider a collection of d chiral superfields Φ^I, and d anti-chiral superfields $\Phi^{\overline{I}}$, where $I, \overline{I} = 1, \cdots, d$. We can define component fields for these superfields as follows:

$$\begin{aligned} \Phi^I| &= x^I, & \Phi^{\overline{I}}| &= x^{\overline{I}}, \\ D_{\alpha,+}\Phi^I| &= \psi^I_{\alpha,+}, & D_{\alpha,-}\Phi^{\overline{I}}| &= \psi^{\overline{I}}_{\alpha,-}, \\ D_{-,+}D_{+,+}\Phi^I| &= F^I_{-+,++}, & D_{+,-}D_{-,-}\Phi^{\overline{I}}| &= F^{\overline{I}}_{+-,--}, \end{aligned} \tag{3.71}$$

where the vertical bar means that we take the component of the superfield with $\theta = 0$. Here, $F^I_{-+,++}$ and $F^{\overline{I}}_{+-,--}$ are auxiliary fields. We can also write down very easily the supersymmetry transformations of the different component fields under the $\mathcal{N}=2$ supersymmetry. For an $\mathcal{N}=2$ superfield the transformation rule takes the form

$$\delta\Phi = \eta^{\alpha a} Q_{\alpha a}\Phi, \tag{3.72}$$

where $\eta^{\alpha a}$ is a constant $\mathcal{N} = 2$ supersymmetry parameter. Projecting onto components, and using the definitions (3.71), one finds

$$\begin{aligned}
\delta x^I &= \eta^{+,+}\psi^I_{+,+} + \eta^{-,+}\psi^I_{-,+}, \\
\delta \psi^I_{+,+} &= \eta^{-,+}F^I_{-+,++} + 2\eta^{+,-}\partial_{\bar{z}}x^I, \\
\delta \psi^I_{-,+} &= -\eta^{+,+}F^I_{-+,++} + 2\eta^{-,-}\partial_z x^I, \\
\delta F^I_{-+,++} &= 2\eta^{-,-}\partial_z\psi^I_{+,+} - 2\eta^{+,-}\partial_{\bar{z}}\psi^I_{-,+}, \\
\delta x^{\bar{I}} &= \eta^{-,-}\psi^{\bar{I}}_{-,-} + \eta^{+,-}\psi^{\bar{I}}_{+,-}, \\
\delta \psi^{\bar{I}}_{-,-} &= \eta^{+,-}F^{\bar{I}}_{+-,--} + 2\eta^{-,+}\partial_z x^{\bar{I}}, \\
\delta \psi^{\bar{I}}_{+,-} &= -\eta^{-,-}F^{\bar{I}}_{-+,--} + 2\eta^{+,+}\partial_{\bar{z}}x^{\bar{I}}, \\
\delta F^{\bar{I}}_{+-,--} &= 2\eta^{+,+}\partial_{\bar{z}}\psi^{\bar{I}}_{-,-} - 2\eta^{-,+}\partial_z\psi^{\bar{I}}_{+,-}.
\end{aligned} \tag{3.73}$$

The transformations under the R-symmetry can be read off from the $U(1)$ indices of the fields.

The supersymmetric sigma model is defined by the following action in superspace for the superfields Φ^I, $\Phi^{\bar{I}}$, $I, \bar{I} = 1, \cdots, d$:

$$S_K = \frac{1}{2}\int \mathrm{d}^2 z\, \mathrm{d}^4\theta\, K(\Phi^I, \Phi^{\bar{I}}). \tag{3.74}$$

Geometrically, this is a sigma model whose target is a Kähler manifold of complex dimension d, with local complex co-ordinates given by x^I, $x^{\bar{I}}$. The Kähler potential is $K(x^I, x^{\bar{I}})$, while the Kähler metric is given by

$$G_{I\bar{J}} = \frac{\partial^2 K}{\partial x^I \partial x^{\bar{J}}}. \tag{3.75}$$

The odd part of the measure can be expressed in terms of covariant derivatives as follows

$$\mathrm{d}^4\theta \to D_{-,+}D_{+,+}D_{-,-}D_{+,-}. \tag{3.76}$$

This makes possible to compute (3.74) efficiently. A simple computation indeed produces the action

$$\begin{aligned}
&\frac{1}{2}\int \mathrm{d}^2 z\Big[G_{I\bar{J}}\big(-F^I_{-+,++}F^{\bar{J}}_{+-,--} - 2\psi^I_{++}D_z\psi^{\bar{J}}_{+-} - 2\psi^{\bar{J}}_{-+}D_z\psi^I_{--} + 2\partial_z x^I\, \partial_{\bar{z}}x^{\bar{J}} + 2\partial_{\bar{z}}x^I\partial_z x^{\bar{J}}\big) \\
&+ \partial_K\partial_{\bar{L}}G_{I\bar{J}}\psi^K_{++}\psi^{\bar{L}}_{--}\psi^I_{-+}\psi^{\bar{J}}_{+-} - \partial_K G_{I\bar{J}}\psi^I_{-+}F^{\bar{J}}_{+-,--}\psi^K_{++} - \partial_{\bar{K}}G_{I\bar{J}}\psi^{\bar{K}}_{+-}F^I_{-+,++}\psi^{\bar{J}}_{--}\Big].
\end{aligned} \tag{3.77}$$

In this equation, D_μ is the covariant derivative on sections of the pull-back of the tangent bundle,

$$D_\mu\psi^I_{-+} = \partial_\mu\psi^I_{-+} + (\partial_\mu x^J)\Gamma^I_{JK}\psi^K_{-+}, \tag{3.78}$$

The action (3.77) is not covariant, but can be made so by redefining the auxiliary fields

$$\begin{aligned}
F^I_{-+,++} &\to F^I_{-+,++} + \Gamma^I_{KJ}\psi^J_{-+}\psi^K_{++}, \\
F^{\bar{I}}_{+-,--} &\to F^{\bar{I}}_{+-,--} + \Gamma^{\bar{I}}_{\bar{K}\bar{J}}\psi^{\bar{J}}_{+-}\psi^{\bar{K}}_{--}.
\end{aligned} \tag{3.79}$$

The action (3.77) then becomes,

$$S_K = \frac{1}{2}\int \mathrm{d}^2 z\Big[G_{I\bar{J}}\big(-4x^I\partial_z\partial_{\bar{z}}x^{\bar{J}} - 2\psi^I_{++}D_z\psi^{\bar{J}}_{+-} - 2\psi^{\bar{J}}_{-+}D_z\psi^I_{--} - F^I_{-+,++}F^{\bar{J}}_{+-,--}\big)$$
$$+ R_{\bar{I}J\bar{K}L}\psi^K_{++}\psi^{\bar{L}}_{--}\psi^I_{-+}\psi^{\bar{J}}_{+-}\Big]. \tag{3.80}$$

This is not the most general action that can be written in superspace, since it only involves D-terms. One can add F-terms, *i.e.* a superpotential $W(X^I)$ for the chiral multiplets:

$$\int d^2 z\big(d^2\theta W(X^I) + \widehat{d^2\theta}\overline{W}(X^{\bar{I}})\big) \tag{3.81}$$

Here, the odd part of the measure is expressed as

$$\begin{aligned}\mathrm{d}^2\theta &\to D_{+,+}D_{-,+},\\ \widehat{\mathrm{d}^2\theta} &\to D_{-,-}D_{+,-}.\end{aligned} \tag{3.82}$$

This leads to Landau–Ginzburg models in two dimensions. One can also add a different set of potential terms that do not admit a superspace representation. We will not consider these more general models here, although they lead to very interesting topological field theories in two dimensions. In particular, Landau–Ginzburg models play a crucial role in many developments of topological string theory.

It is also possible to consider a superpotential for *twisted* chiral superfields. In that case, the odd part of the measure reads

$$\begin{aligned}\mathrm{d}^2\theta &\to D_{+,+}D_{-,-},\\ \widehat{\mathrm{d}^2\theta} &\to D_{-,+}D_{+,-}.\end{aligned} \tag{3.83}$$

3.5 Topological twist

The supersymmetric sigma model that we defined in the last section can be *twisted* in two different ways to produce two inequivalent topological quantum field theories in two dimensions. These two inequivalent twisting procedures are called the A-twist and the B-twist, and they give rise to the topological type-A sigma model and the topological type-B sigma model, respectively. The A-twisting was introduced by Witten in [4], while the B-twisting was introduced in [5, 6, 9].

The twisting procedure amounts to a redefinition of the spin of the fields (equivalently, of the energy-momentum tensor of the theory) by using the internal F_V or F_A currents. In the A-twist, one redefines the spin current as follows:

$$\mathrm{A-twist}: \quad \widetilde{J} = J + F_V, \tag{3.84}$$

while in the B-twist the redefinition is given by

$$\mathrm{B-twist}: \quad \widetilde{J} = J + F_A. \tag{3.85}$$

There are other possible (albeit equivalent) conventions. The convention used here is the one in [8].

The above redefinition means that we are replacing the $U(1)_E$ Lorentz symmetry by the diagonal embedding $U(1)'_E \subset U(1)_E \times U(1)_{V,A}$ for the A- and the B-twist, respectively. It is very illuminating to make a table where we write down the quantum numbers of all the components

	$U(1)_V$	$U(1)_A$	$U(1)_E$	A-twist $U(1)'_E$	B-twist $U(1)'_E$
$Q_{+,+}$	$+1/2$	$+1/2$	$+1/2$	-1	-1
$Q_{-,+}$	$+1/2$	$-1/2$	$-1/2$	0	$+1$
$Q_{+,-}$	$-1/2$	$-1/2$	$+1/2$	0	0
$Q_{-,-}$	$-1/2$	$+1/2$	$-1/2$	$+1$	0

Table 1. Quantum numbers of $Q_{\alpha a}$ under the different $U(1)$ symmetries.

of $Q_{\alpha a}$ under the different symmetries involved. We see that in both cases one gets two *scalar* supercharges, and one vector-valued supercharge. This suggests defining the following operator, also called the *topological charge*,

$$\begin{aligned} \mathrm{A-twist:} \quad & \mathcal{Q} = Q_{+,-} + Q_{-,+}, \\ \mathrm{B-twist:} \quad & \mathcal{Q} = Q_{+,-} + Q_{-,-}, \end{aligned} \tag{3.86}$$

which is a scalar in the resulting theories. $\mathcal{Q}$ is a Grassmannian, scalar charge, so twisted theories violate the spin-statistics theorem. We will also define a vector charge G_μ through the following equations:

$$\begin{aligned} \mathrm{A-twist:} \quad & G_z = Q_{-,-}, \quad G_{\bar z} = Q_{+,+}, \\ \mathrm{B-twist:} \quad & G_z = Q_{-,+}, \quad G_{\bar z} = Q_{+,+}. \end{aligned} \tag{3.87}$$

One can check that, as a consequence of the supersymmetry algebra, the topological charge is nilpotent,

$$\mathcal{Q}^2 = 0, \tag{3.88}$$

and also that

$$\{\mathcal{Q}, G_\mu\} = P_\mu. \tag{3.89}$$

These are the most important relations characterizing the so-called topological algebra that is obtained by the twist of the $\mathcal{N}=2$ algebra.

The transformations of the fields under the topological charge $\mathcal{Q}$ can be simply obtained from the supersymmetry transformations (3.73). In the case of the A-twists, it is convenient to define the fields

$$\begin{aligned} \chi^I &= \psi^I_{-,+}, & \rho^I_{\bar z} &= \psi^I_{+,+}, & \tilde F^I_{\bar z} &= F^I_{-+,++}, \\ \chi^{\bar I} &= \psi^{\bar I}_{+,-}, & \rho^{\bar I}_z &= \psi^{\bar I}_{-,-}, & \tilde F^{\bar I}_z &= F^{\bar I}_{+-,--}. \end{aligned} \tag{3.90}$$

The action (3.80) becomes, after the twisting,

$$\begin{aligned} S = \frac{1}{2}\int_{\Sigma_g} \mathrm{d}^2 z \Big[& G_{I\bar J}\big(-\tilde F^I_{\bar z}\tilde F^{\bar J}_z - 2\rho^I_{\bar z} D_z \chi^{\bar J} - 2\rho^{\bar J}_z D_{\bar z}\chi^I + 2\partial_{\bar z}x^I\partial_z x^{\bar J} + 2\partial_z x^I \partial_{\bar z} x^{\bar J}\big) \\ & + \partial_K\partial_{\bar L} G_{I\bar J}\rho^K_{\bar z}\rho^{\bar L}_z \chi^I\chi^{\bar J} - \partial_K G_{I\bar J}\chi^I \tilde F^{\bar J}_z \rho^K_{\bar z} - \partial_{\bar K} G_{I\bar J}\chi^{\bar K}\tilde F^I_{\bar z}\rho^{\bar J}_z \Big]. \end{aligned} \tag{3.91}$$

The topological charge $\mathcal{Q}$ acts as follows on the fields:

$$\begin{aligned} [\mathcal{Q}, x^i] &= \chi^i, & \{\mathcal{Q}, \rho^I_{\bar z}\} &= 2\partial_{\bar z}x^I + \tilde F^I_{\bar z}, & [\mathcal{Q}, \tilde F^I_{\bar z}] &= -2\partial_{\bar z}\chi^I, \\ \{\mathcal{Q}, \chi^i\} &= 0, & \{\mathcal{Q}, \rho^{\bar I}_z\} &= 2\partial_z x^{\bar I} + \tilde F^{\bar I}_z, & [\mathcal{Q}, \tilde F^{\bar I}_z] &= -2\partial_z\chi^{\bar I}. \end{aligned}$$

In particular, we can write

$$S_\sigma = S_K - \mathrm{i}\int_{\Sigma_g} x^*(B) \tag{3.92}$$

as

$$S_\sigma = \int \mathrm{d}^2 z\, \{Q, v_G\} - x^*(\mathcal{J}), \tag{3.93}$$

where

$$\begin{aligned} v_G = {}& \frac{1}{2} G_{I\bar{J}}\left[\rho^I_{\bar{z}}\partial_z x^{\bar{J}} + \rho^{\bar{J}}_z \partial_{\bar{z}} x^I - \frac{1}{2}\rho^I_{\bar{z}}\tilde{F}^{\bar{J}}_z - \frac{1}{2}\rho^{\bar{J}}_z \tilde{F}^I_{\bar{z}}\right] \\ & + \frac{1}{4}\Big(\partial_K G_{I\bar{J}}\chi^K \rho^{\bar{J}}_z \rho^I_{\bar{z}} + \partial_{\overline{K}} G_{I\bar{J}}\chi^{\overline{K}}\rho^I_{\bar{z}}\rho^{\bar{J}}_z\Big). \end{aligned} \tag{3.94}$$

Later on it will be useful to think regard v_G as a functional defined for an arbitrary $(1,1)$ tensor, and not only the metric $G_{I\bar{J}}$. In order to obtain (3.22) we redefine the auxiliary fields as in (3.79):

$$F^I_{\bar{z}} = \tilde{F}^I_{\bar{z}} + \Gamma^I_{KJ}\chi^J \rho^K_{\bar{z}}, \quad F^{\bar{I}}_z = \tilde{F}^{\bar{I}}_z + \Gamma^{\bar{I}}_{\overline{KJ}}\chi^{\bar{J}}\rho^{\overline{K}}_z. \tag{3.95}$$

Finally, we point out that the ghost number current of the A model is just $-2F_A$.

To make contact with the B-model introduced above it is convenient to define the following fields:

$$\begin{aligned} &\rho^I_z = 2\psi^I_{-,+}, \qquad \chi^{\bar{I}} = \psi^{\bar{I}}_{+,-}, \qquad F^I = 2F^I_{-+,++}, \\ &\rho^I_{\bar{z}} = 2\psi^I_{+,+} \qquad \overline{\chi}^{\bar{I}} = \psi^{\bar{I}}_{-,-}, \qquad F^{\bar{I}} = 2F^{\bar{I}}_{+-,--}. \end{aligned} \tag{3.96}$$

as well as the following redefinition suggested by Witten in [6]:

$$\begin{aligned} \eta^{\bar{I}} &= \chi^{\bar{I}} + \overline{\chi}^{\bar{I}}, \\ \theta_I &= G_{I\bar{J}}(\chi^{\bar{J}} - \overline{\chi}^{\bar{J}}). \end{aligned} \tag{3.97}$$

In this way one recovers the type B topological sigma model. We also point out that the ghost number current of the model is now $-2F_V$, and that the refined ghost numbers which were used in (3.58) come precisely from F_L, F_R.

3.6 Topological sigma models as twisted superconformal field theories

The $\mathcal{N} = (2,2)$ supersymmetric sigma model turns out to be conformally invariant precisely in the Calabi–Yau case $c_1(X) = 0$. This is due to the fact that the β function is proportional to the Ricci tensor. Therefore, the supersymmetry algebra (3.65) gets promoted in this case to the full $\mathcal{N} = (2,2)$ superconformal algebra. This algebra has two isomorphic sectors (L and R), and we will put a bar over the generators in the R sector. In the L sector we have two supersymmetric currents $G^\pm(z)$, an energy momentum tenson $T(z)$, and an R-current $J(z)$. The modes of these currents satisfy the superconformal algebra

$$
\begin{aligned}
\{G^-_r, G^+_s\} &= 2L_{r+s} - (r-s)J_{r+s} + \frac{c}{3}(r^2 - 1/4)\delta_{r+s,0}, \\
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \\
[L_n, G^\pm_r] &= \left(\frac{n}{2} - r\right)G^\pm_{n+r} \\
[L_n, J_m] &= -mJ_{n+m}, \\
[J_n, J_m] &= \frac{c}{3}n\delta_{n+m,0}, \\
[J_n, G^\pm_r] &= \pm G^\pm_{n+r},
\end{aligned}
\tag{3.98}
$$

where c is the central charge. There is a similar set of relations for the R sector. The currents $G^\pm$ have conformal weight $h = 3/2$, and we point out that in a *unitary* theory $G^+ = (G^-)^\dagger$. We will consider the NS sector of the algebra where $m, n \in \mathbb{Z}$ while $r, s \in \mathbb{Z} + \frac{1}{2}$.

The supersymmetric algebra is present in the superconformal algebra by doing the following identifications for the supercharges:

$$
\begin{aligned}
Q_{+,+} &= \overline{G}^-_{-1/2}, \qquad Q_{-,+} = G^+_{-1/2}, \\
Q_{+,-} &= \overline{G}^+_{-1/2}, \qquad Q_{-,-} = G^-_{-1/2}.
\end{aligned}
\tag{3.99}
$$

For the bosonic generators we have

$$
\begin{aligned}
P_z &= L_{-1}, \qquad P_{\bar{z}} = \overline{L}_{-1}, \\
F_L &= \frac{1}{2}J_0, \qquad F_R = -\frac{1}{2}\overline{J}_0
\end{aligned}
\tag{3.100}
$$

while the spin J is given by

$$
J = L_0 - \overline{L}_0.
$$

Let us now discuss the structure of the twisted theory after taking into account the extra structure provided by conformal symmetry. We will discuss the A-model in detail. A similar, "mirror" discussion, applies to the B-model.

After the A-twisting, two of the G currents have conformal weight $h = 1$, and lead to

$$
G^+_{-1/2}, \quad \overline{G}^+_{-1/2}, \rightarrow Q(z), \quad \overline{Q}(\bar{z}),
\tag{3.101}
$$

which add up to $\mathcal{Q}$. The other two currents have spin 2 and lead to

$$
G^-_{-1/2}, \quad \overline{G}^-_{-1/2} \rightarrow G(z), \quad \overline{G}(\bar{z}),
\tag{3.102}
$$

which are the operators introduced in (3.87). Since the conformal weight of the fields have changed, their modding is changed as well. We have,

$$
\begin{aligned}
G(z) &= \sum_{n \in \mathbb{Z}} G_n z^{-n-2}, \qquad \overline{G}(\bar{z}) = \sum_{n \in \mathbb{Z}} \overline{G}_n \bar{z}^{-n-2}, \\
Q(z) &= \sum_{n \in \mathbb{Z}} Q_n z^{-n-1}, \qquad \overline{Q}(\bar{z}) = \sum_{n \in \mathbb{Z}} \overline{Q}_n \bar{z}^{-n-1},
\end{aligned}
\tag{3.103}
$$

therefore the twisting leads to the following change of modding in the superconformal algebra,

$$\begin{aligned} G_s^+ &\to Q_m, \\ s &\to m - \frac{1}{2}, \end{aligned} \tag{3.104}$$

and

$$\begin{aligned} G_r^- &\to G_n, \\ s &\to n + \frac{1}{2}, \end{aligned} \tag{3.105}$$

Notice that, since $Q(z)$ and $\overline{Q}(\bar{z})$ have spin 1, we can form two conserved, *scalar* charges

$$Q_0 = \oint \mathrm{d}z\, Q(z), \qquad \overline{Q}_0 = \oint \mathrm{d}z\, \overline{Q}(z), \tag{3.106}$$

and the scalar supercharge (3.86) is given by

$$\mathcal{Q} = Q_0 + \overline{Q}_0. \tag{3.107}$$

In order to have a consistent conformal algebra, we have to modify the energy-momentum tensor of the theory as follows

$$T(z) \to T(z) + \frac{1}{2}\partial J(z) \tag{3.108}$$

and similarly for the R sector. It follows that the spin generator becomes

$$J = \oint \mathrm{d}z(zT(z) - \bar{z}T(\bar{z})) \to L_0 - \overline{L}_0 + \frac{1}{2}(J_0 - \overline{J}_0) = J + F_V, \tag{3.109}$$

which was our previous definition of twist (3.85) for the A model. Finally, exactness of the energy-momentum tensor means in this language that

$$T(z) = \{\mathcal{Q}, G(z)\}. \tag{3.110}$$

One can check that the new currents form an algebra, sometimes called the *conformal topological algebra* [10]. The CFT point of view makes clear that there are in fact *two* different, independent topological charges (one for each chiral sector). Equivalently, together with $\mathcal{Q}$ there is another nilpotent scalar charge

$$M = Q_0 - \overline{Q}_0 \tag{3.111}$$

as well as two other charges G_z, $G_{\bar{z}}$. A perturbation of the action in CFT is given by

$$\sum_a t^a \int \mathrm{d}^2 z\, \phi_a^{(2)} + \sum_a \bar{t}^a \int \mathrm{d}^2 z\, \overline{\phi}_a^{(2)}, \tag{3.112}$$

where

$$\phi_a^{(2)} = \{G^-, [\overline{G}^-, \phi_a^{(0)}]\}, \qquad \overline{\phi}_a^{(2)} = \{G^+, [\overline{G}^+, \overline{\phi}_a^{(0)}]\}. \tag{3.113}$$

In CFT, the fields $\phi_a^{(0}$, $\overline{\phi}_a^{(0)}$ are chiral (respectively, antichiral) fields. Indeed, one can see that a (q,q) field becomes a $\mathcal{Q}$-closed operator in the A model, while in the B-model it is a (a,c) field that becomes a topological observable. Also, in CFT, the condition for the perturbation to be *marginal* is that the operator $\phi_a^{(0}$ has conformal weights $(h,\tilde{h}) = (1/2,1/2)$. For an (a,a) field this means that its $J_0, \overline{J}_0$ charges must be $(1,1)$. In the A-model, this corresponds precisely to the operators (3.40).

