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An introduction to topological quantum field theory

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Abstract: Notes

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1 QFT in zero dimensions

1.1 Preliminaries

The starting point for TQFT is a Riemannian or pseudo-Riemannian manifold M of dimension d. The fields of the TQFT are in general sections of bundles on M. It could be for example functions defined on M,

$$x : M \to \mathbb{R}. \tag{1.1}$$

A TQFT is a quantum field theory in which at least a subset of the quantities that one computes is independent of the metric of the manifold on which the theory is defined.

If $d = 0$ and the manifold is simply a point, $M = \text{pt}$, the resulting QFT is obviously topological. One could think that QFTs in $d = 0$ are trivial, but this is not the case, as we will see. For example, the theory of random matrices, which is a very rich subject, is a particular case of $d = 0$ QFTs.

QFTs are defined by a set of fields and a functional of the fields, or *action*. We will start with the simplest case in which the field is of the form (1.1). When M is a point, X is just a real variable, and any function $S(x)$ can be regarded as an action for the theory. One virtue of zero dimensions is that path integrals collapse to ordinary integrals. The basic object of a zero-dimensional QFT is the *partition function*

$$Z = \int_{\mathbb{R}} \mathrm{d}x \, \mathrm{e}^{-\frac{1}{\hbar}S(x)}, \tag{1.2}$$

where in analogy with Quantum Mechanics we have introduced a "coupling constant" $\hbar$. We will assume that this integral exists, although one might have more complicated situations in which appropriate integration contours have to be chosen to guarantee convergence.

We can regard the integrand of a zero-dimensional path integral as a probability distribution for the random variable x:

$$\mathrm{d}\mu(x) = \mathrm{d}x\, \mathrm{e}^{-\frac{1}{\hbar}S(x)}. \tag{1.3}$$

Correlation functions in the QFT are averages of functionals of X with this probability distribution,

$$\langle f(x)\rangle = \frac{1}{Z}\int_{\mathbb{R}} \mathrm{d}x\, f(x)\, \mathrm{e}^{-\frac{1}{\hbar}S(x)}. \tag{1.4}$$

In QFT we are interested in evaluating partition functions and correlation functions. A typical procedure to do this is to use *perturbation theory*. This means that we evaluate these quantities as *formal* power series in a parameter which is regarded as small.

To be concrete, let us consider the following action:

$$S(x) = \frac{1}{2}x^2 + \frac{x^4}{4}. \tag{1.5}$$

Let us first change variables

$$x \to \hbar^{1/2}x, \tag{1.6}$$

so that

$$\frac{S(x)}{\hbar} = \frac{1}{2}x^2 + \frac{\hbar}{4}x^4. \tag{1.7}$$

To evaluate

$$Z = \hbar^{1/2}\int_{\mathbb{R}} \mathrm{d}x \exp\left(-\frac{1}{2}x^2 - \frac{\hbar}{4}x^4\right) \tag{1.8}$$

we note that, when $\hbar \to 0$, the integral is a Gaussian and can be explicitly evaluated. If we regard $\hbar$ as a small parameter, we can try to evaluate Z as a formal power series in $\hbar$. This is simply done by expanding inside the integral,

$$\mathrm{e}^{-\hbar x^4/4} = \sum_{k\geq 0}\frac{(-4)^{-k}x^{4k}}{k!}\hbar^k. \tag{1.9}$$

We now exchange integration and the series expansion. This is *not* guaranteed to give a convergent power series for Z, but it gives an *asymptotic* expansion. Indeed, one finds the formal power series

$$Z \sim \sqrt{2\pi\hbar}\sum_{k\geq 0} a_k\hbar^k, \tag{1.10}$$

where

$$a_k = \frac{(-4)^{-k}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}z\frac{z^{4k}}{k!}\mathrm{e}^{-z^2/2} = (-4)^{-k}\frac{(4k-1)!!}{k!}. \tag{1.11}$$

We will see in the exercise section that this series has zero radius of convergence. This is the typical outcome of a perturbative calculation in QFT.

1.2 Correlation functions and Wick's lemma

We will now formalize in some detail this procedure. This will allow us to derive various important results in perturbation theory, which appear almost unchanged in quantum mechanics and quantum field theory. First of all, we will consider a field with values in $\mathbb{R}^p$,

$$\boldsymbol{x} = (x_1, \cdots, x_p) \in \mathbb{R}^p. \tag{1.12}$$

A *Gaussian theory* is defined by an action given by a positive definite quadratic form A:

$$S_G(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T \mathsf{A}\boldsymbol{x} = \frac{1}{2}\sum_{\mu,\nu=1}^{p} x_\mu A_{\mu\nu} x_\nu. \tag{1.13}$$

Here, $A_{\mu\nu}$ is a symmetric matrix with positive eigenvalues. The partition function of such a theory is well-defined (due to positive definiteness) and it can be computed immediately by diagonalizing A:

$$Z_G = \int \mathrm{d}\boldsymbol{x}\, \exp\left(-\frac{1}{2\hbar}\sum_{\mu,\nu=1}^{p} x_\mu A_{\mu\nu} x_\nu\right) = \frac{(2\pi\hbar)^{p/2}}{\sqrt{\det(\mathsf{A})}}. \tag{1.14}$$

We are now interested in computing correlation functions of products of x_μ. This is answered by Wick's lemma.

Lemma 1.1. *(Wick) We have the following results for the correlation functions:*

1. $$\langle x_{\mu_1}\cdots x_{\mu_{2n+1}}\rangle = 0. \tag{1.15}$$

2. $$\langle x_\mu x_\nu\rangle = \hbar\,\mathsf{A}^{-1}_{\mu\nu}. \tag{1.16}$$

3. $$\langle x_{\mu_1}\cdots x_{\mu_{2n}}\rangle = \frac{\hbar^n}{2^n n!}\sum_{\sigma\in S_{2n}} \mathsf{A}^{-1}_{\mu_{\sigma(1)}\mu_{\sigma(2)}}\cdots \mathsf{A}^{-1}_{\mu_{\sigma(2n-1)}\mu_{\sigma(2n)}} \tag{1.17}$$

   where the sum is over all permutations σ of $2n$ elements.