Notice that, in the twisted topological theory, the *holomorphic* perturbation which couples to t^a becomes

$$\{G_z, [G_{\bar z}, \phi_a]\} \tag{3.114}$$

which is nothing but the two-form descendant $(\phi_a^{(2)})_{z\bar z}$. The *antiholomorphic* perturbation which couples to $\bar t^a$ can be written as

$$\{Q_0, [\overline{Q}_0, \overline{\phi}_a^{(0)}]\} = -\frac{1}{2}\{\mathcal{Q}, [M, \overline{\phi}_a^{(0)}]\}. \tag{3.115}$$

and it is $\mathcal{Q}$ exact, therefore it should decouple from correlation functions. It turns out that this is not the case, and this leads to the famous *holomorphic anomaly*, as we will discuss.

Let us work out the extended topological algebra in the A model. The action of the M, $G_{z,\bar z}$ operators is easiliy obtained from (3.73) and reads,

$$\begin{aligned}
&[M, x^I] = -\chi^I, && [M, x^{\bar I}] = \chi^{\bar I},\\
&[G_z, x^I] = 0, && [G_z, x^{\bar I}] = \frac{1}{2}\rho_z^{\bar I},\\
&[G_{\bar z}, x^I] = \frac{1}{2}\rho_{\bar z}^I, && [G_{\bar z}, x^{\bar I}] = 0,
\end{aligned} \tag{3.116}$$

$$\begin{aligned}
&\{M, \chi^I\} = 0, && \{M, \chi^{\bar I}\} = 0,\\
&\{G_z, \chi^I\} = \partial_z x^I, && \{G_z, \chi^{\bar I}\} = -\frac{1}{2}\tilde F_z^{\bar I},\\
&\{G_{\bar z}, \chi^I\} = -\frac{1}{2}\tilde F_{\bar z}^I, && \{G_{\bar z}, \chi^{\bar I}\} = \partial_{\bar z} x^{\bar I},
\end{aligned} \tag{3.117}$$

$$\begin{aligned}
&\{M, \rho_{\bar z}^I\} = 2\partial_{\bar z} x^I - \tilde F_{\bar z}^I, && \{M, \rho_z^{\bar I}\} = -2\partial_z x^{\bar I} + \tilde F_z^{\bar I},\\
&\{G_z, \rho_{\bar z}^I\} = 0, && \{G_z, \rho_z^{\bar I}\} = 0,\\
&\{G_{\bar z}, \rho_{\bar z}^I\} = 0, && \{G_{\bar z}, \rho_z^{\bar I}\} = 0,
\end{aligned} \tag{3.118}$$

$$\begin{aligned}
&[M, \tilde F_z^I] = -2\partial_z \chi^I, && [M, \tilde F_{\bar z}^{\bar I}] = 2\partial_{\bar z}\chi^{\bar I},\\
&[G_z, \tilde F_z^I] = 0, && [G_z, \tilde F_{\bar z}^{\bar I}] = \partial_z \rho_{\bar z}^{\bar I},\\
&[G_{\bar z}, \tilde F_z^I] = \partial_{\bar z}\rho_z^I, && [G_{\bar z}, \tilde F_{\bar z}^{\bar I}] = 0.
\end{aligned} \tag{3.119}$$

Here, $\tilde F_z^{\bar I}$, $\tilde F_{\bar z}^I$ are the auxililary fields introduced in (3.90).

We can now compute the perturbation (3.113). The first perturbation in (3.113) is nothing but the descendant field $\phi^{(2)}$ associated to an operator ϕ_0. We will assume that $\phi^{(0)}$ satisfies

$$[Q_0, \phi^{(0)}] = [\overline{Q}_0, \phi^{(0)}] = 0 \tag{3.120}$$

so that, from the point of view of $\mathcal{N} = 2$ supersymmetry, it corresponds to a *twisted* chiral operator. Let us consider operators of the form (3.40) where (3.41) satisfies $\partial A = \overline{\partial} A = 0$ to comply with (3.120). A simple computation produces the following result for the canonical descendant,

$$
\begin{aligned}
\phi_A^{(2)} &= \{G_z, [G_{\bar z}, \phi^{(0)}]\} \\
&= -\frac{1}{4}\Big[A_{I\bar J}\big(-\tilde F^I_{\bar z}\tilde F^{\bar J}_z - 2\rho^I_{\bar z} D_z\chi^{\bar J} - 2\rho^{\bar J}_z D_{\bar z}\chi^I + 4\partial_z x^I\partial_{\bar z}x^{\bar J}\big) \\
&\quad + \partial_K\partial_{\bar L}A_{I\bar J}\rho^K_{\bar z}\rho^{\bar L}_z\chi^I\chi^{\bar J} - \partial_K A_{I\bar J}\chi^I\tilde F^{\bar J}_z\rho^K_{\bar z} - \partial_{\bar K}A_{I\bar J}\chi^{\bar K}\tilde F^I_{\bar z}\rho^{\bar J}_z\Big],
\end{aligned}
\tag{3.121}
$$

which can be written as

$$
\phi_A^{(2)} = -A_{I\bar J}\Big(\partial_z x^I\partial_{\bar z}x^{\bar J} - \partial_{\bar z}x^I\partial_z x^{\bar J}\Big) - \frac{1}{2}\{\mathcal{Q}, v_A\},
\tag{3.122}
$$

where v_A is the operator introduced in (3.94), but expressed in terms of the tensor $A_{I\overline{J}}$ instead of $G_{I\overline{J}}$. Therefore, up to a $\mathcal{Q}$-exact piece, this is the two-form (3.42) on Σ_g associated to A, as it should be since (3.122) is just the canonical solution to the descent equations.

What is $\overline{\phi}^{(0)}$? This should correspond to a twisted antichiral field, i.e. it has to satisfy

$$
\{G_z, \overline{\phi}^{(0)}\} = \{G_{\bar z}, \overline{\phi}^{(0)}\} = 0.
\tag{3.123}
$$

A natural candidate is the operator

$$
\overline{\phi}_A^{(0)} = \frac{1}{4}A_{I\overline{J}}\rho^I_{\bar z}\rho^{\overline{J}}_z = -\frac{\mathrm{i}}{8}\epsilon^{\mu\nu}A_{I\overline{J}}\rho^I_\mu\rho^{\overline{J}}_\nu
\tag{3.124}
$$

which is a two-form. These operators can be regarded as observables of the *antitopological* theory [11].

Exercise 3.3. Check that the operator $\overline{\phi}^{(0)}$ indeed satisfies (3.123). For this one needs that

$$
\Gamma^I_{JK}\rho^J_{\bar z}\rho^K_{\bar z} = 0
\tag{3.125}
$$

due to the symmetry of the Christoffel symbol.

The perturbed operator corresponding to (3.124) can be easily computed. First notice that

$$
[M, \overline{\phi}_A^{(0)}] = v_A,
\tag{3.126}
$$

therefore

$$
\begin{aligned}
\overline{\phi}_A^{(2)} &= -\frac{1}{2}\{\mathcal{Q}, v_A\} \\
&= -\frac{1}{4}\Big[A_{I\bar J}\big(-\tilde F^I_{\bar z}\tilde F^{\bar J}_z - 2\rho^I_{\bar z} D_z\chi^{\bar J} - 2\rho^{\bar J}_z D_{\bar z}\chi^I + 4\partial_{\bar z} x^I\partial_z x^{\bar J}\big) \\
&\quad + \partial_K\partial_{\bar L}A_{I\bar J}\rho^K_{\bar z}\rho^{\bar L}_z\chi^I\chi^{\bar J} - \partial_K A_{I\bar J}\chi^I\tilde F^{\bar J}_z\rho^K_{\bar z} - \partial_{\bar K}A_{I\bar J}\chi^{\bar K}\tilde F^I_{\bar z}\rho^{\bar J}_z\Big].
\end{aligned}
\tag{3.127}
$$

A consequence of (3.121) and (3.127) is that (3.93) can be written as

$$
S_\sigma = \int \mathrm{d}^2 z\,\{Q_0, [\overline{Q}_0, \overline{\phi}^{(0)}_{-\overline{\mathcal{G}}}]\} + \int \mathrm{d}^2 z\,\{G_z, \{G_{\bar z}, \phi^{(0)}_{-\mathcal{G}}\}\},
\tag{3.128}
$$

where $\mathcal{G}$, $\overline{\mathcal{G}}$ where introduced in (3.31). Therefore, perturbations of the form (3.121) correspond to perturbations of the complexified Kähler form, and they are *holomorphic* in $\mathcal{J}$, or equivalently, in the parameters t^a introduced in (3.38). Perturbations of the form (3.127) correspond to perturbations of the *conjugate* complexified Kähler form, and they are *antiholomorphic* on $\mathcal{J}$, so they depend on $\bar{t}^a$. A similar conclusion holds for the B model. Of course, in the physical, unitary supersymmetric sigma model, one has

$$t^{a*} = \bar{t}^a, \tag{3.129}$$

but in the topological field theory t^a, $\bar{t}^a$ can be regarded as independent variables, and they indeed play a different role: the perturbations on t^a lead to topological perturbations of the action, while the perturbations on $\bar{t}^a$ are $\mathcal{Q}$-exact and in principle decouple.

4 Special geometry and the moduli space of complex structures

Good review references for this section are [12–15].

4.1 The moduli space of complex structures

The theory of deformation of complex structures was started by Kodaira and Spencer. A standard review can be found in the book by Kodaira [16]. A complex manifold M can be considered as a set of domains $\{\mathcal{U}_j\}$ in $\mathbb{C}^n$ glued by coordinate transformations f_{jk} which are biholomorphic functions. A deformation of the complex structure of X is a variation of these transition functions depending on some complex parameters $z = (z^1, \cdots, z^s$. This deformation generates a family of complex manifolds that will be denoted by $\mathcal{X}$. It follows from this definition that infinitesimal deformations of X are always elements in the sheaf cohomology group $H^1(X, TX)$, which is isomorphic to the Dolbeault group $H^1_{\bar{\partial}}(X, TX)$. However, the converse is not true. In other words, topological obstructions to integrate an infinitesimal deformation are found in general, and therefore not every $\bar{\partial}$-closed vector $(0,1)$-form is associated to a family of complex manifolds. The right condition for a vector $(0,1)$-form $\phi(z)$ to represent a deformation of the complex structure is that it verifies the *Kodaira-Spencer equation*

$$\bar{\partial}\phi(z) = \frac{1}{2}[\phi(z), \phi(z)], \tag{4.1}$$

and the initial condition $\phi(0) = 0$. In (4.1) the bracket between a vector $(0,p)$-form $\phi = \sum_\alpha \phi^\alpha \partial_\alpha$ and a vector $(0,q)$-form $\psi = \sum_\alpha \psi^\alpha \partial_\alpha$ is defined by:

$$[\phi, \psi] = \sum_{\alpha,\beta} \Big(\phi^\alpha \wedge \partial_\alpha \psi^\beta - (-1)^{pq} \psi^\alpha \wedge \partial_\alpha \phi^\beta\Big)\partial_\beta. \tag{4.2}$$

In coordinates, we can write

$$\phi = \phi_{\bar{I}}^J \mathrm{d}x^{\bar{I}} \frac{\partial}{\partial x^J} \tag{4.3}$$

and the Kodaira–Spencer equation reads

$$\partial_{[\bar{I}} \phi_{\bar{J}]}^L = \phi_{[\bar{I}}^K \partial_K \phi_{\bar{J}]}^L. \tag{4.4}$$

Notice that, at linear order, we have that indeed ϕ has to be in $H^1(X, TX)$.

Another, more intuitive way of understanding Kodaira–Spencer theory is to regard the deformation of complex structures as a deformation of the Dolbeault $\overline{\partial}$-operator

$$\overline{\partial}'_{\bar{I}} = \overline{\partial}_{\bar{I}} - \phi_{\bar{I}}^J \partial_J, \tag{4.5}$$

Requiring now that the new Dolbeault operator is nilpotent, $(\overline{\partial}')^2 = 0$, leads immediately to (4.4). It is then easy to see that a vector $(0,1)$-form V is closed under the new Dolbeault operator if it satisfies

$$\overline{\partial} V = [\phi, V]. \tag{4.6}$$

This description of Kodaira–Spencer theory has a nice field theory realization [7] obtained by considering deformations of the type B model, as suggested in [6].

The problem of topological obstructions can be now formulated as follows (we assume for simplicity that there is a single parameter z for the complex structures): one can show that every $\phi(z)$ representing a complex deformation verifies

$$\left(\frac{\partial \phi(z)}{\partial z}\right)_{z=0} \in H^1(X, TX) \tag{4.7}$$

so an infinitesimal deformation $\phi_1 \in H^1(X, TX)$ is unobstructed if one can find a solution $\phi(z)$ to (4.1) such that

$$\left(\frac{\partial \phi(z)}{\partial z}\right)_{z=0} = \phi_1. \tag{4.8}$$

It has been proven by Tian [17] and Todorov [18] that, when X is a Calabi-Yau manifold, every infinitesimal deformation is unobstructed. More precisely, given $\phi_1 \in H^1(X, TX)$, there is a power expansion in the parameter z

$$\phi(z) = \phi_1 z + \phi_2 z^2 + \cdots \tag{4.9}$$

such that $\phi(z)$ satisfies (4.1) and therefore corresponds to a deformation of the complex structure. In fact, the vector $(0,1)$-forms appearing in this series are obtained by solving inductively the Kodaira–Spencer equation at order n in t:

$$\bar{\partial}\phi_n = \frac{1}{2}\sum_{i=1}^{n-1}[\phi_i, \phi_{n-i}]. \tag{4.10}$$

In this way, when X is CY, deformations of the complex structure are in one to one correspondence with $\bar{\partial}$-closed vector $(0,1)$-forms.

From this brief review we have two consequences: first of all, in the CY case, the moduli space of complex structures is of dimension $h^{2,1}(X)$. This is easily seen by taking into account that, due to the invertible map (3.60), $\bar{\partial}$-closed vector $(0,1)$-forms are in one-to-one correspondence with $\bar{\partial}$-closed $(2,1)$ forms. We will denote by z^a, $a = 1, \cdots, h^{2,1}(X)$, local complex coordinates for this space. Second, the operators (3.56) for $p = q = 1$ correspond to infinitesimal deformations of the complex structure of the CY X.

It turns out that, in the case of CY manifolds, the best way to understand the variation of complex structures is by considering the holomorphic 3-form Ω as a function of the complex deformation parameters t^a. A basic result in the theory of deformation of complex structures is that

$$\frac{\partial \Omega}{\partial z^a} = \phi_a \wedge \Omega + k_a \Omega \in H^{2,1}(X) \oplus H^{3,0}(X), \tag{4.11}$$

where ϕ_a is a closed, vector $(0,1)$ form, and k_a might depend on the moduli but not on the coordinates of X. Using the map (3.60) one can write this as

$$\frac{\partial \Omega}{\partial z^a} = \chi_a + k_a \Omega \tag{4.12}$$

where $\chi_a = \phi_a'$ is a $(2,1)$ form. The result (4.11) can be easily understood by taking into account that a deformation (4.5) induces a deformation

$$\mathrm{d}x^I \to \mathrm{d}_\phi x^I = \mathrm{d}x^I + \phi^I_{\overline{J}} \mathrm{d}x^{\overline{J}} \tag{4.13}$$

of the basis of holomorphic differentials. Therefore, a closed $(3,0)$ form becomes a linear combination of a closed $(3,0)$ form and a closed $(2,1)$ form.

By the same token, one finds that

$$\frac{\partial^2 \Omega}{\partial z^a \partial z^b} \in H^{3,0}(X) \oplus H^{2,1}(X) \oplus H^{1,2}(X) \tag{4.14}$$

and the $(2,1)$ part of this is given by

$$\phi_b \wedge \partial_a \Omega = \phi_a \wedge \phi_b \wedge \Omega = (\phi_a \wedge \phi_b)'. \tag{4.15}$$

Due to type considerations, it follows that

$$\int_X \Omega \wedge \frac{\partial \Omega}{\partial z^a} = 0 \tag{4.16}$$

and also that

$$\int_X \Omega \wedge \frac{\partial^2 \Omega}{\partial z^a \partial z^b} = 0. \tag{4.17}$$

If we now take a derivative of this last equation, we find

$$\int_X \Omega \wedge \frac{\partial^3 \Omega}{\partial z^a \partial z^b \partial z^c} = -\int_X \frac{\partial \Omega}{\partial z^c} \wedge \frac{\partial^2 \Omega}{\partial z^a \partial z^b} \tag{4.18}$$

The term in the r.h.s. is nothing but (3.62), therefore we find the following alternative expression for the Yukawa coupling

$$C_{abc} = \int_X \Omega \wedge \frac{\partial^3 \Omega}{\partial z^a \partial z^b \partial z^c}. \tag{4.19}$$

Remark 4.1. This is denoted by $-\kappa_{abc}$ in [12]. This introduces certain minus signs in the equations, but preserves the relation to the third derivative of the prepotential, as we will see.