Proof: Correlation functions can be easily obtained from the generating functional

$$Z[\boldsymbol{j}] = \int \mathrm{d}\boldsymbol{x}\, \exp\left\{-\frac{1}{2\hbar}\sum_{\mu,\nu=1}^{p} x_\mu A_{\mu\nu} x_\nu + \sum_{\mu=1}^{p} j_\mu x_\mu\right\}, \tag{1.18}$$

which is the partition function in a theory with a modified "action" depending on an arbitrary source $\boldsymbol{j}$. Note that $Z[0]$ is the Gaussian partition function (1.14). The partition function $Z[\boldsymbol{j}]$ can be computed by Gaussian integration, and one finds

$$Z[\boldsymbol{j}] = Z[0]\exp\left(\frac{\hbar}{2}\sum_{\mu,\nu=1}^{p} j_\mu \mathsf{A}^{-1}_{\mu\nu} j_\nu\right). \tag{1.19}$$

We can now use this explicit formula for $Z[\boldsymbol{j}]$ to calculate correlation functions. Let us expand the exponential in (1.19)

$$\begin{aligned}
\exp\left(\frac{\hbar}{2}\sum_{\mu,\nu=1}^{p} j_\mu \mathsf{A}^{-1}_{\mu\nu} j_\nu\right) &= \sum_{n=0}^{\infty}\frac{\hbar^n}{2^n n!}\left(\sum_{\mu,\nu=1}^{p} j_\mu \mathsf{A}^{-1}_{\mu\nu} j_\nu\right)^n \\
&= \sum_{n=0}^{\infty}\frac{\hbar^n}{2^n n!}\sum_{\mu_1,\cdots,\mu_{2n}} j_{\mu_1}\cdots j_{\mu_{2n}}\mathsf{A}^{-1}_{\mu_1\mu_2}\cdots\mathsf{A}^{-1}_{\mu_{2n-1}\mu_{2n}} \\
&= \sum_{n=0}^{\infty}\frac{\hbar^n}{2^n n!}\sum_{\mu_1,\cdots,\mu_{2n}} j_{\mu_1}\cdots j_{\mu_{2n}}\frac{1}{(2n)!}\sum_{\sigma\in S_{2n}} \mathsf{A}^{-1}_{\mu_{\sigma(1)}\mu_{\sigma(2)}}\cdots\mathsf{A}^{-1}_{\mu_{\sigma(2n-1)}\mu_{\sigma(2n)}}.
\end{aligned} \tag{1.20}$$

In going to the last line, we have used the fact that $\mathsf{A}^{-1}_{\mu_1\mu_2}\cdots\mathsf{A}^{-1}_{\mu_{2n-1}\mu_{2n}}$ can be symmetrized w.r.t. the indices $\mu_1,\cdots,\mu_{2n}$, since it multiplies a symmetric function. Therefore, we sum over all possible permutations σ of the indices in the permutation group of $2n$ elements S_{2n}. On the other hand, (1.20) is computing the generating functional (1.18),

$$Z[\boldsymbol{j}] = Z[0]\sum_{M=0}^{\infty}\frac{1}{M!}\sum_{\mu_1,\cdots,\mu_M} j_{\mu_1}\cdots j_{\mu_M}\langle x_{\mu_1}\cdots x_{\mu_M}\rangle. \tag{1.21}$$

Since (1.20) with (1.21) are equal, and since the source $\boldsymbol{j}$ is arbitrary, we conclude first of all that the correlation functions with an odd number of terms vanish, as expected from the symmetry $\boldsymbol{x}\leftrightarrow-\boldsymbol{x}$ of the theory. When $M=2n$ is even, we find that

$$\langle x_{\mu_1}\cdots x_{\mu_{2n}}\rangle = \frac{\hbar^n}{2^n n!}\sum_{\sigma\in S_{2n}}\mathsf{A}^{-1}_{\mu_{\sigma(1)}\mu_{\sigma(2)}}\cdots\mathsf{A}^{-1}_{\mu_{\sigma(2n-1)}\mu_{\sigma(2n)}}. \tag{1.22}$$

We notice however that in the r.h.s. many terms are equal, namely, terms which differ by a permutation of the two indices of $\mathsf{A}^{-1}_{\mu\nu}$, or by a permutations of the n groups of paired indices. For example, if $n=2$, the terms $\mathsf{A}^{-1}_{\mu_1\mu_2}\mathsf{A}^{-1}_{\mu_3\mu_4}$ and $\mathsf{A}^{-1}_{\mu_2\mu_1}\mathsf{A}^{-1}_{\mu_4\mu_3}$ give the same contribution in the sum, as does the term $\mathsf{A}^{-1}_{\mu_3\mu_4}\mathsf{A}^{-1}_{\mu_1\mu_2}$. We will say that two permutations of $2n$ elements are equivalent if they give the same contribution in the r.h.s. of (1.22). An equivalence class of permutations is called a *pairing*. Each pairing corresponds to $2^n n!$ different permutations, which is precisely the factor in front of the sum in (1.22). The different pairings are obtained by grouping the $2n$ labels in groups of 2, irrespectively of the ordering of the groups and the order of the labels inside each pair. We conclude that (1.22) can be written as a sum over pairings, as follows:

$$\langle x_{\mu_1}\cdots x_{\mu_{2n}}\rangle = \hbar^n\sum_{\text{pairings } P}\mathsf{A}^{-1}_{\mu_{P(1)}\mu_{P(2)}}\cdots\mathsf{A}^{-1}_{\mu_{P(2n-1)}\mu_{P(2n)}}. \tag{1.23}$$

This important result is *Wick's theorem*. Let us note that there are

$$\frac{(2n)!}{2^n n!} \tag{1.24}$$

inequivalent pairings contributing to the r.h.s. of (1.23). For example, when $n=2$ there are $4!/4\cdot 2! = 3$ different pairings, and one finds

$$\begin{aligned}\langle x_{\mu_1}x_{\mu_2}x_{\mu_3}x_{\mu_4}\rangle = {}& \langle x_{\mu_1}x_{\mu_2}\rangle\langle x_{\mu_3}x_{\mu_4}\rangle + \langle x_{\mu_1}x_{\mu_3}\rangle\langle x_{\mu_2}x_{\mu_4}\rangle \\ & + \langle x_{\mu_1}x_{\mu_4}\rangle\langle x_{\mu_2}x_{\mu_3}\rangle.\end{aligned} \tag{1.25}$$

Wick's theorem applies to any Gaussian probability measure and it is the basis of the perturbation theory of path integrals. □

1.3 Perturbation theory and Feynman diagrams

Most multi-dimensional integrals can not be calculated exactly, so one is obliged to perform some type of approximation. One possibility is to develop a perturbative approach in a small parameter, as we mentioned before. Let us then assume that the action is of the form

$$S(\boldsymbol{x}) = S_{\mathrm{G}}(\boldsymbol{x}) + \lambda S_{\mathrm{I}}(\boldsymbol{x}), \tag{1.26}$$

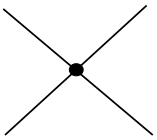

Figure 1. Diagram representing the quartic vertex.

where $S_{\rm G}$ is a quadratic action of the form (1.13), λ is a small parameter, and $S_{\rm I}$ is an interaction term. A typical example is a quartic interaction of the form

$$S_{\rm I}(\boldsymbol{x}) = \sum_{\mu_1,\cdots,\mu_4} g_{\mu_1\cdots\mu_4} x_{\mu_1}\cdots x_{\mu_4}, \tag{1.27}$$

where $g_{\mu_1\cdots\mu_4}$ is totally symmetric in its indices. The perturbative evaluation of the partition function consists in expanding the interacting part in a formal power series in λ,

$$\begin{aligned} Z(\lambda) &= \int \mathrm{d}\boldsymbol{x}\, \mathrm{e}^{-S_{\rm G}(\boldsymbol{x})-\lambda S_{\rm I}(\boldsymbol{x})} = \sum_{N=0}^{\infty} \frac{(-\lambda)^N}{N!} \int \mathrm{d}\boldsymbol{x}\, \mathrm{e}^{-S_{\rm G}(\boldsymbol{x})} \left(S_{\rm I}(\boldsymbol{x})\right)^N \\ &= Z(0) \sum_{N=0}^{\infty} \frac{(-\lambda)^N}{N!} \langle (S_{\rm I}(\boldsymbol{x}))^N \rangle. \end{aligned} \tag{1.28}$$

In this equation, the argument of the partition function is the coupling constant λ. If the interaction is polynomial in x_μ, the averages appearing here can be computed by using Wick's theorem, and we obtain in this way a formal power series in λ. If λ is small enough, this series might give us some understanding of the interacting theory. As we noted above, the information carried by this series is limited, since the series is not even convergent. Nevertheless, this approximation scheme, known as perturbation theory, is widely used in quantum mechanics, quantum field theory and many-body physics.

It turns out that the information contained in each term of the above series can be put in diagrammatic form. Let us assume for definiteness that we have a quartic interaction. We have to evaluate terms of the form

$$\left\langle \left(g_{\mu_1\cdots\mu_4} x_{\mu_1}\cdots x_{\mu_4}\right)^N \right\rangle \tag{1.29}$$

with Wick's theorem. We have used Einstein's convention that repeated indices are summed over. Let us represent each power of the interaction term by a *vertex* with four edges, or quartic vertex, as in figure Fig. 1. The N-th order term (1.29) leads to N vertices. According to Wick's theorem, we have to pair the edges in all possible ways. Each of these pairings can be represented by a diagram called *Feynman diagram*. Consider for example $N = 1$. We have to compute

$$\langle g_{\mu_1\cdots\mu_4} x_{\mu_1}\cdots x_{\mu_4} \rangle \tag{1.30}$$

which leads to three possible pairings, and we find

$$\begin{aligned} \langle g_{\mu_1\cdots\mu_4} x_{\mu_1}\cdots x_{\mu_4} \rangle &= g_{\mu_1\cdots\mu_4} \left(\mathsf{A}^{-1}_{\mu_1\mu_2}\mathsf{A}^{-1}_{\mu_3\mu_4} + \mathsf{A}^{-1}_{\mu_1\mu_3}\mathsf{A}^{-1}_{\mu_2\mu_4} + \mathsf{A}^{-1}_{\mu_1\mu_4}\mathsf{A}^{-1}_{\mu_2\mu_3}\right) \\ &= 3 g_{\mu_1\cdots\mu_4} \mathsf{A}^{-1}_{\mu_1\mu_2}\mathsf{A}^{-1}_{\mu_3\mu_4}. \end{aligned} \tag{1.31}$$

The three equivalent pairings correspond to the Feynman diagram shown in Fig. 2.

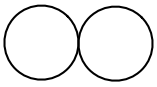

Figure 2. Feynman diagram obtained by pairing all the legs in a single quartic vertex.

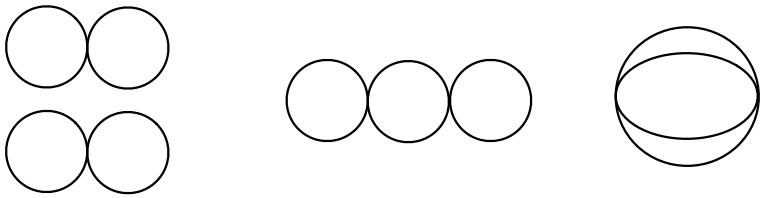

Figure 3. Feynman diagrams obtained at $N = 2$.

Life becomes more interesting when $N = 2$, and one has to compute

$$\left\langle \left(g_{\mu_1\cdots\mu_4} x_{\mu_1}\cdots x_{\mu_4}\right)^2 \right\rangle = \left\langle \left(g_{\mu_1\cdots\mu_4} x_{\mu_1}\cdots x_{\mu_4}\right)\left(g_{\nu_1\cdots\nu_4} x_{\nu_1}\cdots x_{\nu_4}\right)\right\rangle . \tag{1.32}$$

The first thing to note is that we have two types of diagrams now: we can consider diagrams in which the edges of the first vertex do not pair with edges in the second vertex. This gives a diagram with two separated pieces, which is a particular example of a *disconnected diagram*. In addition, when $N = 2$ there are diagrams in which at least one edge of one vertex pairs with an edge of the other vertex. This leads to diagrams which can be not be separated into two pieces without cutting an edge. They are examples of *connected* diagrams. It is easy to see that for $N = 2$ there are two different types of connected diagrams, which are shown in Fig. 3.