The moduli space of complex structures, whose infinitesimal deformations can be identified with the elements in $H^{1,2}(X)$, has a very rich geometric structures. We define the *Kähler potential* as

$$K(X, \overline{X}) = -\log\left(\mathrm{i} \int_X \Omega \wedge \overline{\Omega}\right). \tag{4.20}$$

Since we have a Kähler potential, the moduli space of complex structures is a *Kähler manifold.* The Kähler metric is given by

$$G_{a\bar{b}} = \partial_a \partial_{\bar{b}} K = \frac{\int_X \partial_a \Omega \wedge \overline{\Omega} \int_X \Omega \wedge \partial_{\bar{b}} \overline{\Omega}}{\left(\int_X \Omega \wedge \overline{\Omega}\right)^2} - \frac{\int_X \partial_a \Omega \wedge \partial_{\bar{b}} \overline{\Omega}}{\int_X \Omega \wedge \overline{\Omega}} \tag{4.21}$$

where we have used the fact that Ω is holomorphic w.r.t. the generic complex coordinates z^a. Using now the key equation (4.12) we obtain

$$\begin{aligned} &\int_X \partial_a \Omega \wedge \overline{\Omega} = k_a \int_X \Omega \wedge \overline{\Omega}, \\ &\int_X \partial_a \Omega \wedge \partial_{\bar{b}} \overline{\Omega} = \int_X (\chi_a + k_a \Omega) \wedge \left(\overline{\chi}_{\bar{b}} + \bar{k}_{\bar{b}} \overline{\Omega}\right) = \int_X \chi_a \wedge \overline{\chi}_{\bar{b}} + k_a \bar{k}_{\bar{b}} \int_X \Omega \wedge \overline{\Omega}, \end{aligned} \tag{4.22}$$

therefore

$$G_{a\bar{b}} = -\frac{\int_X \chi_a \wedge \overline{\chi}_{\bar{b}}}{\int_X \Omega \wedge \overline{\Omega}}. \tag{4.23}$$

As in any Kähler manifold, we have the following formulae for the Christoffel symbols

$$\Gamma^a_{bc} = G^{a\bar{a}} \partial_b G_{c\bar{a}}, \qquad \Gamma^{\bar{a}}_{\bar{b}\bar{c}} = G^{a\bar{a}} \bar{\partial}_{\bar{b}} G_{a\bar{c}}. \tag{4.24}$$

There is an additional structure on $\mathcal{M}$ which is related to the freedom in rescaling Ω,

$$\Omega \longrightarrow \mathrm{e}^{f(X)}\Omega. \tag{4.25}$$

This induces the following transformation on the Kähler form,

$$K(X, \overline{X}) \longrightarrow K(X, \overline{X}) - f(X) - \overline{f}(\overline{X}). \tag{4.26}$$

These transformations should be regarded as a $U(1)$ gauge symmetry. Therefore, there is a $U(1)$ gauge bundle, i.e. a line bundle $\mathcal{L}$ (called the Hodge bundle) over the moduli space of complex structures $\mathcal{M}$, and Ω is then a section of $\mathcal{L}$. Since there is a gauge symmetry, there should be also a gauge connection on $\mathcal{L}$. It is given by

$$\mathcal{A}_a = \partial_a K. \tag{4.27}$$

Notice that, with this connection, the covariant derivative

$$\mathcal{D}_a = \partial_a + \partial_a K \tag{4.28}$$

of the section Ω transforms well under (4.25) and (4.26):

$$\mathcal{D}_a \Omega \to \mathrm{e}^f \mathcal{D}_a \Omega. \tag{4.29}$$

Let us now go back to (4.12). We can write

$$\frac{\partial}{\partial z^a} \mathrm{e}^{-K} = \mathrm{i} \frac{\partial}{\partial z^a} \int_X \Omega \wedge \overline{\Omega} = \mathrm{i} \int_X \frac{\partial \Omega}{\partial z^a} \wedge \overline{\Omega} = \mathrm{i} k_a \int_X \Omega \wedge \overline{\Omega} = k_a \mathrm{e}^{-K}, \tag{4.30}$$

and we conclude

$$k_a = -K_a. \tag{4.31}$$

In particular, we can write (4.12) as

$$D_a \Omega = \chi_a. \tag{4.32}$$

In general, we will have sections Ψ of the bundle $\mathcal{L}^n \otimes \overline{\mathcal{L}}^m$, whose covariant derivatives are

$$D_a \Psi = (\partial_a + n K_a)\, \Psi, \qquad D_{\bar{a}} \Psi = (\partial_{\bar{a}} + m K_{\bar{a}})\, \Psi. \tag{4.33}$$

We note that, in view of the transformation (4.26), the exponential of the Kähler parameter e^K has charges $(-1, -1)$.

It follows from the above considerations that

$$[D_a, D_{\bar{b}}]\Omega = (\partial_a + K_a)\, (\partial_{\bar{b}} \Omega) - \partial_{\bar{b}}\, (\partial_a \Omega + K_a \Omega) \tag{4.34}$$

i.e.

$$[D_a, D_{\bar{b}}]\Omega = -G_{a\bar{b}} \Omega. \tag{4.35}$$

The commutator in the l.h.s. defines the curvature of the bundle $\mathcal{L}$, and we conclude that

$$\mathcal{F}_{a\bar{b}} = -G_{a\bar{b}}. \tag{4.36}$$

It follows from this that the Kähler form on $\mathcal{M}$ is 2π times the Chern class of $\mathcal{L}$. In this geometric setting, the Yukawa coupling C_{abc} has to be understood as a holomorphic section of the bundle

$$\mathrm{Sym}^3(T^*\mathcal{M}) \otimes \mathcal{L}^2. \tag{4.37}$$

There is a natural induced connection for tensors in $T^*\mathcal{M} \otimes \mathcal{L}^n$, namely the product connection of the Riemannian connection on $\mathcal{M}$ and the connection on $\mathcal{L}$. The corresponding covariant derivative is

$$D_a = \mathcal{D}_a + n\partial_a K, \tag{4.38}$$

where $\mathcal{D}_a$ is the covariant derivative on $T^*\mathcal{M}$, i.e.

$$[\mathcal{D}_a]_b^c = \delta_b^c \partial_a - \Gamma_{ab}^c. \tag{4.39}$$

Let us now derive some further important properties of the geometry of moduli space. We want to calculate the covariant derivatives of χ_a and $\overline{\chi}_{\bar{b}}$. We note that they are sections of $\mathcal{L}$ and $\overline{\mathcal{L}}$, respectively. Let us start with the easier one,

$$D_a \overline{\chi}_{\bar{b}} = \partial_a \overline{\chi}_{\bar{b}} = \partial_a \left(\partial_b \Omega + K_b \Omega \right), \tag{4.40}$$

and we conclude that

$$D_a \overline{\chi}_{\bar{b}} = G_{a\bar{b}} \Omega. \tag{4.41}$$

Let us now calculate $D_a \chi_b$. It should be a form in $H^{2,1}(X) \oplus H^{1,2}(X)$, so let us write

$$D_a \chi_b = A_{ab}{}^c \chi_c + B_{ab}{}^{\bar{c}} \overline{\chi}_{\bar{c}}. \tag{4.42}$$

To make notations lighter, let us introduce the inner product of forms

$$\langle \alpha, \beta \rangle = -\mathrm{i} \int_X \alpha \wedge \beta. \tag{4.43}$$

In this notation, the equality (4.23) can be written as

$$\langle \chi_a, \overline{\chi}_{\bar{b}} \rangle = G_{a\bar{b}}\, \mathrm{e}^{-K}. \tag{4.44}$$

Let us now take the inner product of equation (4.42) with $\overline{\chi}_{\bar{d}}$:

$$\langle D_a \chi_b, \overline{\chi}_{\bar{d}} \rangle = A_{ab}{}^c \langle \chi_c, \overline{\chi}_{\bar{d}} \rangle = A_{ab\bar{c}} \mathrm{e}^{-K}. \tag{4.45}$$

On the other hand,

$$\langle D_a \chi_b, \overline{\chi}_{\bar{d}} \rangle = D_a \langle \chi_b, \overline{\chi}_{\bar{d}} \rangle - \langle \chi_b, D_a D_{\bar{d}} \overline{\Omega} \rangle. \tag{4.46}$$

The first term in the r.h.s. vanishes. Indeed, we have

$$D_a \langle \chi_b, \overline{\chi}_{\bar{d}} \rangle = (\mathcal{D}_a + K_a) \left(\mathrm{e}^{-K} G_{b\bar{d}} \right) = \mathrm{e}^{-K} \mathcal{D}_a \left(G_{b\bar{d}} \right) = 0, \tag{4.47}$$

since a Riemannian metric is covariantly constant. The second term also vanishes, since

$$- \langle \chi_b, D_a D_{\bar{d}} \overline{\Omega} \rangle = -\langle \chi_b, [D_a, D_{\bar{d}}] \overline{\Omega} \rangle = -\langle \chi_b, G_{a\bar{d}} \overline{\Omega} \rangle = 0, \tag{4.48}$$

since it is the product of a $(2,1)$ form with a $(0,3)$ form. We conclude that $A_{ab}{}^{c}=0$. Let us now calculate the remaining coefficient. We have, after taking the inner product with χ_d,

$$-\,\mathrm{e}^{-K}\,B_{abd}=\langle D_a\chi_b,\chi_d\rangle=\langle D_a\chi_b,D_d\Omega\rangle=D_d\langle D_a\chi_b,\Omega\rangle-\langle D_dD_a\chi_b,\Omega\rangle \tag{4.49}$$

Let us evaluate these terms. We have

$$\langle D_a\chi_b,\Omega\rangle=\langle D_aD_b\Omega,\Omega\rangle=D_a\langle D_b\Omega,\Omega\rangle-\langle D_b\Omega,D_a\Omega\rangle=D_a\langle\chi_b,\Omega\rangle-\langle\chi_b,\chi_a\rangle=0, \tag{4.50}$$

where all pairings vanish due to degree considerations. We conclude that

$$\mathrm{e}^{-K}B_{abd}=\langle D_dD_aD_b\Omega,\Omega\rangle=\langle\partial^3_{abd}\Omega,\Omega\rangle=-\langle\Omega,\partial^3_{abd}\Omega\rangle=\mathrm{i}C_{abc}. \tag{4.51}$$

Here, we have taken into account that the only term in $D_dD_aD_b\Omega$ which has a non-zero pairing with Ω is the triple derivative.

Let us now summarize the properties of the $(2,1)$ forms in the following list:

$$\begin{aligned}
D_a\Omega&=\chi_a,\\
D_a\chi_b&=\mathrm{i}\,\mathrm{e}^{K}C_{ab}{}^{\bar c}\overline{\chi}_{\bar c},\\
D_a\overline{\chi}_{\bar b}&=G_{a\bar b}\Omega,\\
D_a\overline{\Omega}&=0.
\end{aligned} \tag{4.52}$$

By using these relations, we can derive an important identity for the Riemann tensor of the moduli space of complex structures. The commutator of covariant derivatives gives property

$$[D_a,D_{\bar b}]\chi_c=-G_{a\bar b}\chi_c+\partial_{\bar b}\Gamma^d_{ac}\chi_d. \tag{4.53}$$

The first term in the r.h.s. is the curvature of the $U(1)$ bundle, as we saw in (4.35), while the second term is the curvature of $T\mathcal{M}$. Let us calculate the commutator:

$$D_aD_{\bar b}\chi_c=D_aG_{c\bar b}\Omega=G_{c\bar b}\chi_a, \tag{4.54}$$

where we used that $G_{c\bar b}$ is covariantly constant. On the other hand,

$$D_{\bar b}D_a\chi_c=D_{\bar b}\left(\mathrm{i}\mathrm{e}^{K}C_{ac}{}^{\bar d}\overline{\chi}_{\bar d}\right)=\mathrm{i}\mathrm{e}^{K}C_{ac}{}^{\bar d}D_{\bar b}\overline{\chi}_{\bar d}=\mathrm{e}^{2K}C_{ac}{}^{\bar d}\overline{C}_{\bar b\bar d}{}^{m}\chi_m. \tag{4.55}$$

Here, we have used (4.52) and the fact that

$$D_{\bar b}\mathrm{e}^{K}=D_{\bar b}C_{pqr}=0. \tag{4.56}$$

Putting everything together, we obtain

$$\partial_{\bar b}\Gamma^d_{ac}=G_{a\bar b}\delta^d_c+G_{c\bar b}\delta^d_a-C_{acm}\overline{C}^{md}_{\bar b}, \tag{4.57}$$

where we have defined

$$\overline{C}^{ij}_{\bar k}=\mathrm{e}^{2K}\,\overline{C}_{\bar k\bar a\bar b}G^{i\bar a}G^{j\bar b}. \tag{4.58}$$

In terms of the Riemann curvature tensor we can write

$$R_{a\bar b c\bar d}=G_{a\bar b}G_{c\bar d}+G_{a\bar d}G_{c\bar b}-C_{acm}\overline{C}^{m}_{\bar b\bar d}. \tag{4.59}$$

The above conditions mean that $\mathcal{M}$, together with the bundle $\mathcal{L}$, is a *special Kähler manifold* [19].

4.2 Coordinates and prepotential

So far we have considered a generic parametrization of the moduli space of complex coordinates. In order to make further progress we need an explicit parametrization of the moduli space. This is done by introducing the *periods* of Ω. We choose a symplectic basis for $H^3(X,\mathbb{C})$

$$(\alpha_I, \beta^I), \qquad I = 0, 1, \cdots, h^{2,1}(X), \tag{4.60}$$

satisfying

$$\int_X \alpha_I \wedge \beta^J = \delta_I^J, \qquad \int_X \beta^J \wedge \alpha_I = -\delta_I^J. \tag{4.61}$$

Similarly, we define Poincaré duals A^I, B_I satisfying

$$\int_{A^I} \alpha_J = \delta_J^I, \qquad \int_{B_I} \beta_J = -\delta_I^J, \tag{4.62}$$

and zero otherwise. We then define the *periods* of the 3-form Ω as

$$X^I = \int_{A^I} \Omega, \qquad \mathcal{F}_I = \int_{B_I} \Omega. \tag{4.63}$$

Notice that, if we decompose Ω in terms of the basis (4.60), we find

$$\Omega = X^I \alpha_I - \mathcal{F}_I \beta^I. \tag{4.64}$$

In the next Chapter we will introduce explicit expressions for X^I, $\mathcal{F}_I$ in terms of Frobenius solutions of a differential equation, called the Picard–Fuchs equation. With our choice of normalization, these periods are not integral. Therefore, they correspond to a basis of three-cycles over $\mathbb{C}$ rather than $\mathbb{Z}$. However, they can be easily modified to obtain integral periods.

It follows from the theory of deformation of complex structures that the X^I are (locally) complex projective coordinates for $\mathcal{M}$. They are called *special projective coordinates*. Since the X^I parametrize $\mathcal{M}$ we deduce that the other set of periods must depend on them, *i.e.* $\mathcal{F}_I = \mathcal{F}_I(X)$. We now define the *projective prepotential* $\mathcal{F}(X^I)$ as

$$\mathcal{F}(X^I) = \frac{1}{2} X^I \mathcal{F}_I. \tag{4.65}$$

One has the following equality

$$\mathcal{F}_I = \frac{\partial \mathcal{F}}{\partial X^I}. \tag{4.66}$$

It follows that the projective prepotential is homogeneous of degree 2. To see this, notice that (4.16) implies

$$\int_X \Omega \wedge \frac{\partial \Omega}{\partial X^I} = 0, \qquad I = 0, \cdots, h^{2,1}(X). \tag{4.67}$$

By plugging the expansion (4.64) here we find

$$\mathcal{F}_I = X^J \frac{\partial \mathcal{F}_J}{\partial X^I} = \frac{\partial}{\partial X^I}(X^J \mathcal{F}_J) - \mathcal{F}_I \tag{4.68}$$

and (4.66) follows. We can use again the expansion (4.64) to obtain an expression for the Kähler potential in terms of projective coordinates,

$$K(X, \overline{X}) = -\log\left[\mathrm{i}(\overline{X}^I \mathcal{F}_I - X^I \overline{\mathcal{F}}_I)\right] = -\log\left(2\,\mathrm{Im}\,(X^I \overline{\mathcal{F}}_{\overline{I}})\right). \tag{4.69}$$

Let us collect some useful results. The second derivative

$$\tau_{IJ} = \mathcal{F}_{IJ} \tag{4.70}$$

is homogeneous of degree zero, therefore

$$X^M C_{MIJ} = 0. \tag{4.71}$$

Equivalently, one has

$$\tau_{IJ} = \frac{\partial^2 \mathcal{F}}{\partial X^I \partial X^J} = \frac{\partial^2}{\partial X^I \partial X^J}\left(\frac{1}{2}X^M \mathcal{F}_M\right) = \tau_{IJ} + \frac{1}{2}X^M C_{MIJ}. \tag{4.72}$$

We can now relate the Yukawa coupling to the prepotential. First, we notice that

$$\int_X \Omega \wedge \frac{\partial^3 \Omega}{\partial X^I \partial X^J \partial X^K} = -X^M \frac{\partial^3 \mathcal{F}_M}{\partial X^I \partial X^J \partial X^K}. \tag{4.73}$$

On the other hand, the triple derivative

$$C_{IJK} = \frac{\partial^3 \mathcal{F}}{\partial X^I \partial X^J \partial X^K} \tag{4.74}$$

is homogeneous of degree -1, therefore

$$X^M \mathcal{F}_{IJKM} = -C_{IJK}. \tag{4.75}$$

Equivalently, we have the identity

$$\frac{\partial^3 (X^M \mathcal{F}_M)}{\partial X^I \partial X^J \partial X^K} = 3\frac{\partial^3 \mathcal{F}}{\partial X^I \partial X^J \partial X^K} + X^M \frac{\partial^3 \mathcal{F}_M}{\partial X^I \partial X^J \partial X^K}, \tag{4.76}$$

and we conclude that

$$\int_X \Omega \wedge \frac{\partial^3 \Omega}{\partial X^I \partial X^J \partial X^K} = C_{IJK}, \tag{4.77}$$

which is the Yukawa coupling in the projective coordinates. We can relate it to the conventional Yukawa coupling (4.19) as follows. By using the decomposition (4.64) in (4.19), we find

$$C_{abc} = \mathcal{F}_I \partial_{abc} X^I - X^I \partial_{abc} \mathcal{F}_I. \tag{4.78}$$

Let us calculate $\partial_{abc}\mathcal{F}_I$. The triple derivative $\partial_{abc}\mathcal{F}_I$ will lead to second, third, and fourth derivatives of $\mathcal{F}$:

$$\begin{aligned} \partial_{abc}\mathcal{F}_I &= \mathcal{F}_{IJKM}\partial_a X^J \partial_b X^K \partial_c X^M + \mathcal{F}_{IJ}\partial^3_{abc} X^J \\ &\quad + \mathcal{F}_{IJK}\partial^2_{ab} X^J \partial_c X^K + \mathcal{F}_{IJK}\partial^2_{ac} X^J \partial_b X^K + \mathcal{F}_{IJK}\partial^2_{bc} X^J \partial_a X^K. \end{aligned} \tag{4.79}$$

After contraction with X^I, all terms in the last line vanish due to (4.71), and we find

$$C_{abc} = -X^I \mathcal{F}_{IJKM}\partial_a X^J \partial_b X^K \partial_c X^M - X^I \mathcal{F}_{IJ}\partial^3_{abc} X^J + \mathcal{F}_I \partial^3_{abc} X^I. \tag{4.80}$$

By using (4.75) and (4.68), we conclude that

$$C_{abc} = \partial_a X^I \partial_b X^J \partial_c X^K C_{IJK}. \tag{4.81}$$

This formula is valid in arbitrary complex coordinates z for the moduli space.

The X^I are projective coordinates for the moduli space, and we have $h^{2,1}(X)+1$ of them. We can obtain $h^{2,1}(X)$ coordinates for X by introducing the affine coordinates corresponding to the projective coordinates X^I. This is done by choosing a nonzero coordinate, say X^0, and considering the quotients

$$t^i = \frac{X^i}{X^0}, \qquad i = 1, \cdots, h^{2,1}(X). \tag{4.82}$$

Since the projective prepotential is homogeneous, we can define a quantity $F_0(t^a)$ called simply the *prepotential* which only depends on the coordinates t^a

$$\mathcal{F}(X_I) = (X^0)^2 F_0(t^i). \tag{4.83}$$

The prepotential in affine coordinates can be obtained by normalizing Ω in such a way that the 0-th period X^0 is set to 1, i.e. we have to set $\Omega \to \Omega/X_0$.

It will be convenient to divide the periods in four different sets, leading to a *period vector* Π in projective coordinates:

$$\Pi = (X^0, X^i, \mathcal{F}_i, \mathcal{F}_0), \qquad i = 1, \cdots, h^{2,1}(X). \tag{4.84}$$

We have the following equalities

$$\begin{aligned} \mathcal{F}_0 &= X^0(2F_0 - t^i \partial_i F_0), \\ \mathcal{F}_i &= X^0 \partial_i F_0, \quad i = 1, \cdots, h^{2,1}(X), \end{aligned} \tag{4.85}$$

We also have the useful equations:

$$\begin{aligned} \tau_{00} &= 2F_0(t^i) - 2t^i \partial_i F_0 + t^i t^j \partial^2_{ij} F_0, \\ \tau_{0i} &= \partial_i F_0 - t^j \partial^2_{ij} F_0, \\ \tau_{ij} &= \partial^2_{ij} F_0, \end{aligned} \tag{4.86}$$

as well as

$$\begin{aligned} C_{000} &= -\frac{1}{X_0} t^i t^j t^k \partial^3_{ijk} F_0, \\ C_{00i} &= \frac{1}{X_0} t^j t^k \partial^3_{ijk} F_0, \\ C_{0ij} &= -\frac{1}{X_0} t^k \partial^3_{ijk} F_0, \\ C_{ijk} &= \frac{1}{X_0} \partial^3_{ijk} F_0. \end{aligned} \tag{4.87}$$

The equation for the Kähler potential can be written now as

$$\mathrm{e}^{-K} = \mathrm{i} \Pi^\dagger V \Pi, \tag{4.88}$$

where V is the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{4.89}$$

In terms of the homogeneous coordinates t^a, we have

$$\Pi^\dagger V \Pi = -X^0 \overline{X}^0 \Big((t^i - \overline{t}^i)(F_i + \overline{F}_i) + 2\overline{F}_0 - 2F_0 \Big), \tag{4.90}$$

where we have used (4.85).