There are many pairings that lead to the same Feynman diagram. Each component of the disconnected diagram is obtained from 3 different pairings, as we saw before, so for the disconnected diagram in Fig. 3 there are $3^2 = 9$ pairings. This diagram contributes

$$\frac{9}{2}\left(g_{\mu_1\cdots\mu_4}\mathsf{A}^{-1}_{\mu_1\mu_2}\mathsf{A}^{-1}_{\mu_3\mu_4}\right)^2 \tag{1.33}$$

to the partition function The first connected diagram shown in Fig. 3 is obtained from 72 different pairings. To calculate this number, note that in each vertex there are 6 ways to pair two legs, and then there are 2 ways to pair the remaining two legs of the two vertices. This gives in total $6^2 \cdot 2 = 72$ pairings. The contribution of this connected diagram to the partition function is

$$\frac{72}{2} g_{\mu_1\cdots\mu_4} g_{\nu_1\cdots\nu_4} \mathsf{A}^{-1}_{\mu_1\mu_2}\mathsf{A}^{-1}_{\nu_1\nu_2}\mathsf{A}^{-1}_{\mu_3\nu_3}\mathsf{A}^{-1}_{\mu_4\nu_4}. \tag{1.34}$$

The second connected diagram comes from $4! = 24$ different pairings (the different permutations of the four connecting legs). Its contribution is

$$\frac{24}{2} g_{\mu_1\cdots\mu_4} g_{\nu_1\cdots\nu_4} \mathsf{A}^{-1}_{\mu_1\nu_1}\mathsf{A}^{-1}_{\mu_1\mu_2}\mathsf{A}^{-1}_{\mu_3\nu_3}\mathsf{A}^{-1}_{\mu_4\nu_4}. \tag{1.35}$$

Note that the counting of the possible pairings associated to a given Feynman diagram might be an intricate combinatorial problem. A useful constraint is that the total number of pairings

must be equal to (1.24). For example, when $n = 4$, one verifies that

$$9 + 72 + 24 = \frac{8!}{2^4 \cdot 4!}. \tag{1.36}$$

Very often we are interested in the computation of the so-called free energy

$$F = -\log Z, \tag{1.37}$$

rather than in the computation of the partition function. It turns out that this quantity is simpler, from the diagrammatic point of view, since it involves just connected Feynman diagrams. This is proved as follows. Let us denote by

$$\langle S_{\rm I}^{\ell} \rangle^{(c)} \tag{1.38}$$

the contribution of connected diagrams to the correlation function involving ℓ vertices. It is clear that a general, not necessarily connected diagram with N vertices, can be partitioned into connected diagrams. Let us denote by m_ℓ the number of connected diagrams with ℓ vertices appearing in a diagram with N vertices in total. Clearly, we must have

$$\sum \ell m_\ell = N. \tag{1.39}$$

We then have,

$$\langle S_{\rm I}^{N} \rangle = {\sum_{\{m_l\}}}' c_{\{m_\ell\}} \prod_{\ell} \left(\langle S_{\rm I}^{N} \rangle^{(c)} \right)^{m_\ell} \tag{1.40}$$

where the sum is over all possible sets $\{m_\ell\}$, satisfying the constraint (1.39). The coefficient $c_{\{m_\ell\}}$ is a combinatorial factor which gives all possible, non-equivalent ways of grouping N vertices into m_ℓ connected groups of ℓ vertices. This factor is given by

$$c_{\{m_\ell\}} = \frac{N!}{\prod_{\ell} m_\ell! \, (\ell!)^{m_\ell}}. \tag{1.41}$$

In this expression, $N!$ gives all possible permutations of the vertices. However, we have to divide the $N!$ by the number of equivalent configurations. These are obtained in two ways: (a) by permuting the labels of the vertices *inside* each connected subdiagram, which gives $(\ell!)^{m_\ell}$ for each ℓ; (b) by permuting the m_ℓ connected diagrams among them, which gives $m_\ell!$ for each ℓ.

Let us now compute (1.28) by using (1.40) and (1.41). The sum over N can be written as a sum over all possible m_ℓ:

$$\begin{aligned} \frac{Z(\lambda)}{Z(0)} &= \sum_{\{m_l\}} \prod_{\ell=1}^{\infty} \frac{1}{m_\ell!} \left((-\lambda)^\ell \frac{\langle S_{\rm I}^{\ell} \rangle^{(c)}}{\ell!} \right)^{m_\ell} = \prod_{\ell=1}^{\infty} \sum_{m_\ell=0}^{\infty} \frac{1}{m_\ell!} \left((-\lambda)^\ell \frac{\langle S_{\rm I}^{\ell} \rangle^{(c)}}{\ell!} \right)^{m_\ell} \\ &= \exp\left(\sum_{\ell=1}^{\infty} (-\lambda)^\ell \frac{\langle S_{\rm I}^{\ell} \rangle^{(c)}}{\ell!} \right). \end{aligned} \tag{1.42}$$

If we regard (1.28) as a generating function of averages in a probability distribution, the logarithm of $Z(\lambda)$ defines the *cumulants* of the distribution, which are given by the connected averages $\langle S_{\rm I}^{n} \rangle^{(c)}$. It is easy to see that

$$\begin{aligned} \langle S_{\rm I} \rangle^{(c)} &= \langle S_{\rm I} \rangle, \\ \langle S_{\rm I}^2 \rangle^{(c)} &= \langle S_{\rm I}^2 \rangle - \langle S_{\rm I} \rangle^2. \end{aligned} \tag{1.43}$$

The result (1.42) is very useful to compute the perturbative expansion of the free energy

$$F = -\log Z = -\sum_{\ell=1}^{\infty} (-\lambda)^{\ell} \frac{\langle S_{\rm I}^{\ell} \rangle^{(c)}}{\ell!}. \tag{1.44}$$

1.4 Matrix models and the large N expansion

Matrix models are a special example of zero dimensional quantum field theories. More precisely, they can be regarded as gauge theories in zero dimensions. This means that the basic fields have the group structure of a gauge connection, i.e. they are matrices in the adjoint representation of a gauge group. If the gauge group is $U(N)$, these are $N \times N$ Hermitian matrices. For simplicity we will consider one-matrix models, i.e. models where there is a single matrix M. Our gauge group will be $U(N)$, so M will be an $N \times N$ Hermitian matrix. In order to define a theory for M, we should specify an action incorporate gauge invariance in a suitable way. The simplest action we can consider is of the form,

$$\frac{1}{g_s} {\rm Tr} V(M), \tag{1.45}$$

where

$$V(M) = \frac{1}{2} M^2 + \sum_{p \geq 3} \frac{g_p}{p} M^p, \tag{1.46}$$

and g_s and g_p are coupling constants. (1.46) is often called the potential of the matrix model, and the action (1.45) is clearly invariant under the symmetry

$$M \to U M U^{\dagger}, \tag{1.47}$$

where U is any $U(N)$ matrix. This can be regarded as the zero-dimensional version of a gauge transformation.

The partition function of the matrix model theory with action (1.45) is defined, as in the previous section, by

$$Z = \frac{1}{{\rm vol}(U(N))} \int {\rm d}M \, {\rm e}^{-\frac{1}{g_s} {\rm Tr} V(M)}. \tag{1.48}$$

Here, ${\rm vol}(U(N))$ is the volume of the gauge group. The measure in integral is given by

$${\rm d}M = 2^{\frac{N(N-1)}{2}} \prod_{i=1}^{N} {\rm d}M_{ii} \prod_{1 \leq i < j \leq N} {\rm dRe}\, M_{ij} {\rm dIm}\, M_{ij}. \tag{1.49}$$

The numerical factor in (1.49) is introduced to obtain a convenient normalization. We note that these models are particular cases of the integrals discussed in the previous section. The natural observables in the matrix model are gauge-invariant operators, i.e. functions of the matrix M, $f(M)$, satisfying

$$f\left(U M U^{\dagger}\right) = f(M), \qquad U \in U(N). \tag{1.50}$$

The (normalized) vev of such an observable is defined as

$$\langle f(M) \rangle = \frac{\int {\rm d}M \, f(M) \, {\rm e}^{-{\rm Tr}\, V(M)/g_s}}{\int {\rm d}M \, {\rm e}^{-{\rm Tr}\, V(M)/g_s}}. \tag{1.51}$$

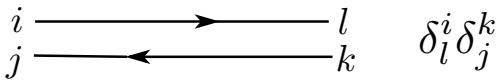

Figure 4. The propagator in the Hermitian matrix model.

A particularly simple example of the one-matrix model is the *Gaussian matrix model*, defined by the potential (1.46) with $g_p = 0$ for $p \geq 3$. Its partition function will be denoted by

$$Z_G = \frac{1}{\mathrm{vol}(U(N))} \int \mathrm{d}M \, \mathrm{e}^{-\frac{1}{2g_s}\mathrm{Tr}\, M^2}, \tag{1.52}$$

and we will also denote by $\langle f(M) \rangle_G$ the normalized vev of a gauge-invariant function $f(M)$. The main interest of the Gaussian matrix model is that, as in any other QFT, it can be taken as the starting point to perform a perturbation expansion. For example, if one wants to evaluate the partition function (1.48) of a general matrix model with action (1.46), one expands the exponential of $\sum_{p\geq 3}(g_p/g_s)\mathrm{Tr}M^p/p$ in (1.48), and computes Z as a power series in the coupling constants g_p. The evaluation of each term of the series involves the computation of vevs in the Gaussian matrix model. This computation can be interpreted again in terms of Feynman diagrams, and the perturbative expansion of the free energy $F = -\log Z$ will only involve connected diagrams.

The basic building block of Wick's theorem is the correlation function of two variables or fields. From the structure of the trace we deduce that

$$\langle M_{ij} M_{kl} \rangle_G = g_s \delta_{il} \delta_{jk}. \tag{1.53}$$

Since each field has two ordered indices, it is convenient to use double-line diagrams as shown in Fig. 4. The vertices of the theory are also very easy to write down. For example, the cubic vertex

$$\frac{g_3}{g_s}\mathrm{Tr}\, M^3 = \frac{g_3}{g_s} \sum_{i,j,k} M_{ij}\, M_{jk}\, M_{ki} \tag{1.54}$$

can be represented in the double line notation as in Fig. 5. A vertex of order p can be represented in a similar way by drawing p double lines joined together.

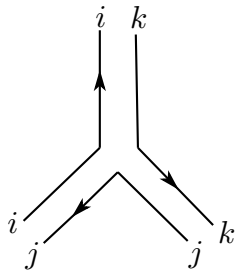

Figure 5. The cubic vertex (1.54) in the double line notation.

For concreteness, let us consider a Hermitian matrix model with a cubic vertex (1.54). If we expand the interaction in power series, the first non-zero Gaussian average that we have to

compute is

$$\langle (\mathrm{Tr}\, M^3)^2 \rangle_G. \tag{1.55}$$

The different Wick contractions give three topologically inequivalent diagrams. We have first the diagram shown in Fig. 6, which contributes

$$3 \sum_{ijkmnp} \langle M_{ij} M_{mn} \rangle \langle M_{jk} M_{pm} \rangle \langle M_{ki} M_{np} \rangle = 3 g_s^3 N^3. \tag{1.56}$$

The second diagram is given by two circles joined by a single propagator. Its contribution is

$$9 \sum_{ijkmnp} \langle M_{ij} M_{ki} \rangle \langle M_{jk} M_{pm} \rangle \langle M_{mn} M_{np} \rangle = 9 g_s^3 N^3. \tag{1.57}$$

The remaining diagram is represented in Fig. 7 and gives a factor

$$3 \sum_{ijkmnp} \langle M_{ij} M_{mn} \rangle \langle M_{jk} M_{np} \rangle \langle M_{ki} M_{pm} \rangle = 3 g_s^3 N. \tag{1.58}$$

Putting everything together, we find

$$\langle (\mathrm{Tr}\, M^3)^2 \rangle_{\mathrm{G}} = g_s^3 (12\, N^3 + 3\, N). \tag{1.59}$$

From this result we obtain the first term in the perturbative expansion of the free energy for the cubic matrix model with $g_p = 0$, $p \geq 4$,

$$F - F_G = \frac{2}{3} g_s g_3^2 N^3 + \frac{1}{6} g_s g_3^2 N + \cdots \tag{1.60}$$

Here, $F_G = \log Z_G$ is the free energy of the Gaussian matrix model.