Example 4.2. The simplest example of special geometry is probably the one coming from the prepotential

$$F_0(t, S) = \frac{1}{2} C_{ab} t^a t^b S, \qquad a, b = 1, \cdots, n, \tag{4.91}$$

with $n + 1$ moduli. Here, C_{ab} is an invertible, symmetric matrix. For $n = 10$ this is the prepotential of the Enriques Calabi–Yau manifold. The only nonzero Yukawa coupling is

$$C_{abS} = C_{ab}. \tag{4.92}$$

The Kähler potential is

$$\mathrm{e}^{-K} = (S + \bar{S})Y, \quad Y = \frac{1}{2} C_{ab}(t^a + \bar{t}^a)(t^b + \bar{t}^b). \tag{4.93}$$

The Kähler metric has components

$$G_{S\bar{S}} = \frac{1}{(S + \bar{S})^2}, \quad G_{a\bar{b}} = -\frac{C_{ab}}{Y} + \frac{\partial_a Y \partial_{\bar{b}} Y}{Y^2}. \tag{4.94}$$

The inverse metric is easy to compute,

$$G^{S\bar{S}} = (S + \bar{S})^2, \quad G^{a\bar{b}} = Y^2 (C^{-1})^{ac} G_{c\bar{d}} (\overline{C}^{-1})^{\bar{d}\bar{b}}. \tag{4.95}$$

We will denote by G the matrix $G_{a\bar{b}}$ in the fiber directions. We have the useful equality

$$C^{-1} G C^{-1} G = \frac{\mathbf{1}_{10}}{Y^2}. \tag{4.96}$$

The Christoffel symbols can be also evaluated. We have,

$$\begin{aligned} \Gamma^S_{SS} &= -\frac{2}{S + \bar{S}}, \\ \Gamma^c_{ab} &= \frac{(t^d + \bar{t}^d)}{Y} (\delta^c_d C_{ab} - \delta^c_b C_{ad} - \delta^c_a C_{bd}). \end{aligned} \tag{4.97}$$

We can check that $D_i C_{jkl} = 0$. For example, for $i = c$, $j = a$, $k = b$, $c = S$, this is

$$-\Gamma^d_{ca} C_{db} - \Gamma^d_{cb} C_{ad} + 2\partial_c K C_{ab}, \tag{4.98}$$

but since

$$\Gamma^d_{ca} C_{db} + \Gamma^d_{cb} C_{ad} = 2C_{ab} \frac{\partial_c Y}{Y}, \tag{4.99}$$

the above vanishes.

4.3 The big moduli space

We can use the projective coordinates X^I to define a "big" moduli space, $\widehat{\mathcal{M}}$, of dimension $h^{2,1}(X)+1$. This space is also Kähler. Its Kähler potential is

$$\mathcal{K} = \frac{\mathrm{i}}{2}(X^K \overline{\mathcal{F}}_K - \overline{X}^K \mathcal{F}_K), \tag{4.100}$$

which leads to the Kähler metric

$$\mathcal{G}_{IJ} = \partial_I \overline{\partial}_J \mathcal{K} = -\frac{\mathrm{i}}{2}(\tau - \overline{\tau})_{IJ} = \mathrm{Im}\, \tau_{IJ}. \tag{4.101}$$

We will need some properties of the metric of the big moduli space, and in particular its relation to the metric of the "small" moduli space. We first note that, from (4.68) and (4.69), we have

$$\mathrm{e}^{-K} = \mathrm{i}(\overline{X}^I X^K \tau_{KI} - X^I \overline{X}^K \overline{\tau}_{KI}) = \mathrm{i} X^I (\tau - \overline{\tau})_{IJ} \overline{X}^J, \tag{4.102}$$

which we will write as

$$\mathrm{i}\, \mathrm{e}^K X^I (\tau - \overline{\tau})_{IJ} \overline{X}^J = 1. \tag{4.103}$$

At this point we introduce an important set of quantities. We first introduce a Greek index $\alpha = (0, a)$, where a are indices for the usual complex coordinates of the "small" moduli space. Then, following [14, 20] we define

$$\chi^I_\alpha = D_\alpha X^I, \qquad \alpha, I = 0, 1, \cdots, h^{2,1}(X), \tag{4.104}$$

with the proviso that

$$D_0 X^I = X^I, \tag{4.105}$$

and we recall that

$$D_a X^I = (\partial_a + K_a) X^I. \tag{4.106}$$

We also note that

$$\chi^I_a = \int_{\alpha_I} \chi_a, \tag{4.107}$$

where χ_a are the forms introduced in (4.12). The matrix χ^I_α is invertible, and its inverse will be denoted by χ^α_I, satisfying

$$\chi^I_\alpha \chi^\alpha_J = \delta^I_J, \qquad \chi^I_\alpha \chi^\beta_I = \delta^\beta_\alpha. \tag{4.108}$$

Let us now take an anti-holomorphic derivative w.r.t. z^b in (4.103). We find,

$$D_{\bar b}\left(\mathrm{e}^K X^I (\tau - \overline{\tau})_{IJ} \overline{X}^J\right) = -\mathrm{e}^K \partial_{\bar b} \overline{\tau}_{IJ} X^I \overline{X}^J + \mathrm{e}^K X^I (\tau - \overline{\tau})_{IJ} D_{\bar b} \overline{X}^J = \mathrm{e}^K X^I (\tau - \overline{\tau})_{IJ} \overline{\chi}^J_{\bar b}, \tag{4.109}$$

where we used that $\overline{C}_{IJK} \overline{X}^K = 0$. We conclude that

$$X^I (\tau - \overline{\tau})_{IJ} \overline{\chi}^J_{\bar b} = 0. \tag{4.110}$$

Let us now apply the holomorphic derivative D_a to this equation. We find,

$$\partial_a \tau_{IJ} X^I \overline{\chi}^J_{\bar b} + (\tau - \overline{\tau})_{IJ} \chi^I_a \overline{\chi}^J_{\bar b} + X^I (\tau - \overline{\tau})_{IJ} D_a \overline{\chi}^J_{\bar b} = (\tau - \overline{\tau})_{IJ} \chi^I_a \overline{\chi}^J_{\bar b} + X^I (\tau - \overline{\tau})_{IJ} G_{a\bar b} \overline{X}^J = 0, \tag{4.111}$$

where we used (4.71) and the third equation in (4.52) (we note that τ_{IJ} has charges $(0,0)$, and it is a scalar from the point of view of the moduli space $\mathcal{M}$, so that $D_a\tau_{IJ} = \partial_a\tau_{IJ}$). If we use now (4.103) in the last term, we conclude that

$$G_{a\bar{b}} = -\mathrm{i}\mathrm{e}^K(\tau - \overline{\tau})_{IJ}\chi^I_a\overline{\chi}^J_{\bar{b}}. \tag{4.112}$$

This will be very useful in the following. It also gives the equation,

$$(\tau - \overline{\tau})_{IJ} = \mathrm{i}\,\mathrm{e}^{-K}G_{a\bar{b}}\chi^a_I\overline{\chi}^{\bar{b}}_J. \tag{4.113}$$

Remark 4.3. The signs here are different to the ones in [14, 20], which can be traced back to the fact that our prepotential has the opposite sign to the one used in [14].

The above results suggest to define a $(h^{2,1}(X)+1)\times(h^{2,1}(X)+1)$ matrix giving an "extended" metric:

$$G_{\alpha\bar{\beta}} = -\mathrm{i}\mathrm{e}^K(\tau - \overline{\tau})_{IJ}\chi^I_\alpha\overline{\chi}^J_{\bar{\beta}}. \tag{4.114}$$

In matrix form, the above results lead to

$$G_{\alpha\bar{\beta}} = \begin{pmatrix} -1 & 0 \\ 0 & G_{a\bar{b}} \end{pmatrix}. \tag{4.115}$$

4.4 Symplectic transformations

An important aspect of the formulation in the big moduli space is that there is an action of the symplectic group $\mathrm{Sp}(2n)$, where $n = h^{2,1}(X)+1$. An element $\Gamma \in \mathrm{Sp}(2n)$ is an invertible $2n\times 2n$ matrix that satisfies

$$\Gamma^{\mathrm{T}}\Omega\Gamma = \Omega, \qquad \Omega = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}. \tag{4.116}$$

If we write it as

$$\Gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{4.117}$$

then the condition (4.116) implies that the $n\times n$ matrices A, B, C, D satisfy

$$A^{\mathrm{T}}D - C^{\mathrm{T}}B = \mathbf{1}_n, \quad A^{\mathrm{T}}C = C^{\mathrm{T}}A, \quad B^{\mathrm{T}}D = D^{\mathrm{T}}B. \tag{4.118}$$

We deduce that the matrix

$$\Gamma^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix} \tag{4.119}$$

satisfies

$$\Gamma^{-1}\Gamma = \mathbf{1}_{2n}, \tag{4.120}$$

therefore it must be the inverse of Γ, since Γ is invertible. By calculating now

$$\Gamma\Gamma^{-1} = \begin{pmatrix} AD^T - BC^T & BA^T - AB^T \\ CD^T - DC^T & DA^T - CB^T \end{pmatrix} \tag{4.121}$$

we deduce additional conditions satisfies by a symplectic matrix,

$$AD^T - BC^T = \mathbf{1}_n, \qquad AB^T = BA^T, \qquad CD^T = DC^T. \tag{4.122}$$

A symplectic transformation Γ acts on the periods as

$$\begin{aligned} X^J_\Gamma &= C^{JI}\mathcal{F}_I + D^J{}_I X^I, \\ \mathcal{F}^\Gamma_J &= A_J{}^I \mathcal{F}_I + B_{JI} X^I. \end{aligned} \tag{4.123}$$

It follows that τ_{IJ} transforms as

$$\tau^\Gamma = (A\tau + B)(C\tau + D)^{-1}. \tag{4.124}$$

It will be important later to note that $(\tau - \bar{\tau})^{-1}$ transforms with a shift:

$$\left[(\tau - \bar{\tau})^{-1\Gamma}\right]^{IJ} = (C\tau + D)^I{}_K \, (C\tau + D)^J{}_L \left[(\tau - \bar{\tau})^{-1\Gamma}\right]^{KL} - C^{IK} \, (C\tau + D)^J{}_K \, . \tag{4.125}$$

We also note that the χ^J_α transform as

$$\chi^{J\Gamma}_\alpha = \left(C^{JK}\tau_{KI} + D^J{}_I\right) \chi^I_\alpha, \tag{4.126}$$

and the inverse as

$$\chi^{\alpha\Gamma}_J = (C\tau + D)^{-1K}{}_J \, \chi^\alpha_K. \tag{4.127}$$

4.5 The holomorphic limit of special geometry

In order to make contact with the A-model, we have to consider in detail the holomorphic limit of special geometry, as explained in [15]. We will assume that we are in the large radius frame, and t^i are flat coordinates (4.82) appropriate for this frame. The holomorphic limit is defined as the limit in which $\bar{t}^i \to \infty$ while keeping t^i fixed. Formally we can take

$$\bar{t}^i \to \bar{s}\bar{t}^i, \qquad \bar{s} \to \infty. \tag{4.128}$$

In order to work out this, we have to be more explicit about the structure of the prepotential in the coordinates appropriate for the large radius frame and limit. It has the form

$$F_0(t^i) = -\frac{1}{6} d_{ijk} t^i t^j t^k + b_i t^i + \frac{c}{2} + F_0^{\rm inst}(t_i). \tag{4.129}$$

Here, d_{ijk}, b_i and c are real coefficients with a topological meaning:

$$d_{ijk} = \int_X J_i \wedge J_j \wedge J_k, \qquad b_i = -(2\pi)^2 \int_X c_2(X) \wedge J_i, \qquad c = -\zeta(3)\chi. \tag{4.130}$$

In the holomorphic limit, the instanton part of the conjugate prepotential is exponentially suppressed, and the conjugate period vector is given by

$$\Pi^\dagger \approx \overline{X}^0 \left(1, \bar{s}\bar{t}^i, -\frac{\bar{s}^2}{2} d_{ijk}\bar{t}^j\bar{t}^k + b_i, \frac{1}{6}\bar{s}^3 d_{ijk}\bar{t}^i\bar{t}^j\bar{t}^k + \bar{s} b_i \bar{t}^i + c\right). \tag{4.131}$$

In this and the following equations the approximate symbol $\approx$ means that the r.h.s. agrees with the l.h.s. up to exponentially small corrections $\mathcal{O}(\mathrm{e}^{-\bar{s}})$. Therefore, we can write

$$\mathrm{e}^{-K} \approx |X^0|^2 \sum_{r=0}^{3} \bar{s}^r \mathcal{C}_r, \tag{4.132}$$

where

$$\mathcal{C}_3 = -\frac{1}{6} d_{ijk} \bar{t}^i \bar{t}^j \bar{t}^k, \qquad \mathcal{C}_2 = \frac{1}{2} d_{ijk} t^i \bar{t}^j \bar{t}^k. \tag{4.133}$$

It is easy to see that the terms $\mathcal{C}_{0,1}$ will give contributions which vanish in the holomorphic limit. We can then write

$$K \approx -\log X^0 - \log \overline{X}^0 - \log\left(\overline{s}^2 M\right) \tag{4.134}$$

where

$$M = \frac{\overline{s}}{6} d_{ijk} \bar{t}^i \bar{t}^j \bar{t}^k - \frac{1}{2} d_{ijk} t^i \bar{t}^j \bar{t}^k + \mathcal{O}(\overline{s}^{-1}). \tag{4.135}$$

We note that[1]

$$\begin{aligned} \partial_a M &= -\frac{1}{2} d_{ijk} \partial_a t^i \bar{t}^j \bar{t}^k + \mathcal{O}(\overline{s}^{-1}), \\ \partial_{\bar{b}} M &= \frac{\overline{s}}{2} d_{ijk} \partial_{\bar{b}} \bar{t}^i \bar{t}^j \bar{t}^k - d_{ijk} t^i \partial_{\bar{b}} \bar{t}^j \bar{t}^k + \mathcal{O}(\overline{s}^{-1}), \\ \partial^2_{a\bar{b}} M &= -d_{ijk} \partial_a t^i \partial_{\bar{b}} \bar{t}^j \bar{t}^k + \mathcal{O}(\overline{s}^{-1}). \end{aligned} \tag{4.136}$$

A first consequence of these results is that

$$K_a \approx -\partial_a \log X^0 - \frac{1}{M} \partial_a M = -\partial_a \log X^0 + \mathcal{O}(\overline{s}^{-1}). \tag{4.137}$$

Therefore, in the holomorphic limit we set

$$K_a \to -\frac{1}{X^0} \partial_a X^0. \tag{4.138}$$

Let us now work out the Kähler metric. We have

$$G_{a\bar{b}} = \partial^2_{a\bar{b}} K \approx \frac{1}{M^2} \partial_a M \partial_{\bar{b}} M - \frac{1}{M} \partial^2_{a\bar{b}} M. \tag{4.139}$$

An explicit evaluation produces the expression

$$G_{a\bar{b}} = \frac{1}{\overline{s}} J^i_a L_{ij} \overline{J}^j_{\bar{b}} + \mathcal{O}(\overline{s}^{-2}) \tag{4.140}$$

where

$$J^i_a = \frac{\partial t^i}{\partial z^a}, \qquad L_{ij} = -\frac{\partial^2}{\partial \bar{t}^i \partial \bar{t}^j} \log \overline{m}, \tag{4.141}$$

and

$$\overline{m} = \frac{1}{6} d_{ijk} \bar{t}^i \bar{t}^j \bar{t}^k \tag{4.142}$$

The most important point here is that the matrix L_{ij} is *purely antiholomorphic*. It follows in particular that

$$\log \det G_{a\bar{b}} \approx \log \det(\partial_a t^i) + \text{antiholomorphic} + \mathcal{O}(\overline{s}^{-1}). \tag{4.143}$$

We also note that the inverse metric is given by

$$G^{a\bar{b}} \approx \overline{s} \left(\overline{J}^{-1}\right)^{\bar{b}}_i \left(L^{-1}\right)^{ij} \left(J^{-1}\right)^a_j + \mathcal{O}(\overline{s}^0). \tag{4.144}$$

[1] To make things clearer, we have used Latin indices at the beginning of the alphabet $a, b, \cdots$ to label arbitrary complex coordinates z^a, while Latin indices from the middle of the alphabet $i, j, \cdots$ refer to flat coordinates t^i.

Let us now determine the holomorphic limit of the Christoffel symbols:

$$\Gamma^c_{ab} = G^{c\bar{d}}\partial_a G_{\bar{d}b} \approx \left(\overline{J}^{-1}\right)_i^{\bar{d}} \left(L^{-1}\right)^{ik} \left(J^{-1}\right)_k^c \partial_a J_b^p L_{pq} \overline{J}_{\bar{d}}^q + \mathcal{O}(\overline{s}^{-1}). \tag{4.145}$$

Therefore, in the holomorphic limit we have

$$\Gamma^c_{ab} \to \frac{\partial z^c}{\partial t^p} \frac{\partial^2 t^p}{\partial z^a \partial z^b}. \tag{4.146}$$

It also follows from this formula that, in the holomorphic limit, the Christoffel symbol vanishes in flat coordinates. Indeed, we can use the standard formula for the transformation of the Christoffel symbols under a change of coordinates,

$$\Gamma^i_{jk} = \frac{\partial z^a}{\partial t^j} \frac{\partial z^b}{\partial t^k} \left(\frac{\partial t^i}{\partial z^c} \Gamma^c_{ab} - \frac{\partial^2 t^i}{\partial z^a \partial z^b} \right), \tag{4.147}$$

which vanishes after plugging in the expression (4.146).

One can consider the holomorphic limit of the various quantities appearing in special geometry. Of special relevance is the matrix χ^I_α defined in (4.104), and its inverse. In the holomorphic limit, by using (4.138), we find

$$\chi^{\ I}_{\alpha,h} = X^0 \begin{pmatrix} 1 & t^a \\ 0 & \partial_k t^a \end{pmatrix}, \tag{4.148}$$

i.e.

$$\chi^{\ 0}_{0,h} = X^0, \qquad \chi^{\ a}_{0,h} = X^0 t^a, \qquad \chi^{\ 0}_{a,h} = 0, \qquad \chi^{\ a}_{k,h} = X^0 \frac{\partial t^a}{\partial z^k}. \tag{4.149}$$

Then, the inverse matrix is given by

$$\chi^{\ \alpha}_{I,h} = X^0 \begin{pmatrix} 1 & -t^b \frac{\partial z^l}{\partial t^b} \\ 0 & \frac{\partial z^l}{\partial t^b} \end{pmatrix} \tag{4.150}$$

i.e. its entries are given by

$$\chi^{\ 0}_{0,h} = \frac{1}{X^0}, \qquad \chi^{\ l}_{0,h} = -\frac{1}{X^0} t^b \frac{\partial z^l}{\partial t^b}, \qquad \chi^{\ 0}_{a,h} = 0, \qquad \chi^{\ l}_{a,h} = \frac{1}{X^0} \frac{\partial z^l}{\partial t^a}. \tag{4.151}$$

4.6 Propagators

As we will see, the genus g free energies of BCOV are written as polynomials in a set of generators called *propagators*. We give here a thorough description of the propagators, building on [14, 15, 20, 21].

We define S^{ab} through the equation

$$\partial_{\bar{c}} S^{ab} = \overline{C}_{\bar{c}}^{ab}, \tag{4.152}$$

where the object in the r.h.s. was defined in (4.58). In addition to S^{ab}, defined in (4.152), we introduce as well

$$\partial_{\bar{c}} S^b = G_{a\bar{c}} S^{ab}, \qquad \partial_{\bar{c}} S = G_{a\bar{c}} S^a. \tag{4.153}$$

We will gather the propagators in a matrix of the form

$$\begin{pmatrix} S^{00} & S^{0a} \\ S^{a0} & S^{ab} \end{pmatrix} = \begin{pmatrix} 2S & -S^a \\ -S^a & S^{ab} \end{pmatrix}. \tag{4.154}$$

Due to the importance of propagators, it is crucial to have explicit expressions for them. We note that (4.152) and (4.153) define them only up to the addition of a holomorphic function. In the case of S^{ab}, an explicit expression can be obtained by using (4.57). Indeed, by using the holomorphy of C_{abc} and the definition of the propagator, we can write (4.57) as

$$\partial_{\bar{b}}\Gamma^r_{ac} = \delta^r_c \partial_{\bar{b}} K_a + \delta^r_a \partial_{\bar{b}} K_c - \partial_{\bar{b}}\left(C_{acp} S^{rp}\right), \tag{4.155}$$

which can be integrated immediately to

$$\Gamma^r_{ac} = \delta^r_c K_a + \delta^r_a K_c - C_{acp} S^{rp} + s^r_{ac}, \tag{4.156}$$

where s^r_{ac} are holomorphic. One can proceed in this way and obtain explicit results for the propagators, up to holomorphic ambiguities. However, it turns out to be more useful to find explicit expressions for them by working in the big moduli space, as first pointed out in [20] and further developed in [14].