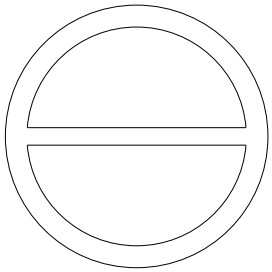

Figure 6. Diagram obtained by contracting two cubic vertices.

The fatgraphs obtained from the potential (1.46) are characterized topologically by the number of propagators or edges E, the number of vertices with p legs V_p, and the number of closed loops h. The total number of vertices is $V = \sum_p V_p$. Each propagator gives a power of g_s, while each interaction vertex with p legs gives a power of g_p/g_s. The fatgraph will then have an overall factor

$$g_s^{E-V} N^h \prod_p g_p^{V_p}. \tag{1.61}$$

We can also regard the fatgraph as a Riemann surface with holes, in which each closed loop represents the boundary of a hole. The genus g of such a surface is determined by the elementary topological relation

$$2g - 2 = E - V - h, \tag{1.62}$$

therefore we can write (1.61) as

$$g_s^{2g-2+h} N^h \prod_p g_p^{V_p} = g_s^{2g-2} t^h \prod_p g_p^{V_p}, \tag{1.63}$$

where we have introduced the so-called *'t Hooft parameter*

$$t = N g_s. \tag{1.64}$$

The fatgraphs with $g = 0$ are called *planar*, while the ones with $g > 0$ are called *non-planar*. The graph in Fig. 6 is planar: it has $E = 3$, $V_3 = 2$ and $h = 3$, therefore $g = 0$, and topologically it corresponds to a sphere with three holes. The graph in Fig. 7 is non-planar: it has $E = 3$, $V_3 = 2$ and $h = 1$, therefore $g = 1$, and represents a torus with one hole (it is easy to see this by drawing the diagram on the surface of a torus).

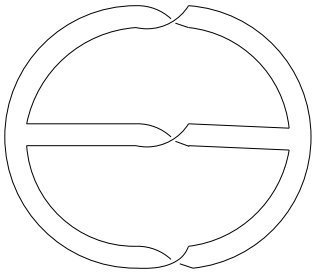

Figure 7. Another diagram obtained by contracting two cubic vertices.

The free energy of a matrix model can then be written, after using fatgraphs, as a double expansion

$$F - F_G = \sum_{g,h} a_{g,h} t^h g_s^{2g-2}. \tag{1.65}$$

We can now try to resum partially the expansion in the 't Hooft parameter t, order by order in the genus expansion. Therefore, one defines

$$F_g(t) = \sum_h a_{g,h} t^h, \tag{1.66}$$

and we can write the total free energy as

$$F - F_G = \sum_{g=0}^{\infty} F_g(t) g_s^{2g-2}. \tag{1.67}$$

It turns out that the $F_g(t)$ are analytic functions at the origin and they can be computed very often in closed form (at least for small g). This is a vast subject which has become very important in modern mathematical physics.

2 Supersymmetric QFT in zero dimensions

2.1 Introducing Grassmann variables

Many TQFTs are of the so-called "cohomological type": they have nilpotent, anticommuting symmetry which is typically inherited from a supersymmetric theory. Therefore, it is important to introduce fermionic fields in the game. This is done through the so-called Grassmann variables.

Definition 2.1. A *Grassmann algebra* with n generators Gr_n is a $\mathbb{C}$-algebra where the generators $\theta_1, \cdots, \theta_n$ satisfy the relations

$$\{\theta_i, \theta_j\} = 0, \qquad i, j = 1, \cdots, n \tag{2.1}$$

In particular, they are nilpotent. □

Gr_n is a polynomial algebra in anticommuting variables. Its elements have a degree which is simply the number of generators. We will denote the degree of an element $\alpha \in \mathrm{Gr}_n$ by $|\alpha|$. As a vector space, we can decompose

$$\mathrm{Gr}_n = \oplus_{k=0}^{n} \mathrm{Gr}_n^k, \tag{2.2}$$

where Gr_n^k are the elements of degree k. As a vector space, Gr_n^k has dimension $\binom{n}{k}$, and Gr_n has in total dimension 2^n. A general element of Gr_n can be written as

$$f(\boldsymbol{\theta}) = \sum_{k=0}^{n} \sum_{1 \le i_1 < \cdots < i_n \le n} f^{i_1 \cdots i_k} \theta_{i_1} \cdots \theta_{i_k}, \qquad f^{i_1 \cdots i_k} \in \mathbb{C}. \tag{2.3}$$

We need to derive and to integrate in a Grasmann algebra. The derivatives

$$\frac{\partial}{\partial \theta_i} : \mathrm{Gr}_n \to \mathrm{Gr}_n \tag{2.4}$$

act as elementary derivatives but are antiderivations w.r.t. to the degree. Their action on a monomial is defined by

$$\frac{\partial}{\partial \theta_i} \theta_{i_1} \cdots \theta_{i_n} = \sum_{l=1}^{k} (-1)^{l-1} \delta_{i i_l} \theta_{i_1} \cdots \hat{\theta}_{i_l} \cdots \theta_{i_k}, \tag{2.5}$$

where $\hat{\theta}_{i_l}$ means that we remove this generator.

Example 2.2. In the Grassmann algebra Gr_n, with $n \ge 2$, we have

$$\frac{\partial}{\partial \theta_1}(\theta_1 \theta_2) = \theta_2, \qquad \frac{\partial}{\partial \theta_2}(\theta_1 \theta_2) = -\theta_1. \tag{2.6}$$

□

We want to define an integral over a Grassmann algebra. In the case of a single generator, integration is defined by

$$\int \mathrm{d}\theta = 0, \qquad \int \mathrm{d}\theta \theta = 1. \tag{2.7}$$

Definition 2.3. The *Berezin integral* is a linear functional

$$\int \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n : \mathrm{Gr}_n \to \mathbb{C} \tag{2.8}$$

defined as follows. Let $f(\boldsymbol{\theta})$ be of the form (2.3). Then

$$\int \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \, f(\boldsymbol{\theta}) = f^{12\cdots n}. \tag{2.9}$$

We will often denote

$$\mathrm{d}\boldsymbol{\theta} = \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n. \tag{2.10}$$

Note that this integration acts as a differentiation. We also note that

$$\int \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \, \theta_{\sigma(1)} \cdots \theta_{\sigma(n)} = (-1)^{\epsilon(\sigma)}, \tag{2.11}$$

where $\sigma \in S_n$ is a permutation of n elements and $\epsilon(\sigma)$ its signature.

Lemma 2.4. *(Change of variables in the Berezin integral) Let* $\{\theta_i\}_{i=1,\cdots,n}$, $\{\widetilde{\theta}_i\}_{i=1,\cdots,n}$ *two sets of generators of* Gr_n*, related by a linear change of variables:*

$$\theta_i = \sum_{j=1}^{n} a_{ij} \widetilde{\theta}_j. \tag{2.12}$$

Then,

$$\int \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \, f(\boldsymbol{\theta}) = \frac{1}{\det(A)} \int \mathrm{d}\widetilde{\theta}_1 \cdots \mathrm{d}\widetilde{\theta}_n \, \widetilde{f}(\widetilde{\boldsymbol{\theta}}), \tag{2.13}$$

where

$$\widetilde{f}(\widetilde{\boldsymbol{\theta}}) = f(\boldsymbol{\theta}). \tag{2.14}$$

Proof: We have

$$\begin{aligned} f^{1\cdots n}\theta_1 \cdots \theta_n &= f^{1\cdots n} \sum_{j_1,\cdots,j_n} a_{1j_1} \cdots a_{nj_n} \widetilde{\theta}_{j_1} \cdots \widetilde{\theta}_{j_n} \\ &= f^{1\cdots n} \widetilde{\theta}_1 \cdots \widetilde{\theta}_n \sum_{\sigma \in S_n} (-1)^{\epsilon(\sigma)} a_{1\sigma(1)} \cdots a_{n\sigma(n)}, \end{aligned} \tag{2.15}$$

therefore

$$\widetilde{f}^{12\cdots n} = \det(A) f^{12\cdots n}, \tag{2.16}$$

which proves the Lemma. □

This lemma can be understood as a formula for the change of variables

$$\mathrm{d}\boldsymbol{\theta} = \frac{1}{\det(A)} \mathrm{d}\widetilde{\boldsymbol{\theta}}. \tag{2.17}$$

After this, we can perform Gaussian integrations. A quadratic form in Grassmann variables is given by

$$\sum_{i,j=1}^{n} a_{ij} \theta_i \theta_j, \tag{2.18}$$

and we can take A to be an antisymmetric matrix. The exponential of such a quadratic form is defined by a power series expansion. However, this series terminates due to the nilpotency of θ_i:

$$\exp\left(\frac{1}{2}\sum_{i,j=1}^{n} a_{ij}\theta_i\theta_j\right) = \sum_{k=0}^{m} \frac{1}{2^k \cdot k!}\left(\sum_{i,j=1}^{n} a_{ij}\theta_i\theta_j\right)^k . \tag{2.19}$$

where

$$m = \left[\frac{n}{2}\right]. \tag{2.20}$$

We want to calculate now the Gaussian Berezin integral

$$\int \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \, \exp\left(\frac{1}{2}\sum_{i,j=1}^{n} a_{ij}\theta_i\theta_j\right). \tag{2.21}$$

If n is odd, this integral vanishes. If $n = 2m$ is even, we have

$$\begin{aligned}\int \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \, \exp\left(\frac{1}{2}\sum_{i,j=1}^{n} a_{ij}\theta_i\theta_j\right) &= \frac{1}{2^m \cdot m!}\int \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \left(\sum_{i,j=1}^{n} a_{ij}\theta_i\theta_j\right)^m \\ &= \frac{1}{2^m \cdot m!}\sum_{\sigma\in S_n} (-1)^{\epsilon(\sigma)} a_{\sigma(1)\sigma(2)}\cdots a_{\sigma(n-1)\sigma(n)}.\end{aligned} \tag{2.22}$$

This is, by definition, the *Pfaffian* of the antisymmetric $2m \times 2m$ matrix A:

$$\mathrm{Pf}(A) = \frac{1}{2^m \cdot m!}\sum_{\sigma\in S_n} (-1)^{\epsilon(\sigma)} a_{\sigma(1)\sigma(2)}\cdots a_{\sigma(n-1)\sigma(n)}. \tag{2.23}$$

Example 2.5. Let us consider the simple example of Pfaffian, for $n = 2$ and

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}. \tag{2.24}$$

We have

$$\mathrm{Pf}(A) = \frac{1}{2}(a_{12} - a_{21}) = a. \tag{2.25}$$

Note that, in this example,

$$(\mathrm{Pf}(A))^2 = \det(A), \tag{2.26}$$

which is true for general antisymmetric A. □

So far we have considered elements in the Grassmann algebra where the coefficients in (2.3) are complex numbers. However, if we have both commuting variables $\boldsymbol{x} = (x_1, \cdots, x_p) \in \mathbb{R}^p$ and Grassmann variables $\theta_1, \cdots, \theta_n$, we can promote the coefficients (2.3) to functions of the commuting variables $\boldsymbol{x}$:

$$f(\boldsymbol{x}, \boldsymbol{\theta}) = \sum_{k=0}^{n} \sum_{1\le i_1 \le \cdots i_n \le n} f^{i_1\cdots i_k}(\boldsymbol{x})\theta_{i_1}\cdots\theta_{i_k}, \qquad f^{i_1\cdots i_k} \in C^\infty(U). \tag{2.27}$$

These objects are usually called *superfunctions*. They can be also regarded as functions defined on the *supermanifold* $\mathbb{R}^{p|n}$ with p commuting variables and n anticommuting variables.