Let us then define the *non-holomorphic propagators* as

$$\mathsf{S}^{\alpha\beta} = -\chi^\alpha_I (\tau - \bar{\tau})^{-1IJ} \chi^\beta_J, \tag{4.157}$$

which we can write as

$$\mathsf{S}^{\alpha\beta} = G^{\alpha\bar{\alpha}} G^{\beta\bar{\beta}} \mathsf{S}_{\bar{\alpha}\bar{\beta}}, \tag{4.158}$$

where

$$\mathsf{S}_{\bar{\alpha}\bar{\beta}} = \mathrm{e}^{2K} (\tau - \bar{\tau})_{IJ} \bar{\chi}^I_{\alpha} \bar{\chi}^J_{\bar{\beta}}. \tag{4.159}$$

We will now show that the non-holomorphic propagators introduced above satisfy the defining properties (4.152) and (4.153). We first calculate

$$\begin{aligned} \partial_{\bar{a}} \mathsf{S} &= \frac{1}{2} D_{\bar{a}} \left(\mathrm{e}^{2K} (\tau - \bar{\tau})_{IJ} \overline{X}^I \overline{X}^J \right) = -\frac{1}{2} \mathrm{e}^{2K} D_{\bar{a}} \bar{\tau}_{IJ} \overline{X}^I \overline{X}^J + \mathrm{e}^{2K} (\tau - \bar{\tau})_{IJ} \overline{X}^I \bar{\chi}^J_{\bar{a}} \\ &= \mathsf{S}_{\bar{0}\bar{a}}, \end{aligned} \tag{4.160}$$

since

$$D_{\bar{a}} \bar{\tau}_{IJ} \overline{X}^I \overline{X}^J = \overline{C}_{IJK} \overline{X}^I \overline{X}^J \bar{\chi}^K_{\bar{a}} = 0 \tag{4.161}$$

due to (4.71). We also note that

$$\mathsf{S}_{\bar{0}\bar{a}} = G_{0\bar{0}} G_{\bar{a}b} \mathsf{S}^{0b} = G_{\bar{a}b} \mathsf{S}^b, \tag{4.162}$$

which is the second equation in (4.153). We now calculate

$$D_{\bar{a}} \mathsf{S}_{\bar{0}\bar{b}} = D_{\bar{a}} \left(\mathrm{e}^{2K} (\tau - \bar{\tau})_{IJ} \bar{\chi}^I_{\bar{b}} \overline{X}^J \right) = \mathrm{e}^{2K} (\tau - \bar{\tau})_{IJ} \left(D_{\bar{a}} \bar{\chi}^I_{\bar{b}} \overline{X}^J + \bar{\chi}^I_{\bar{b}} \bar{\chi}^J_{\bar{a}} \right). \tag{4.163}$$

We note that, by using the conjugate of the third equation in (4.52), we find

$$(\tau - \bar{\tau})_{IJ} D_{\bar{a}} \bar{\chi}^I_{\bar{b}} \overline{X}^J \propto \overline{C}_{\bar{a}\bar{b}}{}^{c} \chi^I_c (\tau - \bar{\tau})_{IJ} \overline{X}^J = 0 \tag{4.164}$$

due to the complex conjugate of (4.110). Therefore,

$$D_{\bar{a}} \mathsf{S}_{\bar{0}\bar{b}} = \mathsf{S}_{\bar{a}\bar{b}}. \tag{4.165}$$

We now evaluate

$$\partial_{\bar{a}} \mathsf{S}^b = \partial_{\bar{a}} \left(G^{b\bar{c}} \mathsf{S}_{\bar{0}\bar{c}} \right) = G^{b\bar{c}} D_{\bar{a}} \mathsf{S}_{\bar{0}\bar{c}} = G^{b\bar{c}} \mathsf{S}_{\bar{a}\bar{c}} = G_{\bar{a}c} \mathsf{S}^{cb}, \tag{4.166}$$

which is the first equation in (4.153).

We can now verify (4.152). We start with

$$D_{\bar{c}}\mathsf{S}_{\bar{a}\bar{b}} = D_{\bar{c}}\left(\mathrm{e}^{2K}(\tau-\bar{\tau})_{IJ}\bar{\chi}^I_{\bar{a}}\bar{\chi}^J_{\bar{b}}\right) = -\mathrm{e}^{2K}\overline{C}_{IJK}\bar{\chi}^I_{\bar{a}}\bar{\chi}^J_{\bar{b}}\bar{\chi}^K_{\bar{c}} - 2\mathrm{e}^{2K}(\tau-\bar{\tau})_{IJ}D_{\bar{c}}\bar{\chi}^I_{\bar{a}}\bar{\chi}^J_{\bar{b}}. \tag{4.167}$$

By using the conjugate of the third equation in (4.52), we find

$$\begin{aligned}-2\mathrm{e}^{2K}(\tau-\bar{\tau})_{IJ}D_{\bar{c}}\bar{\chi}^I_{\bar{a}}\bar{\chi}^J_{\bar{b}} &= -2\mathrm{e}^{2K}(\tau-\bar{\tau})_{IJ}\mathrm{i}\mathrm{e}^{K}\overline{C}_{\bar{c}\bar{a}\bar{m}}G^{\bar{m}l}\chi^I_l\bar{\chi}^J_{\bar{b}} = 2\mathrm{e}^{2K}G_{l\bar{b}}\overline{C}_{\bar{c}\bar{a}\bar{m}}G^{\bar{m}l}\\ &= 2\mathrm{e}^{2K}\overline{C}_{\bar{c}\bar{a}\bar{b}},\end{aligned} \tag{4.168}$$

where we used (4.112). We now note that (4.81) can be written as

$$C_{abc} = C_{IJK}\chi^I_a\chi^J_b\chi^K_c. \tag{4.169}$$

This is because the difference between (4.81) and (4.169) involves terms of the form $C_{IJK}X^K = 0$ due to (4.71). We conclude that

$$D_{\bar{c}}\mathsf{S}_{\bar{a}\bar{b}} = \mathrm{e}^{2K}C_{\bar{a}\bar{b}\bar{c}}. \tag{4.170}$$

From this it follows that

$$\partial_{\bar{c}}\mathsf{S}^{ab} = \mathrm{e}^{2K}G^{a\bar{a}}G^{\bar{b}}C_{\bar{a}\bar{b}\bar{c}}. \tag{4.171}$$

This is precisely the condition (4.152).

One of the most important properties of the propagators defined by (4.157) is that they form a closed ring upon taking (covariant) derivatives, and we have the following formulae:

$$\begin{aligned}D_a\mathsf{S}^{bc} &= \delta^b_a\mathsf{S}^c + \delta^c_a\mathsf{S}^b - C_{ade}\mathsf{S}^{db}\mathsf{S}^{ec},\\ D_a\mathsf{S}^b &= 2\delta^b_a\mathsf{S} - C_{ade}\mathsf{S}^d\mathsf{S}^{eb},\\ D_a\mathsf{S} &= -\frac{1}{2}C_{abc}\mathsf{S}^b\mathsf{S}^c,\\ D_iK_j &= -K_iK_j - C_{ijk}\mathsf{S}^k + C_{ijk}\mathsf{S}^{kl}K_l + g_{ij},\end{aligned} \tag{4.172}$$

where g_{ij} is a holomorphic function.

Let us first establish the first three equations. To do this, we need some preliminary results. From (4.114) we obtain

$$\mathrm{i}\mathrm{e}^{K}\bar{\chi}^J_{\bar{\beta}} = -G_{\alpha\bar{\beta}}(\tau-\bar{\tau})^{-1JK}\chi^\alpha_K. \tag{4.173}$$

We can now combine this equation with the third one in (4.52) to obtain

$$\begin{aligned}D_r\chi^I_s &= -C_{rs}{}^{\bar{q}}G_{\bar{q}m}(\tau-\bar{\tau})^{-1IK}\chi^m_K = -C_{rsm}(\tau-\bar{\tau})^{-1IK}\chi^m_K\\ &= C_{rsm}\mathsf{S}^{mn}\chi^I_n.\end{aligned} \tag{4.174}$$

We can now use this result to calculate $D_i\chi^\alpha_I$. By taking derivatives in $\chi^\alpha_I\chi^J_\alpha = \delta^J_I$, we have

$$\chi^J_\alpha D_i\chi^\alpha_I = -\chi^\alpha_I D_i\chi^J_\alpha = -\chi^m_I D_i\chi^J_m - \chi^J_i\chi^0_I = -C_{irm}\mathsf{S}^{mn}\chi^J_n\chi^r_I - \chi^J_i\chi^0_I. \tag{4.175}$$

We conclude that

$$D_i\chi^\beta_I = -C_{irm}\mathsf{S}^{m\beta}\chi^r_I - \delta^\beta_i\chi^0_I. \tag{4.176}$$

Let us now write the propagator (4.157) as

$$\mathsf{S}^{\alpha\beta} = -\mathrm{i}\mathrm{e}^{K}\mathfrak{G}^{IJ}\chi^\alpha_I\chi^\beta_J, \tag{4.177}$$

where

$$\mathfrak{G}_{KL} = \mathrm{ie}^K (\tau - \bar{\tau})_{KL}, \qquad \mathfrak{G}^{KL} = -\mathrm{ie}^{-K} (\tau - \bar{\tau})^{-1KL}, \tag{4.178}$$

are, up to an overall factor, the Kähler metric (4.101) and its inverse. We easily calculate

$$D_a \mathfrak{G}_{KL} = \mathrm{ie}^K D_a \tau_{KL} = \mathrm{ie}^K C_{KLM} \chi_a^M = \mathrm{ie}^K C_{abc} \chi_K^b \chi_L^c. \tag{4.179}$$

For the inverse one finds,

$$D_a \mathfrak{G}^{LM} = \mathrm{ie}^{-K} C_{abc} (\tau - \bar{\tau})^{-1LI} \chi_I^b (\tau - \bar{\tau})^{-1MJ} \chi_J^c. \tag{4.180}$$

We now have all the ingredients to calculate the covariant derivative of the propagator. We find,

$$D_c \mathsf{S}^{\alpha\beta} = -\mathrm{ie}^K D_c \mathfrak{G}^{IJ} \chi_I^\alpha \chi_J^\beta - \mathrm{ie}^K \mathfrak{G}^{IJ} D_c \chi_I^\alpha \chi_J^\beta - \mathrm{ie}^K \mathfrak{G}^{IJ} D_c \chi_I^\beta \chi_J^\alpha, \tag{4.181}$$

i.e. the last term is obtained from the second one by exchanging α and β. By using the results above, we obtain

$$\begin{aligned} D_c \mathsf{S}^{\alpha\beta} &= C_{cpq} \mathsf{S}^{\alpha p} \mathsf{S}^{\beta q} + \mathrm{ie}^K \mathfrak{G}^{IJ} C_{crm} \mathsf{S}^{m\alpha} \chi_I^r \chi_J^\beta + \mathrm{ie}^K \mathfrak{G}^{IJ} \delta_j^\alpha \chi_I^0 \chi_J^\beta + (\alpha \leftrightarrow \beta) \\ &= -C_{cpq} \mathsf{S}^{\alpha p} \mathsf{S}^{\beta q} - \delta_j^\alpha \mathsf{S}^{0\beta} - \delta_j^\beta \mathsf{S}^{0\alpha}. \end{aligned} \tag{4.182}$$

This gives the first three equations of (4.172), as we consider the different possibilities for the indices.

We want to show now that

$$D_i K_j = -K_i K_j - C_{ijk} \mathsf{S}^k + C_{ijk} \mathsf{S}^{kl} K_l + g_{ij}. \tag{4.183}$$

We first need the result:

$$\partial_{\bar{k}} \chi_I^a = 0, \quad a \neq 0, \tag{4.184}$$

in other wordds, χ_I^a is holomorphic. In particular, it can be evaluated by (4.151). In addition, we have

$$\partial_{\bar{k}} \chi_I^0 = -\partial_{\bar{k}} \left(K_m \chi_I^m \right). \tag{4.185}$$

The holomorphic property (4.184) is easily proved by using (4.108):

$$\partial_{\bar{k}} \chi_I^\alpha = -\chi_I^\beta \chi_J^\alpha \partial_{\bar{k}} \chi_\beta^J = -\chi_I^\beta \chi_J^\alpha \partial_{\bar{k}} K_\beta \chi_0^J = -\chi_I^\beta \chi_J^\alpha G_{\bar{k}\beta} \chi_0^J = -\chi_I^\beta G_{\bar{k}\beta} \delta_0^\alpha, \tag{4.186}$$

where we set $K_0 = 0$. This vanishes if $\alpha = i$, and for $\alpha = 0$ it gives

$$\partial_{\bar{k}} \chi_I^0 = -\chi_I^\beta G_{\bar{k}\beta} = -\chi_I^\beta \partial_{\bar{k}} K_\beta = -\partial_{\bar{k}} \left(-\chi_I^\beta K_\beta \right) = -\partial_{\bar{k}} \left(-\chi_I^m K_m \right). \tag{4.187}$$

From the second equality in (4.184) we deduce that

$$\chi_I^0 = -\chi_I^m K_m + h_I, \tag{4.188}$$

where h_I is a *holomorphic* function.

Example 4.4. One can use the results (4.151) to write an explicit formula for h_I, in terms of periods and their derivatives. Since h_I is holomorphic, in the definition (4.188) we can evaluate χ_I^α in the holomorphic limit, and we find

$$
\begin{aligned}
h_0 &= \frac{1}{X^0} + \frac{t^b}{(X^0)^2}\frac{\partial X^0}{\partial z^m}\frac{\partial z^m}{\partial t^b}, \\
h_a &= -\frac{1}{(X^0)^2}\frac{\partial X^0}{\partial z^m}\frac{\partial z^m}{\partial t^a},
\end{aligned}
\tag{4.189}
$$

where summation over repeated indices is understood. An alternative expression can be found in the one modulus case. We have

$$
\chi_\alpha{}^I = \begin{pmatrix} X^0 & X^1 \\ D_z X^0 & D_z X^1 \end{pmatrix}, \qquad \chi_I{}^\alpha = \frac{1}{W}\begin{pmatrix} D_z X^1 & -X^1 \\ D_z X^0 & X^0 \end{pmatrix}, \tag{4.190}
$$

where

$$
W = X^0 (X^1)' - (X^0)' X^1, \tag{4.191}
$$

and the $'$ denotes derivative w.r.t. z. We deduce that

$$
\chi_0{}^z = -\frac{X^1}{W}, \qquad \chi_1{}^z = \frac{X^0}{W}, \tag{4.192}
$$

which are holomorphic as required by (4.184). Let us also note that, in terms of $t = X^1/X^0$, we have

$$
\chi_0{}^z = -\frac{t}{X^0}\frac{\partial z}{\partial t}, \qquad \chi_1{}^z = \frac{1}{X^0}\frac{\partial z}{\partial t}, \tag{4.193}
$$

in agreement with the result in (4.151). We also find that

$$
h = \frac{1}{W}\begin{pmatrix} (X^1)' \\ -(X^0)' \end{pmatrix}, \tag{4.194}
$$

which can be seen to agree with the general expression (4.189). □

Let us now prove (4.183). The starting point is (4.103). By taking a derivative we obtain

$$
-\,\mathrm{e}^{-K} K_j = \mathrm{i}\overline{X}^I \partial_j X^K (\tau - \bar\tau)_{IK}, \tag{4.195}
$$

where we used (4.71). Taking a second derivative gives

$$
\mathrm{e}^{-K}\left(-\partial_i K_j + K_i K_j\right) = \mathrm{i}\partial^2_{ij} X^I (\tau - \bar\tau)_{IJ} \overline{X}^J + \mathrm{i}\partial_i X^K \partial_j X^L C_{IKL} \overline{X}^I. \tag{4.196}
$$

On the other hand, by definition of S^m, we have

$$
\begin{aligned}
C_{ijm}\mathsf{S}^m &= C_{ijm}\chi_I^m (\tau - \bar\tau)^{-1IJ} \chi_J^0 = \mathrm{i}\mathrm{e}^K C_{ijm}\chi_I^m \bar\chi_{\bar 0}^I \\
&= \mathrm{i}\mathrm{e}^K C_{ijm}\chi_I^m \overline{X}^I,
\end{aligned}
\tag{4.197}
$$

where we used the definition of S^m and (4.173). By using now (4.169), we get

$$
\begin{aligned}
C_{ijm}\mathsf{S}^m &= \mathrm{i}\mathrm{e}^K\, C_{PQR}\chi_i^P \chi_j^Q \chi_m^R \chi_J^m \overline{X}^J = \mathrm{i}\mathrm{e}^K\, C_{PQR}\chi_i^P \chi_j^Q \left(\delta_J^R - \chi_0^R \chi_J^0\right) \overline{X}^J \\
&= \mathrm{i}\mathrm{e}^K\, C_{PQJ}\chi_i^P \chi_j^Q \overline{X}^J = \mathrm{i}\mathrm{e}^K\, C_{PQJ}\partial_i X^P \partial_j X^Q \overline{X}^J
\end{aligned}
\tag{4.198}
$$

where we use (4.71) repeteadly. This is essentially the last term in (4.196). To reconstruct the first term, we use (4.113) to write

$$\begin{aligned}\mathrm{i}\partial_{ij}^2 X^I(\tau-\bar\tau)_{IJ}\overline{X}^J &= -\mathrm{e}^{-K}\partial_{ij}^2 X^I\overline{X}^J G_{\alpha\bar\beta}\chi_I^\alpha\bar\chi_J^{\bar\beta} = -\mathrm{e}^{-K}\partial_{ij}^2 X^I G_{\alpha\bar\beta}\bar\chi_{\bar 0}^J\bar\chi_J^{\bar\beta}\chi_I^\alpha \\ &= -\mathrm{e}^{-K}\partial_{ij}^2 X^I G_{\alpha\bar 0}\chi_I^\alpha = \mathrm{e}^{-K}\partial_{ij}^2 X^I\chi_I^0\end{aligned} \tag{4.199}$$

We now go back to (4.196) with all these results. We find

$$\mathrm{e}^{-K}\left(-\partial_i K_j + K_i K_j\right) = \mathrm{e}^{-K}C_{ijm}\mathsf{S}^m + \mathrm{e}^{-K}\partial_{ij}^2 X^I\chi_I^0. \tag{4.200}$$

If we use (4.188) we can write

$$\begin{aligned}\partial_i K_j - K_i K_j &= -C_{ijm}\mathsf{S}^m - \partial_{ij}^2 X^I\left(-\chi_I^m K_m + h_I\right) \\ &= -C_{ijm}\mathsf{S}^m + \partial_{ij}^2 X^I\chi_I^m K_m - \partial_{ij}^2 X^I h_I \\ &= -C_{ijm}\mathsf{S}^m + f_{ij}^m K_m + g_{ij}\end{aligned} \tag{4.201}$$

where

$$f_{ij}^m = \partial_{ij}^2 X^I\chi_I^m, \qquad g_{ij} = -\partial_{ij}^2 X^I h_I \tag{4.202}$$

are holomorphic. We note that these quantities enter into the holomorphic derivatives of χ_I^m, h_I, and one finds after some work,

$$\partial_i\chi_I^k = -\delta_i^k h_I - f_{im}^k\chi_I^m, \qquad \partial_i h_I = g_{im}\chi_J^m. \tag{4.203}$$

We now can rederive from this a more precise version of (4.156). To do this, we consider

$$\partial_i K_j - K_i K_j = -C_{ijm}\mathsf{S}^m + f_{ij}^m K_m + g_{ij} \tag{4.204}$$

and take a derivative w.r.t. $\bar z^{\bar m}$. By using the first equation in (4.152), we obtain

$$\partial_i G_{j\bar m} - G_{i\bar m}K_j - G_{j\bar m}K_i = -C_{ijl}G_{\bar m b}\mathsf{S}^{bl} + f_{ij}^k G_{\bar m k}. \tag{4.205}$$

After multiplying it by $G^{a\bar m}$, we get

$$\Gamma_{ij}^a = \delta_i^a K_j + \delta_j^a K_i - C_{ijl}\mathsf{S}^{al} + f_{ij}^a. \tag{4.206}$$

This gives a concrete realization of (4.156) with an explicit expression for the propagators and also for s_{ij}^a. To derive now (4.183), we just have to recall that

$$D_i K_j = \partial_i K_j - \Gamma_{ij}^k K_k \tag{4.207}$$

and use (4.204) and (4.206).

The non-holomorphic propagators we have introduced in (4.157) are not invariant under symplectic transformations. Their transformation rule can be deduced from the definition, together with (4.125) and (4.127). One easily finds,

$$\mathsf{S}^{\alpha\beta\Gamma} = \mathsf{S}^{\alpha\beta} + \chi_K^\alpha\left[(C\tau+D)^{-1}C\right]^{KJ}\chi_J^\beta. \tag{4.208}$$

We note that the matrix $(C\tau+D)^{-1}C$ is symmetric. Indeed, this is equivalent to

$$C(C\tau+D)^T = (C\tau+D)C^T, \tag{4.209}$$

which follows from the symmetry of τ and the last condition in (4.122).

We now want to construct *invariant* propagators by adding an additional piece to the non-holomorphic propagators. More precisely, we define the full propagators

$$S^{\alpha\beta} = \mathsf{S}^{\alpha\beta} + \Delta S^{\alpha\beta} \tag{4.210}$$

in such a way that

$$S^{\alpha\beta\Gamma} = S^{\alpha\beta}. \tag{4.211}$$

Let us now implement these conditions to obtain additional information on $\Delta S^{\alpha\beta}$. We will start with S^{jk}. In this case, the additional piece ΔS^{jk} is purely holomorphic, and we write

$$\Delta S^{jk} = \mathcal{S}^{jk} \tag{4.212}$$

which gives the holomorphic limit of the propagator S^{jk}. We want to preserve the condition (4.156). By taking into account (4.206), we find that

$$\Gamma^k_{ij} = \delta^k_i K_j + \delta^k_j K_i - C_{ijl}\mathsf{S}^{lk} - C_{ijl}\mathcal{S}^{lk} + s^k_{ij}, \tag{4.213}$$

where

$$s^k_{ij} = f^k_{ij} + C_{ijk}\mathcal{S}^{lk}. \tag{4.214}$$

A typical choice is that s^k_{ij} are rational functions of the moduli z^α, therefore invariant under symplectic transformations by construction. By using the explicit expression for the f^k_{ij} in (4.202) we deduce the following constraint on the holomorphic part of the propagators:

$$C_{ijl}\mathcal{S}^{lk} = s^k_{ij} - \partial^2_{ij} X^I \chi^k_I. \tag{4.215}$$

Remark 4.5. By considering the holomorphic limit of (4.206), we find as well

$$\Gamma^k_{ij,h} = \delta^k_i K_{j,h} + \delta^k_j K_{i,h} - C_{ijl}\mathcal{S}^{lk} + s^k_{ij}, \tag{4.216}$$

where the holomorphic limits of the Christoffel symbol and the derivative of the Kähler potential are given by (4.146) and (4.138), respectively. For this equation to be compatible with (4.215), we must have the following relation:

$$\Gamma^k_{ij,h} - \delta^k_i K_{j,h} - \delta^k_j K_{i,h} = \partial^2_{ij} X^I \chi^k_I. \tag{4.217}$$

This is easily checked by taking into account (4.151).

Let us now consider the additional pieces that have to be added to the other propagators. Their structure is determined by requiring that the relations (4.153) still hold. This leads to

$$\begin{aligned} S^k &= \mathsf{S}^k + \mathcal{S}^{kl} K_l + \mathcal{S}^k, \\ S &= \mathsf{S} + \frac{1}{2}\mathcal{S}^{kl} K_k K_l + \mathcal{S}^k K_k + \mathcal{S}, \end{aligned} \tag{4.218}$$

where $\mathcal{S}^k$, $\mathcal{S}$ are *holomorphic*. Indeed, one has

$$\partial_{\bar{a}} S^k = \partial_{\bar{a}} \mathsf{S}^k + \mathcal{S}^{kl}\partial_{\bar{a}} K_l = G_{\bar{a}l}\mathsf{S}^{kl} + G_{\bar{a}l}\mathcal{S}^{kl} = G_{\bar{a}l} S^{kl}. \tag{4.219}$$

Similarly,

$$\partial_{\bar{a}} S = \partial_{\bar{a}} \mathsf{S} + \mathcal{S}^{kl} G_{\bar{a}l} K_k + G_{\bar{a}k} \mathcal{S}^k = G_{\bar{a}l} \left(\mathsf{S}^l + \mathcal{S}^{lk} K_k + \mathcal{S}^l \right) = G_{\bar{a}l} S^l. \tag{4.220}$$

We can now obtain a constraint for $\mathcal{S}^k$ by requiring that (4.204) holds with invariant versions of f_{ij}^m, g_{ij}:

$$\partial_i K_j - K_i K_j = -C_{ijk} S^k + s_{ij}^m K_m + h_{ij}. \tag{4.221}$$

By plugging the above equation for S^k we obtain that indeed (4.221) holds with

$$h_{ij} = g_{ij} + C_{ijk} \mathcal{S}^k. \tag{4.222}$$

Equivalently, we have

$$C_{ijk} \mathcal{S}^k - h_{ij} = h_I \partial_{ij}^2 X^I. \tag{4.223}$$

The holomorphic propagators $\mathcal{S}^{\alpha\beta}$ play a crucial role, and in particular we will need their transformations under the action of the symplectic group. These follow from their definition, the transformation rule (4.208), and the invariance of the full propagator (4.211). One finds that the holomorphic propagators transform as follows under a symplectic transformation:

$$\begin{aligned} \mathcal{S}^{kl\Gamma} &= \mathcal{S}^{kl} - [(C\tau + D)^{-1} C]^{IJ} \chi_I^k \chi_J^l, \\ \mathcal{S}^{k\Gamma} &= \mathcal{S}^k + [(C\tau + D)^{-1} C]^{IJ} \chi_I^k h_J, \\ \mathcal{S}^{\Gamma} &= \mathcal{S} - \frac{1}{2} [(C\tau + D)^{-1} C]^{IJ} h_I h_J. \end{aligned} \tag{4.224}$$

The first line follows simply from the fact that $S^{jk} = \mathsf{S}^{jk} + \mathcal{S}^{jk}$, so the transformation rule of $\mathcal{S}^{jk}$ is just the opposite of S^{jk}. To obtain the second line we note that

$$\mathsf{S}^k \to \mathsf{S}^k - [(C\tau + D)^{-1} C]^{IJ} \chi_J^0 \chi_I^k, \tag{4.225}$$

and

$$\mathcal{S}^{kl} K_l \to \mathcal{S}^{kl} K_l - [(C\tau + D)^{-1} C]^{IJ} \chi_J^l K_l \chi_I^k, \tag{4.226}$$

therefore invariance of S^k requires

$$\mathcal{S}^k \to \mathcal{S}^k + [(C\tau + D)^{-1} C]^{IJ} \chi_I^k (\chi_J^0 + \chi_J^l K_l) = \mathcal{S}^k + [(C\tau + D)^{-1} C]^{IJ} \chi_I^k h_J, \tag{4.227}$$

where we used the definition of h_J in (4.188). Similarly, for $\mathcal{S}$ we find that the total shift of the first three terms in the definition of S, in the second line of (4.218), involve the combination

$$\frac{1}{2} \chi_I^0 \chi_J^0 - \frac{1}{2} \chi_I^k \chi_J^l K_k K_l + \chi_I^k K_k h_J = \frac{1}{2} h_I h_J \tag{4.228}$$

where the first term appears in the shift of S, the second one in the shift of $\mathcal{S}^{kl} K_k K_l / 2$, and the third one in the shift of $\mathcal{S}^k K_k$.

As a consistency check, let us note that the transformation rules for $\mathcal{S}^{jk}$, $\mathcal{S}^k$ are compatible with the expressions (4.215), (4.223). Under a symplectic transformation, f_{ij}^k transforms as

$$f_{ij}^{k\Gamma} = f_{ij}^k + \left[(C\tau + D)^{-1} C \right]^{PJ} C_{JLK} \chi_i^L \chi_j^K \chi_P^k, \tag{4.229}$$

One the other hand, from (4.224) we find

$$\begin{aligned} C_{ijl}\mathcal{S}^{lk} &\to C_{ijl}\mathcal{S}^{lk} - C_{ijl}\chi_I^l \left[(C\tau+D)^{-1}C\right]^{IJ}\chi_J^k \\ &= C_{ijl}\mathcal{S}^{lk} - C_{PQI}\chi_i^P\chi_j^Q\chi_J^k \left[(C\tau+D)^{-1}C\right]^{IJ}, \end{aligned} \tag{4.230}$$

where we used (4.169). The second term in the second line is minus the shift due to f_{ij}^k. A similar argument can be applied to $\mathcal{S}^k$.

At this point it is useful to introduce the shifted propagators [21]:

$$\begin{aligned} \tilde{S}^{ij} &= S^{ij}, \\ \tilde{S}^i &= S^i - S^{ij}K_j, \\ \tilde{S} &= S - S^iK_i + \frac{1}{2}S^{ij}K_iK_j. \end{aligned} \tag{4.231}$$

In terms of the non-holomorphic propagators $\mathsf{S}^{\alpha\beta}$ and the holomorphic functions $\mathcal{S}^{\alpha\beta}$, we find

$$\begin{aligned} \tilde{S}^i &= \mathsf{S}^i - \mathsf{S}^{ij}K_j + \mathcal{S}^i, \\ \tilde{S} &= \mathsf{S} - \mathsf{S}^iK_i + \frac{1}{2}\mathsf{S}^{ij}K_iK_j + \mathcal{S}. \end{aligned} \tag{4.232}$$

From their expression it is easy to see that the non-holomorphic propagators $\mathsf{S}^{\alpha\beta}$ vanish in the holomorphic limit $\bar{\tau}\to\infty$. Therefore,

$$\mathcal{S}^{\alpha\beta} = \tilde{\mathcal{S}}^{\alpha\beta} \tag{4.233}$$

give the holomorphic limit of the tilded propagators. We can then write,

$$\tilde{S}^{\alpha\beta} = \tilde{\mathsf{S}}^{\alpha\beta} + \tilde{\mathcal{S}}^{\alpha\beta}, \tag{4.234}$$

where

$$\begin{aligned} \tilde{\mathsf{S}}^{ij} &= \mathsf{S}^{ij}, \\ \tilde{\mathsf{S}}^i &= \mathsf{S}^i - \mathsf{S}^{ij}K_j, \\ \tilde{\mathsf{S}} &= \mathsf{S} - \mathsf{S}^iK_i + \frac{1}{2}\mathsf{S}^{ij}K_iK_j. \end{aligned} \tag{4.235}$$

We can now use (4.172) to obtain formulae for the covariant derivatives of the invariant propagators. One obtains,

$$\begin{aligned} D_iS^{jk} &= \delta_i^jS^k + \delta_i^kS^j - C_{imn}S^{mj}S^{nk} + h_i^{jk}, \\ D_iS^j &= 2\delta_i^jS - C_{imn}S^mS^{nj} + h_i^{jk}K_k + h_i^j, \\ D_iS &= -\frac{1}{2}C_{imn}S^mS^n + \frac{1}{2}h_i^{mn}K_mK_n + h_i^jK_j + h_i, \end{aligned} \tag{4.236}$$

where h_i^{jk}, h_i^j, h_i are holomorphic functions. They are explicitly given as follows:

$$\begin{aligned} h_i^{kl} &= \mathcal{D}_i^f\mathcal{S}^{kl} - \delta_i^k\mathcal{S}^l - \delta_i^l\mathcal{S}^k + C_{imn}\mathcal{S}^{mk}\mathcal{S}^{nl}, \\ h_i^k &= \mathcal{D}_i^f\mathcal{S}^k - 2\delta_i^k\mathcal{S} + h_{im}\mathcal{S}^{mk}, \\ h_i &= \partial_i\mathcal{S} - \frac{1}{2}C_{imn}\mathcal{S}^m\mathcal{S}^n + h_{in}\mathcal{S}^n. \end{aligned} \tag{4.237}$$

In these equations, we have denoted

$$\mathcal{D}_i^f = \partial_i + f_{i*}^* \tag{4.238}$$

i.e. a (covariant) derivative where f_{ij}^k is treated as a connection[2]. In addition, one has

$$D_i K_j = -K_i K_j - C_{ijk} S^k + C_{ijk} S^{kl} K_l + h_{ij}. \tag{4.239}$$

From these equations one obtains the following ones, involving the conventional derivatives of the tilded propagators:

$$\begin{aligned}
\partial_i S^{jk} &= C_{imn} S^{mj} S^{nk} + \delta_i^j \tilde{S}^k + \delta_i^k \tilde{S}^j - s_{im}^j S^{mk} - s_{im}^k S^{mj} + h_i^{jk}, \\
\partial_i \tilde{S}^j &= C_{imn} S^{mj} S^n + 2\delta_i^j \tilde{S} - s_{im}^j \tilde{S}^m - h_{ik} S^{kj} + h_i^j, \\
\partial_i \tilde{S} &= \frac{1}{2} C_{imn} \tilde{S}^m \tilde{S}^n - h_{ij} \tilde{S}^j + h_i, \\
\partial_i K_j &= K_i K_j - C_{ijn} S^{mn} K_m + s_{ij}^m K_m - C_{ijk} \tilde{S}^k + h_{ij}.
\end{aligned} \tag{4.240}$$

The first three equations in (4.240) can be used to check (4.237). Indeed, remember that $\mathcal{S}^{\alpha\beta}$ are the holomorphic limits of the tilded propagators. Therefore, the holomorphic limit of the first three equations in (4.240) must reproduce the equations in (4.237). For $\mathcal{S}$ the agreement is immediate. For the other two propagators, we have to use (4.214).

As a final check of the formalism, we can show that the second equation in (4.237), when combined with the first two equations of (4.224), leads to the transformation rule for $\mathcal{S}$. Let us abbreviate $M = (C\tau + D)^{-1} C$. Then,

$$\partial_i M^{IJ} = -M^{IP} C_{PQR} M^{QJ} \chi_i^R. \tag{4.241}$$

It follows that

$$\partial_i \mathcal{S}^{k\Gamma} = \partial_i \mathcal{S}^k - M^{IP} C_{PQR} M^{QJ} \chi_i^R \chi_I^k h_J + M_{IJ} \left(h_J \partial_i \chi_I^k + \chi_I^k \partial_i h_J \right). \tag{4.242}$$

The other ingredients we need are

$$f_{ij}^{k\Gamma} \mathcal{S}^{j\Gamma} = f_{ij}^k \mathcal{S}^{j\Gamma} + M^{IJ} C_{JLR} \chi_i^L \chi_j^R \chi_I^k \mathcal{S}^j + f_{ij}^k M^{IJ} \chi_I^j h_J + M^{IP} C_{PQR} M^{QJ} \chi_i^R \chi_I^k h_J, \tag{4.243}$$

where we simplified the last term by using (4.71), and

$$h_{im} \mathcal{S}^{mk\Gamma} = h_{im} \mathcal{S}^{mk} - M^{IJ} h_{im} \chi_I^i \chi_J^m. \tag{4.244}$$

By using (4.81), we write (4.223) as

$$C_{IPR} \chi_i^I \chi_j^R \mathcal{S}^j = \chi_I^m (h_{im} - g_{im}). \tag{4.245}$$

Therefore, the second term in the r.h.s. of (4.243) combines with the second term in the r.h.s. of (4.244) and we get at the end

$$2\delta_i^k (\mathcal{S}^\Gamma - \mathcal{S}) = M^{IJ} \left(h_J \partial_i \chi_I^k + \chi_I^k \partial_i h_J + f_{ij}^k \chi_I^j h_J - \chi_I^m \chi_J^k g_{im} \right) = -M^{IJ} h_I h_J \delta_i^k, \tag{4.246}$$

after using (4.203).

[2]The second and third equations in (4.237), as well as (4.239), are stated erroneously in [14].

We note that the equations (4.224) follow then from (4.215), (4.223) and the structural equations (4.240), once we assume a transformation of the form

$$X^{I\Gamma} = C^{IJ}\mathcal{F}_J + D^I{}_J X^J, \tag{4.247}$$

where C, D satisfy $CD^T = DC^T$. We don't need to assume that we have a fully-fledged symplectic transformation to use these equations, and there are cases in the literature where the transformation that we need in order to go from one frame to the other is slightly more general than symplectic[3]. The only subtlety in this derivation is that, from (4.215) and (4.223), one finds the first two equations of (4.224), but multiplied by matrix of Yukawa couplings, i.e. we find

$$C_{ijk}(\mathcal{S}^{kl\Gamma} - \mathcal{S}^{kl}) = -[(C\tau + D)^{-1}C]^{IJ} C_{ijk}\chi_I^k \chi_J^l, \tag{4.248}$$

and similarly for $\mathcal{S}^k$ (multiplied by C_{ijk}). Therefore, in order to deduce (4.224), one has to "invert" the Yukawa coupling matrix. A similar issue is discussed in [15], section 6.3, where it is argued that this can be done.

5 Mirror symmetry

5.1 Statement and examples

Mirror symmetry is a surprising statement. It says that, given a CY manifold X, there is another CY manifold $\tilde{X}$ such that the A model on X is equivalent to the B model on $\tilde{X}$, and viceversa. The underlying reason for this equivalence is the fact that both models are obtained from the same supersymmetric sigma model by a "twisting" procedure. One consequence of this equivalence is that the prepotentials of both theories are the same,

$$F_0^{\mathrm{A}}(t_A; X) = F_0^{\mathrm{B}}(t_B; \tilde{X}), \tag{5.1}$$

In order for this statement to be meaningful, we need an appropriate identification between the Kähler parameters t_A^a appearing in the A-model prepotential (see (3.49) and the moduli that parametrize the complex structures. This identification is called the *mirror map*. It turns out that the Kähler parameters t_A^a have to be identified with the quotient of periods in (4.82), which we will denote t_B^a

$$t_B^a = t_A^a. \tag{5.2}$$

Notice that for mirror symmetry to make sense, one needs

$$h^{1,1}(X) = h^{1,2}(\tilde{X}). \tag{5.3}$$

Mirror symmetry makes possible to compute the complicated instanton expansion of the A model prepotential (3.49) in terms of period integrals in the B model, and it has been extensively studied in the last fifteen years, see [8] for a detailed exposition. We will now explain this computation and check some examples of mirror symmetry.

There are mirror symmetry constructions for many different Calabi–Yau manifolds. A good survey can be found in [8, 23, 24]. A nice example is the quintic CY. Its mirror must have one

[3] For example, in the case of one-parameter CY manifolds with three singular points, a natural basis of periods at the conifold is introduced in [22], and it is shown that the transformation going from the standard large radius periods to that basis is not symplectic, but a generalization thereof.

single complex structure and 101 two-classes. This uniparametric mirror family can be regarded as a subfamily of the quintic hypersufaces in $\mathbb{P}^4$

$$\sum_{i=0}^{4} x_i^5 - \psi \prod_{i=0}^{4} x_i = 0, \tag{5.4}$$

modded out by a symmetry group $\mathbb{Z}_5^3$ (which has fixed points) and resolving the orbifold singularities (see the seminal paper [25] for details). This family has a single complex deformation parameter, namely ψ. In the next section we will see how to compute the periods for this mirror quintic.

The Enriques CY is an interesting example since it is self-mirror. Of course, this is consistent with its Hodge numbers (2.40). In this way, the prepotential (4.91) can be identified with the prepotential of the A model after identifying

$$t^a = \int_{\eta^a} \mathcal{J}, \qquad S = \int_{\eta_S} \mathcal{J}, \tag{5.5}$$

where η^a, $a = 1, \cdots, 10$, η_S is a basis of two cycles corresponding respectively to the two-cycles of the Enriques surface and the torus. Notice that there are no instanton corrections to the prepotential in this case [26].

5.2 Computing the periods

We have now a very powerful framework to reformulate the computation of the correlation functions in the type B model. We have reduced the problem to the problem of computing the periods of the CY manifold. We now address how to obtain these.

The most powerful method is the use of *Picard–Fuchs equations*. It turns out that the periods of the CY manifold satisfy a certain set of differential equations that can be solved in closed form. The basis of solutions of these differential equations provide a linear basis for the periods.

Let us consider the quintic mirror (5.4). The PF equations will depend on

$$z = \psi^{-5}. \tag{5.6}$$

If we introduce the operator

$$\theta = z\partial_z \tag{5.7}$$

the PF equation reads

$$\Big[\theta^4 - 5z(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4)\Big]\Pi = 0 \tag{5.8}$$

where Π denotes a generic period. The solutions to this differential equation can be generated by Frobenius method, which applies to systems of differential equations of the above type with various variables z_i. One first introduces

$$\varpi_0(z, \rho) = \sum_{n\geq 0} a_n(\rho) z^{n+\rho}, \tag{5.9}$$

where we denoted

$$z^{n+\rho} = \prod_i z_i^{n_i+\rho_i} \tag{5.10}$$

as well as the derivative operators

$$\mathcal{D}_{i_1\cdots i_n} = \frac{1}{n!}\frac{\partial^n}{\partial\rho_{i_1}\cdots\partial\rho_{i_n}}\bigg|_{\rho=0}. \tag{5.11}$$

Then, we define

$$\varpi_i^{(1)}(z) = \mathcal{D}_i\varpi_0(z), \tag{5.12}$$

which has the structure

$$\varpi_i^{(1)}(z) = \widetilde{\varpi}_i^{(1)}(z) + \varpi_0(z)\log\, z_i, \tag{5.13}$$

where

$$\widetilde{\varpi}_i^{(1)}(z) = \sum_{n\geq 0} d_n z^n, \qquad d_n = \frac{\mathrm{d}a_n(\rho)}{\mathrm{d}\rho_i}\bigg|_{\rho=0}. \tag{5.14}$$

The Kähler parameters of the mirror manifold are identified with the above solution,

$$-\,t^i(z) = \frac{\varpi_i^{(1)}(z)}{\varpi_0(z)} = \log\, z_i + \frac{\widetilde{\varpi}_i^{(1)}(z)}{\varpi_0(z)}. \tag{5.15}$$

This is often called the *mirror map*, since it relates the Kähler parameters of the mirror manifold to the complex coordinate which appears in the algebraic equations describing the family of CY manifolds. By comparing to the definition of the flat coordinates, we have

$$X^0(z) = \varpi_0(z), \qquad X^i(z) = -\varpi_i^{(1)}(z). \tag{5.16}$$

Let us now define

$$\varpi_i^{(2)} = \kappa_{ijk}\mathcal{D}_{jk}\varpi_0(z), \qquad \varpi^{(3)} = \kappa_{ijk}\mathcal{D}_{ijk}\varpi_0(z) \tag{5.17}$$

where κ_{ijk} is the classical intersection number. The functions

$$\varpi_0(z), \qquad \varpi_i^{(1)}(z), \qquad \varpi_i^{(2)}(z), \qquad \varpi^{(3)}(z), \tag{5.18}$$

with $i = 1, \cdots, h^{2,1}(X)$, provide a basis for the periods (over $\mathbb{C}$). We will sometimes call this the *Frobenius basis.* We can also obtain the prepotential F_0 of the CY manifold from the equation

$$\frac{\partial F_0}{\partial t^i} = \frac{1}{\varpi_0(z)}\left\{-\varpi_i^{(2)}(z) - \frac{(2\pi)^2}{24}\left(\int_X c_2\wedge J_i\right)\right\}, \tag{5.19}$$

or equivalently

$$\mathcal{F}_i = -\varpi_i^{(2)}(z) - \frac{(2\pi)^2}{24}\left(\int_X c_2\wedge J_i\right). \tag{5.20}$$

The third period calculates $\mathcal{F}_0$, up to a global factor, and we have

$$2F_0 - 2t^i\partial_i F_0 = \frac{1}{\varpi_0(z)}\left\{-\varpi_i^{(3)}(z) + \frac{(2\pi)^2}{24}\left(\int_X c_2\wedge J_i\right)\varpi_i^{(1)}(z) - \zeta(3)\chi\right\}, \tag{5.21}$$

where χ is the Euler characteristic of the CY manifold. Equivalently, we have

$$\mathcal{F}_0 = -\varpi_i^{(3)}(z) + \frac{(2\pi)^2}{24}\left(\int_X c_2\wedge J_i\right)\varpi_i^{(1)}(z) - \zeta(3)\chi. \tag{5.22}$$

It is useful to write this in matrix form,

$$\Pi = \begin{pmatrix} \mathcal{F}_0 \\ \mathcal{F}_j \\ X^0 \\ X^j \end{pmatrix} = \begin{pmatrix} -\zeta(3)\chi & (2\pi)^2 \frac{c_2^k}{24} & 0 & -1 \\ -(2\pi)^2 \frac{c_2^j}{24} & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varpi_0 \\ \varpi_k^{(1)} \\ \varpi_k^{(2)} \\ \varpi^{(3)} \end{pmatrix}, \tag{5.23}$$

where we have denoted

$$c_2^k = \int_X c_2 \wedge J_k. \tag{5.24}$$

These results were first obtained for the quintic CY in [25], and they were generalized to more general CY manifolds in e.g. [27]. The resulting structure for the prepotential is

$$F_0(t^i) = -\frac{\kappa_{ijk} t^i t^j t^k}{6} - \frac{(2\pi)^2}{24} \left(\int_X c_2(X) \wedge J_i \right) t^i - \frac{\chi}{2} \zeta(3) + F_0^{\text{inst}}(t^i), \tag{5.25}$$

where

$$F_0^{\text{inst}}(t^i) = \sum_\beta N_{0,\beta} Q^\beta \tag{5.26}$$

is due to worldsheet instantons[4]. Note that, as compared to (3.49), the expression above contains additional constant and linear terms in t^i. These are allowed since the prepotential is defined by its third derivative, as in (3.50). Sometimes one adds to the prepotential a further imaginary term, quadratic in the flat coordinates t^i

$$F_0(t) = -\pi \mathrm{i} \sigma_{ij} t^i t^j + \cdots \tag{5.27}$$

In the one-modulus case one has (see e.g. [22, 27])

$$\sigma = \frac{\kappa}{2} \bmod 1. \tag{5.28}$$

Let us apply this general formalism to the quintic CY. The solution for $\varpi_0(z,\rho)$ is obtained immediately from the PF equation. By plugging the power series ansatz, we find the recursion

$$(n+\rho) a_n = 5(5\rho + 5n - 4) \cdots (5\rho + 5n - 1) a_{n-1}. \tag{5.29}$$

and setting the initial condition $a_0 = 1$ we find,

$$a_n(\rho) = \frac{\prod_{m=1}^{5n} (m + 5\rho)}{\prod_{m=1}^{n} (m+\rho)^5}, \tag{5.30}$$

therefore

$$\varpi_0(z) = \sum_{n=0}^\infty \frac{(5n)!}{(n!)^5} z^n. \tag{5.31}$$

To compute d_n, we write

$$a_n(\rho) = \frac{\Gamma(5n + 5\rho + 1) \Big(\Gamma(\rho+1) \Big)^5}{\Gamma(5\rho + 1) \Big(\Gamma(n + \rho + 1) \Big)^5} \tag{5.32}$$

[4] In comparing this convention with the one in e.g. [22], we note that $t = -2\pi \mathrm{i} t^{\text{BKSZ}}$ and $F_0^{\text{BKSZ}} = -F_0$.

therefore

$$\frac{\mathrm{d}a_n(\rho)}{\mathrm{d}\rho} = a_n(\rho)\Big[5\psi(5n+5\rho+1) + 5\psi(\rho+1) - 5\psi(5\rho+1) - 5\psi(n+\rho+1)\Big] \tag{5.33}$$

and

$$d_n = 5\frac{(5n)!}{(n!)^5}\Big[\psi(5n+1) - \psi(n+1)\Big]. \tag{5.34}$$

Using the standard identity

$$\psi(x+n) - \psi(x) = \sum_{k=0}^{n-1}\frac{1}{x+k}, \tag{5.35}$$

we can write the mirror map in the form

$$Q = \mathrm{e}^{-t} = z\exp\left[\frac{5}{\varpi_0(z)}\sum_{n=1}^{\infty}\frac{(5n)!}{(n!)^5}\Big(\sum_{j=n+1}^{5n}\frac{1}{j}\Big)z^n\right]. \tag{5.36}$$

The first few terms are

$$Q = z + 770\,z^2 + 1014275\,z^3 + 1703916750\,z^4 + 3286569025625\,z^5 + \cdots, \tag{5.37}$$

which can be inverted to give

$$z = Q - 770\,Q^2 + 171525\,Q^3 - 81623000\,Q^4 - 35423171250\,Q^5 + \cdots \tag{5.38}$$

We can also compute

$$\begin{aligned}\frac{\mathrm{d}^2a_n(\rho)}{\mathrm{d}\rho^2}\Big|_{\rho=0} &= a_n(0)\Big[25(\psi(1+5n)-\psi(1+n))^2 + 25\psi'(1+5n) - 5\psi'(1+n) - 20\psi'(1)\Big]\\ &= a_n(0)\left[25\Big(\sum_{j=n+1}^{5n}\frac{1}{j}\Big)^2 - 25\sum_{j=1}^{5n}\frac{1}{j^2} + 5\sum_{j=1}^{n}\frac{1}{j^2}\right],\end{aligned} \tag{5.39}$$

where we used

$$\psi'(n) = \frac{\pi^2}{6} - \sum_{k=1}^{n-1}\frac{1}{k^2}, \quad n>1, \qquad \psi'(1) = \frac{\pi^2}{6}. \tag{5.40}$$

Using the value of the classical intersection number

$$\kappa = 5 \tag{5.41}$$

we can also compute $\partial_t F_0$. Notice that this will be expressed in terms of z, but using the inverse mirror map (5.38) we will be able to express it in terms of Q. Finally, we obtain for the prepotential

$$\begin{aligned}F_0(t) &= -\frac{5}{6}t^3 - (2\pi)^2\frac{50}{24}t + 100\zeta(3)\\ &\quad + 2875\,Q + \frac{4876875\,Q^2}{8} + \frac{8564575000\,Q^3}{27} + \frac{15517926796875\,Q^4}{64} + \cdots\end{aligned} \tag{5.42}$$

which is the famous prepotential of the quintic Calabi–Yau manifold as computed by Candelas, Green, Parkes and de la Ossa [25]. The rational numbers appearing here are the genus zero

Gromov–Witten invariants $N_{0,\beta}$ of the quintic. In particular, there are 2875 holomorphic spheres of degree one in the quintic. To extract integer numbers from $N_{0,\beta}$ one has to take into account some subtle topological effects. What happens is roughly that, given a "primitive" map of the sphere into the quintic, in the homology class labelled by β, one can consider multicoverings of the same map of degree d. It turns out that each multicovering contributes

$$\frac{1}{d^3} Q^{d\beta}. \tag{5.43}$$

Therefore, the sum over Gromov–Witten invariants becomes

$$\sum_\beta N_{0,\beta} Q^\beta = \sum_\beta n_{0,\beta} \mathrm{Li}_3(Q^\beta). \tag{5.44}$$

The numbers $n_{0,\beta}$ turn out to be integers. Formulae (5.43), (5.44) were first noted in [25], and justified later in the context of Gromov–Witten theory by various mathematicians, see [28] for a proof and references to previous work. The general picture of how to extract integer invariants from Gromov–Witten invariants was put forward by Gopakumar and Vafa in [29], and for this reason the invariants $n_{0,\beta}$ are often called *Gopakumar–Vafa invariants* (in this case, of genus zero). From (5.44) and (5.42) we find that

$$n_{0,2} = \frac{4876875}{8} - \frac{2875}{8} = 609250. \tag{5.45}$$

Exercise 5.1. Local $\mathbb{P}^2$ has one single Kähler parameter, associated to the complexified area of $\mathbb{P}^1 \subset \mathbb{P}^2$. The PF equation describing the periods of the mirror of local $\mathbb{P}^2$ is

$$\Big[\theta^3 + 3z(3\theta+2)(3\theta+1)\theta\Big]\Pi = 0. \tag{5.46}$$

Notice that in this example $\varpi_0(z) = 1$. Compute the prepotential by using the classical intersection number $\kappa = -1/3$ (the fact that this number is fractional is related to the noncompactness of this CY).

It is important sometimes to have an *integral* basis of periods, corresponding to integrals over integral homology cycles. These can be obtained from the Frobenius basis as follows:

$$\Pi_{\mathbb{Z}} = \begin{pmatrix} \Phi_0 \\ \Phi_j \\ J^0 \\ J^j \end{pmatrix} = \begin{pmatrix} \frac{\zeta(3)\chi}{(2\pi \mathrm{i})^3} & \frac{c_2^k}{24(2\pi \mathrm{i})} & 0 & (2\pi \mathrm{i})^{-3} \\ \frac{c_2^j}{24} & \frac{\sigma_{jk}}{2\pi \mathrm{i}} & -\frac{1}{(2\pi \mathrm{i})^2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & (2\pi \mathrm{i})^{-1}\delta_k^j & 0 & 0 \end{pmatrix} \begin{pmatrix} \varpi_0 \\ \varpi_k^{(1)} \\ \varpi_k^{(2)} \\ \varpi^{(3)} \end{pmatrix} \tag{5.47}$$

Note that

$$\Phi_0 = -\frac{1}{(2\pi \mathrm{i})^3}\mathcal{F}_0, \qquad \Phi_i = \frac{1}{(2\pi \mathrm{i})^2}\mathcal{F}_i - \frac{\sigma_{ij}}{2\pi \mathrm{i}} X^j, \qquad J^i = -\frac{1}{2\pi \mathrm{i}} X^i. \tag{5.48}$$

In the literature on mirror symmetry, integral periods are often used to define the prepotential. This convention introduces factors $2\pi \mathrm{i}$ and is not always convenient if one wants e.g. to extract Gromov–Witten invariants, hence our simpler choice for X^I, $\mathcal{F}_I$.

6 Topological strings

6.1 Coupling to gravity and holomorphic anomaly equations

What we have described so far are the topological field theories/CFTs that are behind topological strings, but in order to obtain a string theory properly speaking we have to couple the models to gravity. It turns out that topological strings behave in this respect like *critical* strings. Indeed, what characterizes critical strings is that the 2d metric field essentially decouples, except for a finite number of moduli. But in the topological sigma models the property (3.2) tells us that a similar decoupling occurs. Indeed, in the BRST quantization of the critical bosonic string, one finds a nilpotent BRST operator, $\mathcal{Q}_{\rm BRST}$, and the energy-momentum tensor turns out to be a $\mathcal{Q}_{\rm BRST}$-commutator

$$T(z) = \{\mathcal{Q}_{\rm BRST}, b(z)\}. \tag{6.1}$$

In addition, there is a ghost number with anomaly $3\chi(\Sigma_g) = 6 - 6g$, in such a way that $\mathcal{Q}_{\rm BRST}$ and $b(z)$ have ghost number 1 and -1, respectively. This structure makes possible to define the partition function of the bosonic string at genus g, as

$$F_g^{\rm bos} = \int_{\overline{M}_g} \left\langle \prod_{k=1}^{6g-6} (b, \mu_k) \right\rangle, \tag{6.2}$$

where

$$(b, \mu_k) = \int_{\Sigma_g} {\rm d}^2 z (b_{zz} (\mu_k)_{\bar z}{}^z + b_{\bar z \bar z} (\overline{\mu}_k)_z{}^{\bar z}), \tag{6.3}$$

and μ_k are the Beltrami differentials, and can be regarded as deformations of the complex structure of the Riemann surface. The insertions of the b field are there to soak up the zero modes of the ghost current.

The $\mathcal{Q}$-exactness of (3.2) is analogue to (6.1), and the composite field $G_{\mu\nu}$ plays the role of an anti-ghost. Furthermore, the anomaly in the ghost current for the twisted sigma model is precisely $6g - 6$ for a Calabi–Yau threefold, and therefore the ghost number symmetry in these models plays exactly the same role as the ghost number symmetry in the bosonic string. Therefore, we can define the higher genus amplitudes or free energies of the topological string as

$$F_g = \int_{\overline{M}_g} \left\langle \prod_{k=1}^{6g-6} (G, \mu_k) \right\rangle, \qquad g \geq 2 \tag{6.4}$$

with G instead of b. In terms of the moduli space of complex structures, F_gs have to be thought as sections of the line bundle $\mathcal{L}^{2-2g}$. In particular, their covariant derivative reads

$$D_a F_g = \partial_a F_g + (2 - 2g) K_a F_g. \tag{6.5}$$

For $g = 1$ there is a slightly different expression [30]

$$F_1 = \frac{1}{2} \int \frac{{\rm d}^2 \tau}{\tau_2} {\rm Tr} \Big[(-1)^{J_0 + \overline{J}_0} J_0 \overline{J}_0 q^{L_0} \bar{q}^{\overline{L}_0} \Big]. \tag{6.6}$$

Here, τ is the modular parameter of the torus, $q = {\rm e}^{2\pi {\rm i} \tau}$, and J_0, $\overline{J}_0$ are the charges associated to the $J, \overline{J}$ currents of the twisted superconformal theory. Of course, they can be written in terms of $F_{L,R}$ following (3.100).

The F_g defined here are computed in a topological theory which includes a general perturbation like (3.112), i.e. we consider the action

$$S(t,\bar{t}) = S_0 + \sum_{a=1}^{\ell} t^a \int \mathrm{d}^2 z\, \phi_a^{(2)} + \sum_{a=1}^{\ell} \bar{t}^a \int \mathrm{d}^2 z\, \overline{\phi}_a^{(2)}, \tag{6.7}$$

where S_0 is the action at a basepoint and $\ell = h^{1,1}(X)$ (respectively, $h^{1,2}(X)$) in the A (B) model.

We can now try to compute the above amplitudes by using the same kind of semiclassical approximation that proved to be so useful in the topological sigma models. However, there is an obstruction to do that. It turns out that the F_g are much more subtle than the topological sigma models we started with, since there are obstructions to use the kind of arguments based on integration by parts in moduli space that we used in (3.6) and (3.8). These obstructions are the content of the *holomorphic anomaly equations* (HAE) of Bershadsky, Cecotti, Ooguri and Vafa (BCOV) [15].

One could think that the perturbation depending on $\bar{t}^a$ does not affect the computation of F_g, since this is a $\mathcal{Q}$-exact operator. in that case, the F_g would be holomorphic functions of the t^a. But this is not the case. If we compute

$$\frac{\partial F_g}{\partial \bar{t}^{\bar{a}}} = \langle \{\mathcal{Q}, [Q_0, \overline{\phi}_a^{(2)}]\}\rangle, \tag{6.8}$$

we would we tempted to think that this is the integral of a total derivative and therefore that it vanishes. But this is not so, since we are now integrating over the moduli space of metrics of a two-dimensional Riemann surface Σ_g and this is known to have boundaries. The computation of (6.8) can be done in detail using the techniques of CFT on Riemann surfaces, and the result reads [15]

$$\bar{\partial}_{\bar{k}} F_g = \frac{1}{2}\overline{C}_{\bar{k}}^{ij}\left(D_i D_j F_{g-1} + \sum_{r=1}^{g-1} D_i F_r D_j F_{g-r}\right) . \tag{6.9}$$

for $g > 1$ and

$$\bar{\partial}_{\bar{k}} \partial_m F_1 = \frac{1}{2}\overline{C}_{\bar{k}}^{ij} C_{mij} - \left(\frac{\chi}{24} - 1\right) G_{\bar{k}m}, \tag{6.10}$$

for $g = 1$. In these equations, χ is the Euler number of the CY X, C_{ijk} is the Yukawa coupling, D_i is the covariant derivative introduced in (4.38), and $\overline{C}_{\bar{k}}^{ij}$ is defined in (4.58).

What are the consequence of the holomorphic anomaly? First of all, as we explained in (3.128), the dependence on $\bar{t}^{\bar{a}}$ is a antiholomorphic dependence on the conjugate deformation parameters (Kähler moduli in the A-model, complex moduli in the B-model). But, as emphasized by Witten [31], this means that the topological string amplitudes depend on a choice of a background point in the moduli space of the model. This is because, even before perturbing, when we write the action as in (3.128), the correlation functions will depend on $\overline{\mathcal{G}}$, which is the antiholomorphic part of our choice of basepoint. By the same token, if we introduce an explicit coupling constant $\hbar$ as we did in (3.8), there is no decoupling and there are corrections to the evaluation by instantons.

There is therefore a $\bar{t}$ dependence

$$F_g(t,\bar{t}) \tag{6.11}$$

in the topological string amplitudes, which in general cannot be computed solely by considering semiclassical configurations. There is however a choice of basepoint where indeed one can compute via semiclassical methods, namely when $\bar{t} \to \infty$. This is because in this limit any configuration which gives a nonzero result for the bosonic part of the first term in (3.128) is exponentially suppressed, and by looking at (3.127) we see that in the A model, for example, this forces maps to be holomorphic. One can indeed verify that in this limit

$$\mathcal{F}_g(t) = \lim_{\bar{t}\to\infty} F_g(t, \bar{t}) \tag{6.12}$$

can be computed as a sum over worldsheet instantons of genus g. For $g = 1$ we have [30]

$$\mathcal{F}_1(t) = \frac{1}{24} \sum_a t^a \int_X c_2(X) \wedge J_a + \sum_\beta N_{1,\beta} Q^\beta, \tag{6.13}$$

while for $g \geq 2$,

$$F_g(t) = c_g \chi(X) + \sum_\beta N_{g,\beta}\, \mathrm{e}^{-\beta\cdot t}. \tag{6.14}$$

In (6.13) the first term involves the second Chern class of X, while in (6.14) the first term is the so-called contribution of constant maps [15]. χ is the Euler characteristic of X, and c_g is given by an integral of the moduli space of Riemann surfaces, whose value can be obtained explicitly by using string dualities [32] or by a direct calculation [33]

$$c_g = \frac{(-1)^{g-1} B_{2g} B_{2g-2}}{4g(2g-2)(2g-2)!}. \tag{6.15}$$

In (6.13), (6.14), $N_{g,\beta}$ are the Gromov–Witten invariants of genus g and in the class β. They are rational numbers that "count" holomorphic maps at genus g in the class n_I, and they can be computed by integrating an appropriate function over the space of collective coordinates of the instanton. This computation is not easy (to put it mildly). As in the case of the Gromov–Witten invariants of genus zero, it is possible to extract from them integer Gopakumar–Vafa invariants.

The holomorphic anomaly actually makes possible to compute $F_g(t, \bar{t})$ (and then its holomorphic limit) when combined with extra information about the geometry of the CY manifold. In fact, in practical terms, the holomorphic anomaly is the only effective, available method to compute F_g for general compact CY targets.

6.2 The genus one free energy

The equation (6.10) can be integrated as follows. From the special geometry equation for the Riemann tensor (4.59) we find the Ricci tensor

$$R_{a\bar{b}} = R_{a\bar{b}c}{}^{c} = G_{a\bar{b}}(m+1) - C_{apq}\overline{C}^{pq}_{\bar{b}} \tag{6.16}$$

where m is the number of moduli. On the other hand,

$$R_{a\bar{b}} = \partial_{\bar{b}} \Gamma^c_{ac} = \partial_{\bar{b}} \left(G^{c\bar{m}} \partial_a G_{c\bar{m}} \right). \tag{6.17}$$

We now recall that the derivative of the determinant of a matrix A is given by

$$\partial_x \det A = \det A \cdot A^{-1\,IJ} \partial_x A_{IJ}, \tag{6.18}$$

therefore we can write

$$R_{a\bar{b}} = \partial_a \partial_{\bar{b}} \log\det G. \tag{6.19}$$

We then find,

$$\tfrac{1}{2}C_{apq}\overline{C}^{pq}_{\bar{b}} = \frac{m+1}{2}G_{a\bar{b}} - \frac{1}{2}\partial_a\partial_{\bar{b}}\log\det G = \partial_a\partial_{\bar{b}}\left(-\frac{1}{2}\log\det G + \frac{m+1}{2}K\right). \tag{6.20}$$

We can then integrate twice w.r.t. a and $\bar{b}$ in (6.10), to obtain

$$F_1 = -\tfrac{1}{2}\log\det G + \left(\frac{m+1}{2} - \frac{\chi}{24} + 1\right)K + f_1(z), \tag{6.21}$$

where $f_1(z)$ is an arbitrary holomorphic function which is usually called the *holomorphic ambiguity*. (there is also in principle an integration constant in (6.21) which is purely anti-holomorphic, but we will not consider it here). Using the results in section 4.5, one can find the holomorphic limit of the F_1 coupling:

$$\mathcal{F}_1 = -\frac{1}{2}\log\det\left(\frac{\partial t}{\partial z}\right) - \left(\frac{m+1}{2} - \frac{\chi}{24} + 1\right)\log(X^0(z)) + f_1(z). \tag{6.22}$$

The holomorphic ambiguity has to be fixed with additional data, typically coming from the geometric meaning of $\mathcal{F}_1$ in (6.13). One additional piece of information one might use is that the sum over Gromov–Witten invariants of genus one can be decomposed as

$$\sum_\beta N_{1,\beta}Q^\beta = \sum_\beta \left(\frac{1}{12}n_{0,\beta} + n_{1,\beta}\right)\mathrm{Li}_1(Q^\beta). \tag{6.23}$$

In this formula $n_{0,\beta}$ is the Gopakumar–Vafa invariant at genus zero which we found in (5.44), while $n_{1,\beta}$ are Gopakumar–Vafa invariants at genus one. The formula (6.23) was first proposed in [30] and later on re-derived on physical grounds in the more general context of [29]. The geometric meaning of this formula is the following. First, of all, given a map from a torus to X, one can obtain additional maps by considering multicoverings. A multiple cover of degree d contributes

$$\frac{1}{d}Q^{d\beta}, \tag{6.24}$$

and this leads to the $\mathrm{Li}_1(Q^\beta)$ appearing in (6.23). In addition, maps from the sphere to X, counted by $n_{0,\beta}$, lead to maps from the torus by "bubbling," i.e. by attaching an infinitesimal handle to the sphere. This leads to the first term in the r.h.s. of (6.23).

Example 6.1. *F_1 for the quintic.* Let us consider the quintic CY manifold. In this case, we have a single modulus and

$$\frac{m+1}{2} - \frac{\chi}{24} + 1 = \frac{31}{3}. \tag{6.25}$$

In this case, (6.22) reads

$$\mathcal{F}_1 = -\frac{31}{3}\log\left(\varpi_0(z)\right) - \frac{1}{2}\log\left(\frac{\mathrm{d}t}{\mathrm{d}z}\right) + f_1(z), \tag{6.26}$$

where we parametrize the holomorphic ambiguity in terms of two constants a, b,

$$f_1(z) = a\log(z) + b\log(1 - 5^5 z). \tag{6.27}$$

The coefficient a can be fixed against the large radius limit of $\mathcal{F}_1$ (6.13). For this manifold one has [30]

$$\int_X c_2(X) \wedge J = 50, \tag{6.28}$$

and the large t asymptotics is given by

$$\mathcal{F}_1 = \frac{25}{12} t + \mathcal{O}(\mathrm{e}^{-t}). \tag{6.29}$$

This fixes

$$a = -\frac{31}{12}. \tag{6.30}$$

The coefficient b can be obtained by using enumerative considerations, like e.g. the vanishing of the Gopakumar–Vafa invariant of genus 1 and degree 1. This gives

$$b = -\frac{1}{12}. \tag{6.31}$$

We then find

$$\mathcal{F}_1 = \frac{25}{12} t + \frac{2875}{12} Q + \frac{407125}{8} Q^2 - \frac{1287042500}{9} Q^3 + \mathcal{O}(Q^4). \tag{6.32}$$

From the expression (6.23) we deduce

$$n_{1,1} = n_{1,2} = 0, \qquad n_{1,3} = 609250, \tag{6.33}$$

and so on. □

6.3 The higher genus free energies

The strategy is to regard (6.9), (6.10) as differential equations for F_g. Since the r.h.s. of the equations only depends on $F_{g'}$ with $g' < g$ and on special geometry data, one can try to use them in a recursion procedure to determine the amplitudes. However, the problem is that, if $F_g(t, \bar{t})$ is a solution to the equations, then

$$F_g(t, \bar{t}) + f_g(t), \tag{6.34}$$

where $f_g(t)$ is *holomorphic* in t, is also a solution. This is called for obvious reasons the *holomorphic ambiguity*. However, in some circumstances this ambiguity can be fixed with physical input. Let us illustrate the procedure with the example of the Enriques CY.

Example 6.2. *The Enriques CY.* To illustrate how to solve the holomorphic anomaly equations, we will consider a particular example, namely the Enriques CY. Most of the data of special geometry were computed in 3.4. Using these data it is elementary to compute the tensor $\bar{C}_{\bar{k}}^{ij}$, which is a section of $\mathcal{L}^{-2} \otimes \mathrm{Sym}^2(T_{\mathcal{M}}) \otimes \overline{T}^*_{\mathcal{M}}$:

$$\begin{aligned} \bar{C}^{ab}_{\bar{S}} &= \bar{C}_{\bar{S}\bar{p}\bar{q}} G^{a\bar{p}} G^{b\bar{q}} \mathrm{e}^{2K} = \frac{(C^{-1})^{ab}}{(S+\bar{S})^2}, \\ \bar{C}^{Sb}_{\bar{a}} &= \bar{C}_{\bar{a}\bar{d}\bar{S}} G^{b\bar{d}} G^{S\bar{S}} \mathrm{e}^{2K} = G_{\bar{a}d} (C^{-1})^{db}, \end{aligned} \tag{6.35}$$

and it is zero otherwise.

Let us now study the holomorphic anomaly equations, starting with $g=1$. Since $\chi=0$, we find

$$\begin{aligned}\partial_S\overline{\partial}_{\bar S}F_1 &= \frac{1}{2}\bar C^{ab}_{\bar S}C_{ab}+G_{S\bar S}=\frac{6}{(S+\bar S)^2},\\ \partial_a\overline{\partial}_{\bar b}F_1 &= \frac{1}{2}\bar C^{Sd}_{\bar b}C_{ad}+G_{a\bar b}=2G_{a\bar b},\end{aligned} \tag{6.36}$$

and the other derivatives vanish. The general solution to this equation is

$$F_1(t,\bar t,S,\bar S)=-2\log Y-6\log(S+\bar S)-f_1(t,S)-\overline{f_1(t,S)}, \tag{6.37}$$

where $f_1(t,S)$ encodes the holomorphic ambiguity. Clearly, without further physical input one cannot make much progress. The first piece of input is the factorization

$$f_1(t,S)=f_1(t)+g(S) \tag{6.38}$$

which makes possible to write the S-dependent piece of $F_1(t,\bar t,S,\bar S)$ as

$$-6\log\Big((S+\bar S)|g^2(S)|^2\Big). \tag{6.39}$$

Now, S is the Kähler parameter of the torus in K3$\times\mathbb{T}^2$, and there is an action of $\mathrm{SL}(2,\mathbb{Z})$ on it. It is natural to assume that the topological string amplitudes F_g are modular forms of a given weight with respect to this modular group. Due to the logarithm, F_1 should be modular invariant. As we will see in a moment, it follows from the holomorphic anomaly equation that F_g has modular weight $2-2g$. This fixes $f(S)$ to be of modular weight 1, and absence of singularities for $S\neq\infty$ fixes

$$g(S)=\eta^2(S),\quad q_S=\mathrm{e}^{-S}, \tag{6.40}$$

where $\eta(S)$ is the Dedekind eta function. This argument was already used in [30] to compute F_1 for an elliptic curve. The other piece of the ambiguity $f_1(t)$ is still not fixed, but can be obtained by using for example heterotic/type II duality [34] or a generalization of the modularity argument [35].

Let us now write the holomorphic anomaly equations for higher genus in this model:

$$\begin{aligned}\partial_{\bar S}F_g &= \frac{1}{2}\frac{(C^{-1})^{ab}}{(S+\bar S)^2}\left(D_aD_bF_{g-1}+\sum_{r=1}^{g-1}D_aF_rD_bF_{g-r}\right),\\ \partial_{\bar a}F_g &= (C^{-1})^{db}G_{\bar a d}\left(D_SD_bF_{g-1}+\sum_{r=1}^{g-1}D_SF_rD_bF_{g-r}\right),\end{aligned} \tag{6.41}$$

For $g=2$, the covariant derivatives acting on F_1 are ordinary derivatives. Notice that

$$\partial_SF_1=\frac{1}{2}\widehat{E}_2(S,\bar S), \tag{6.42}$$

where

$$\widehat{E}_2(S,\bar S)=E_2(S)-\frac{12}{S+\bar S} \tag{6.43}$$

is the covariant Eisenstein series, which is a *nonholomorphic* modular form of weight 2. This confirms that the modular weight of F_2 with respect to $\mathrm{SL}(2,\mathbb{Z})$ is 2. The ring of almost holomorphic modular forms is generated by $\widehat{E}_2$, E_4 and E_6, therefore we should expect F_2 to be given by

$$F_2(t,\bar t,S,\bar S)=\widehat{E}_2(S,\bar S)c(t,\bar t). \tag{6.44}$$

We can now plug this ansatz in the first equation of (6.41) and we obtain

$$c(t, \bar{t}) = \frac{1}{24} (C^{-1})^{ab} \left(D_a D_b F_1^E + D_a F_1^E D_b F_1^E \right), \tag{6.45}$$

where we have denoted

$$F_1^E(t, \bar{t}) = -2 \log Y + f_1(t) + f_1(\bar{t}). \tag{6.46}$$

Therefore, if we know $f_1(t)$ (the holomorphic ambiguity for genus one) we can obtain the exact value of F_2. This procedure can be extended to obtain nontrivial results for F_g at low genus, see [20] for more details. □

For a general CY we don't have the special structures of the Enriques CY example. One can however find a systematic method to solve the HAEs by introducing the propagators of section 4.6. It is easy to see that the F_gs can be written as polynomials in the propagators and the derivatives K_a, and we can regard these propagators and K_a as generators of a polynomial ring in which the F_gs take values. In particular, the non-holomorphic dependence of F_g is contained in these generators. Then, the HAE can be thought as recursive equations determining the polynomial structure of the F_gs in these generators [20, 21, 36]. We will follow in particular the formulation in [21]. To write the equations in that form, we use generic complex parameters z^a, $\bar{z}^{\bar{a}}$, and we translate the anti-holomorphic dependence of F_g in its dependence w.r.t. the generators. One obtains in this way,

$$\begin{aligned} \bar{\partial}_{\bar{a}} F_g &= \frac{\partial F_g}{\partial S^{cd}} \overline{C}_{\bar{a}}^{cd} + \frac{\partial F_g}{\partial K_c} G_{c\bar{a}} + \frac{\partial F_g}{\partial S^b} G_{e\bar{a}} S^{eb} + \frac{\partial F_g}{\partial S} G_{b\bar{a}} S^b \\ &= \frac{\partial F_g}{\partial S^{cd}} \overline{C}_{\bar{a}}^{cd} + G_{\bar{a}b} \left(\frac{\partial F_g}{\partial K_b} + \frac{\partial F_g}{\partial S^c} S^{cb} + \frac{\partial F_g}{\partial S} S^b \right). \end{aligned} \tag{6.47}$$

By comparing this expression to (6.9), we find the two equations

$$\begin{aligned} \frac{\partial F_g}{\partial S^{ab}} &= \frac{1}{2} D_a D_b F_{g-1} + \frac{1}{2} \sum_{g_1+g_2=g} D_a F_{g_1} D_b F_{g_2}, \\ 0 &= \frac{\partial F_g}{\partial K_a} + S^a \frac{\partial F_g}{\partial S} + S^{ab} \frac{\partial F_g}{\partial S^b}. \end{aligned} \tag{6.48}$$

It turns out convenient to use the tilded generators (4.231). Then, the second equation in (6.48) becomes simply

$$\frac{\partial F_g}{\partial K_a} = 0, \tag{6.49}$$

while the first equation reads

$$\frac{\partial F_g}{\partial S^{ab}} - \frac{1}{2} \frac{\partial F_g}{\partial \tilde{S}^a} K_b - \frac{1}{2} \frac{\partial F_g}{\partial \tilde{S}^b} K_a + \frac{1}{2} \frac{\partial F_g}{\partial \tilde{S}} K_a K_b = \frac{1}{2} D_a D_b F_{g-1} + \frac{1}{2} \sum_{g_1+g_2=g} D_a F_{g_1} D_b F_{g_2}. \tag{6.50}$$

We want to write down this equation explicitly. To do this, we note that the double covariant derivative as

$$D_i D_j F_{g-1} = \left(\partial_i \delta_j^k - \Gamma_{ij}^k + (4-2g) K_i \right) (\partial_k + (4-2g) K_k) F_{g-1}. \tag{6.51}$$

The key idea the expression for the Christoffel symbol in terms of the propagators (4.156), as well as the last equation in (4.240). We also have to take into account that F_1 is slightly different from the higher F_gs, and it satisfies

$$D_i F_1 = C_i - \left(\frac{\chi}{24} - 1\right) K_i, \qquad C_i = \frac{1}{2} C_{ijk} S^{jk} + f_i(z). \tag{6.52}$$

We can think about F_1 as having the derivative

$$\partial'_i F_i = C_i, \tag{6.53}$$

and $U(1)$ charge $-\chi/24 + 1$. We now take into account that, if $g > 2$, we have

$$\begin{aligned} D_i D_j F_{g-1} = \Big(&\partial_j \partial_k + (C_{jkm} S^{ml} - s^l_{jk})\partial_l + (4-2g)(h_{jk} - C_{jkl}\tilde{S}^l) \\ &+ (3-2g) K_j\, \partial_k + (3-2g) K_k\, \partial_j + (4-2g)(3-2g) K_k K_j \Big) F_{g-1}, \end{aligned} \tag{6.54}$$

where we used (4.156) and the last equation in (4.240). If $g = 2$, we have

$$\begin{aligned} D_j D_k F_1 = {} & \partial_j C_k + (C_{jkm} S^{ml} - s^l_{jk}) C_l - \left(\frac{\chi}{24} - 1\right)(h_{jk} - C_{jkl}\tilde{S}^l) \\ &+ (3-2g) K_j C_k + (3-2g) K_k C_j - \left(\frac{\chi}{24} - 1\right)(3-2g) K_k K_j. \end{aligned} \tag{6.55}$$

We can unify both equations if we define, as in [13],

$$\partial'_k F_g = \begin{cases} \partial_k F_g, & \text{if } g \geq 2, \\ C_k, & \text{if } g = 1, \end{cases} \tag{6.56}$$

and[5]

$$c_g = \begin{cases} (2-2g) F_g, & \text{if } g \geq 2, \\ -\left(\frac{\chi}{24} - 1\right), & \text{if } g = 1. \end{cases} \tag{6.57}$$

Then, we can write

$$\begin{aligned} D_j D_k F_{g-1} = {} & \Big(\partial_j \partial'_k + (C_{jkm} S^{ml} - s^l_{jk})\partial'_l + (3-2g) K_j\, \partial'_k + (3-2g) K_k\, \partial'_j \Big) F_{g-1} \\ &+ \Big(h_{jk} - C_{jkl}\tilde{S}^l + K_k K_j\Big) c_{g-1}. \end{aligned} \tag{6.58}$$

By using the same notation, we find

$$\sum_{h=1}^{g-1} D_j F_{g-h} D_k F_h = \sum_{h=1}^{g-1} \Big\{ \partial'_j F_{g-h} \partial'_k F_h + K_k c_h \partial'_j F_{g-h} + K_j c_{g-h} \partial'_k F_h + K_j K_k c_h c_{g-h} \Big\}. \tag{6.59}$$

With these ingredients, we can write down the explicit expression for (6.50). Since F_g is independent of K_i due to (6.49), we can equate both sides of the equation according to the powers

[5]Note that we define c_g with the opposite sign to the one defined in [13].

of K_j. We then obtain the following set of three equations:

$$\begin{aligned}
\frac{\partial F_g}{\partial \tilde{S}^j} &= -(3-2g)\partial'_j F_{g-1} - \sum_{h=1}^{g-1} c_h \partial'_j F_{g-h}, \\
\frac{\partial F_g}{\partial \tilde{S}} &= (3-2g)c_{g-1} + \sum_{h=1}^{g-1} c_h c_{g-h}, \\
\frac{\partial F_g}{\partial S^{jk}} &= \frac{1}{2}\left(\partial_j \partial'_k + (C_{jkm}S^{ml} - s^l_{jk})\partial'_l\right) F_{g-1} + \frac{1}{2}(h_{jk} - C_{jkl}S^l)c_{g-1} \\
&\quad + \frac{1}{2}\sum_{h=1}^{g-1} \partial'_j F_{g-h} \partial'_k F_h.
\end{aligned} \tag{6.60}$$

Example 6.3. F_1 *redux.* Let us now use the formalism of propagators to rederive the expression (6.21). By using the very definition of the propagator (4.152) and of the Kähler potential (4.21), we can integrate (6.10) a first time to obtain

$$\partial_i F_1 = \frac{1}{2} S^{kl} C_{kli} - \left(\frac{\chi}{24} - 1\right) K_i + f_i^{(1)}. \tag{6.61}$$

Now, one uses (4.156) to write

$$S^{kl} C_{kli} = -\Gamma^l_{il} + (m+1) K_i + s^l_{il}, \tag{6.62}$$

where m is the number of moduli. Let us now consider the holomorphic limit. By using (4.146), we find

$$\Gamma^l_{il} = \frac{\partial^2 t^a}{\partial z^i \partial z^l} \frac{\partial z^l}{\partial t^a} = \frac{\partial}{\partial z_i}\left\{\operatorname{Tr}\log\left(\frac{\partial t^a}{\partial z^k}\right)\right\} = \frac{\partial}{\partial z_i}\left\{\log\det\left(\frac{\partial t^a}{\partial z^k}\right)\right\}. \tag{6.63}$$

We can then integrate (6.61) as

$$F_1 = \left(\frac{m+1}{2} - \frac{\chi}{24} + 1\right) K - \frac{1}{2}\left\{\log\det\left(\frac{\partial t^a}{\partial z^k}\right)\right\} + f_1(z), \tag{6.64}$$

where $f_1(z)$ is such that

$$\partial_i f_1(z) = f_i^{(1)}(z) + \frac{1}{2} s^l_{il}. \tag{6.65}$$

Example 6.4. F_2 *for an arbitrary one-modulus CY.* In the one-modulus case there are four generators, namely S^{zz}, $\tilde{S}^z$, $\tilde{S}$ and K_z. We will denote the holomorphic three-point function by C_z. The relevant derivatives are

$$\begin{aligned}
\partial_z S^{zz} &= C_z \left(S^{zz}\right)^2 + 2\tilde{S}^z - 2 s^z_{zz} S^{zz} + h^{zz}_z, \\
\partial_z K_z &= K_z^2 - C_z \tilde{S}^z - S^{zz} C_z K_z + s^z_{zz} K_z + h_{zz}.
\end{aligned} \tag{6.66}$$

The starting point of the recursion is (6.61), which in the one modulus case reads

$$D_z F_1 = \frac{1}{2} C_z S^{zz} + \left(1 - \frac{\chi}{24}\right) K_z + f_z^{(1)}. \tag{6.67}$$

In this case, the r.h.s. of (6.50) reads

$$\begin{aligned}\frac{\partial F_2}{\partial S^{zz}} - \frac{\partial F_2}{\partial \tilde{S}^z} K_z + \frac{1}{2}\frac{\partial F_2}{\partial \tilde{S}} K_z^2 = &\frac{5}{8} C_z^2 \left(S^{zz}\right)^2 + \frac{1}{4}\left(C_z' - 3C_z s_{zz}^z + 4C_z f_z^{(1)}\right) S^{zz} \\ &+ \frac{1}{4} h_z^{zz} C_z + \frac{1}{2}\partial_z f_z^{(1)} + \frac{1}{2} f_z^{(1)}\left(f_z^{(1)} - s_{zz}^z\right) + \frac{1}{2}\left(1 - \frac{\chi}{24}\right) h_{zz} \\ &+ \frac{\chi}{48} C_z \tilde{S}^z + \frac{\chi}{48}\left(\frac{\chi}{24} - 1\right) K_z^2 - \frac{\chi}{48}\left(C_z S^{zz} + 2 f_z^{(1)}\right) K_z\end{aligned} \tag{6.68}$$

By using that F_2 is independent of K_z, this can be integrated to obtain

$$\begin{aligned}F_2 = &\frac{5}{24} C_z^2 \left(S^{zz}\right)^3 + \frac{1}{8}\left(C_z' - 3C_z s_{zz}^z + 4C_z f_z^{(1)}\right)\left(S^{zz}\right)^2 \\ &+ \left(\frac{1}{4} h_z^{zz} C_z + \frac{1}{2}\partial_z f_z^{(1)} + \frac{1}{2} f_z^{(1)}\left(f_z^{(1)} - s_{zz}^z\right) + \frac{1}{2}\left(1 - \frac{\chi}{24}\right) h_{zz}\right) S^{zz} \\ &+ \frac{\chi}{48}\left(C_z S^{zz} + 2 f_z^{(1)}\right)\tilde{S}^z + \frac{\chi}{24}\left(\frac{\chi}{24} - 1\right)\tilde{S} + f^{(2)}.\end{aligned} \tag{6.69}$$

□

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