We will now assume that $p = n$. In that case, superfunctions are closely related to differential forms. Given a differential form on U of degree k,

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} \omega_{i_1 \cdots i_k}(\boldsymbol{x}) \mathrm{d}x_{i_1} \wedge \cdots \mathrm{d}x_{i_k}, \tag{2.28}$$

we obtain a superfunction

$$f_\omega(\boldsymbol{x}, \boldsymbol{\theta}) = \sum_{1 \le i_1 < \cdots < i_k \le n} \omega_{i_1 \cdots i_k}(\boldsymbol{x}) \theta_{i_1} \cdots \theta_{i_n}. \tag{2.29}$$

Conversely, a superfunction can be regarded as a sum of differential forms of all degrees. In order for this identification to hold, we postulate that, under a change of coordinates

$$\widetilde{x}_i = \widetilde{x}_i(\boldsymbol{x}), \qquad i = 1, \cdots, n, \tag{2.30}$$

the Grasmann coordinates $\theta_1, \cdots, \theta_n$ transform as the differentials $\mathrm{d}x_1, \cdots, \mathrm{d}x_n$, i.e.

$$\theta_i = \sum_{j=1}^{n} \frac{\partial x_i}{\partial \widetilde{x}_j} \widetilde{\theta}_j. \tag{2.31}$$

We can now define a *superintegral* as an integration over both $\boldsymbol{x}$ and $\boldsymbol{\theta}$:

$$\int_{\mathbb{R}^{n|n}} \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{\theta} f(\boldsymbol{x}, \boldsymbol{\theta}) = \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{x}\, f^{1 \cdots n}(\boldsymbol{x}). \tag{2.32}$$

This equivalence leads to the following important principle: *Berezin integrals of superfunctions are equivalent to integrals of differential forms*:

$$\int_{\mathbb{R}^{n|n}} \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{\theta}\, f_\omega(\boldsymbol{x}, \boldsymbol{\theta}) = \int_{\mathbb{R}^n} \omega. \tag{2.33}$$

We also note that the measure of the superintegration is invariant under orientation-preserving diffeomorphisms, since

$$\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\theta} = \frac{\left|\det\left(\frac{\partial x_i}{\partial y_j}\right)\right|}{\det\left(\frac{\partial x_i}{\partial y_j}\right)}. \tag{2.34}$$

It is easy to generalize the construction of $\mathbb{R}^{n|n}$ to a general manifold M by working on coordinate patches. This defines the so-called supermanifold ΠTM, where TM stands for the tangent bundle of M. This is because we can regard the θ_i, locally, as a basis of sections of the tangent bundle. More details on supermanifolds can be found in e.g. [1].

2.2 Introducing supersymmetry

The example presented in this section is obtained by a dimensional reduction of supersymmetric quantum mechanics, and it is discussed in detail in [2].

We can now enlarge our QFTs in zero dimensions by adding Grassmann variables. Let us consider the simplest case, in which we have two Grassmann variables $\theta_{1,2}$. The action is now a superfunction which is even (i.e. its expansion involves only an even number of θs). The most general action involving two conventional variables x, D and two Grassmann variables is then

$$S(x, D, \boldsymbol{\theta}) = S_0(x, D) - \theta_1 \theta_2 S_1(x, D). \tag{2.35}$$

The partition function corresponding to this action is

$$Z = \frac{1}{2\pi}\int \mathrm{d}x\mathrm{d}D\mathrm{d}\boldsymbol{\theta}\, \mathrm{e}^{-S_0(x,D)+\theta_1\theta_2 S_1(x,D)} = \frac{1}{2\pi}\int \mathrm{d}x\mathrm{d}D\, \mathrm{e}^{-S_0(x,D)} S_1(x,D). \tag{2.36}$$

There are special choices of $S_{0,1}(x)$ in which the theory has a symmetry exchanging bosonic and Grassmann variables. Let $h(x)$ be a real, smooth function, and let us consider

$$S_0(x,D) = \frac{1}{2}D^2 + \mathrm{i}h'(x)D, \qquad S_1(x) = h''(x). \tag{2.37}$$

Let us consider the following transformation of the variables:

$$\begin{aligned} \delta x &= \epsilon_1\theta_1 + \epsilon_2\theta_2, \\ \delta\theta_1 &= -\mathrm{i}D\epsilon_2, \\ \delta\theta_2 &= \mathrm{i}D\epsilon_1, \\ \delta D &= 0. \end{aligned} \tag{2.38}$$

Here, $\epsilon_{1,2}$ are Grassmann variables which can be regarded as infinitesimal parameters for the transformation. We have in fact two different transformations, corresponding to the two Grassmann parameters, which we can write as

$$\begin{aligned} \delta_1 x &= \theta_1, \\ \delta_1\theta_1 &= 0, \\ \delta_1\theta_2 &= -\mathrm{i}D, \\ \delta_1 D &= 0. \end{aligned} \tag{2.39}$$

and

$$\begin{aligned} \delta_2 x &= \theta_2, \\ \delta_2\theta_1 &= \mathrm{i}D, \\ \delta_2\theta_2 &= 0, \\ \delta_2 D &= 0. \end{aligned} \tag{2.40}$$

The transformations $\delta_{1,2}$ have two important properties:

1. They change the Grassmann character of the variables (they exchange "bosonic" and "fermionic" coordinates). They are "odd" symmetries. Note that these symmetries act like exterior differentials, namely

$$\delta_k(f_1 f_2) = \delta_k(f_1) f_2 + (-1)^{|f_1|}\delta_k(f_2). \tag{2.41}$$

2. They are nilpotent and anticommuting:

$$\delta_1^2 = \delta_2^2 = \{\delta_1, \delta_2\} = 0. \tag{2.42}$$

3. They are indeed symmetries of the action. The best way of showing this is to realize that the action is actually "exact", namely that

$$S = \delta_1\delta_2\left(-h + \frac{1}{2}\theta_1\theta_2\right). \tag{2.43}$$

Therefore,

$$\delta_1 S = \delta_2 S = 0. \tag{2.44}$$

The exactness of the action has a very important consequence. Let us assume that we make an infinitesimal variation of the function h: $h \to h + \delta h$. Then, we have

$$\delta_h S(x, D, \boldsymbol{\theta}) = \delta_1 \delta_2 \left(-\delta h\right), \tag{2.45}$$

so this is also $\delta_{1,2}$ exact. In particular, we find

$$\begin{aligned} \delta_h Z &= -\frac{1}{2\pi} \int \mathrm{d}x \, \mathrm{d}D \, \mathrm{d}\boldsymbol{\theta} \, \delta_h S(x, D, \boldsymbol{\theta}) \, \mathrm{e}^{-S(x,D,\boldsymbol{\theta})} = -\frac{1}{2\pi} \int \mathrm{d}x \, \mathrm{d}D \, \mathrm{d}\boldsymbol{\theta} \, \delta_1 \delta_2 \left(-\delta h\right) \, \mathrm{e}^{-S(x,D,\boldsymbol{\theta})} \\ &= \frac{1}{2\pi} \int \mathrm{d}x \, \mathrm{d}D \, \mathrm{d}\boldsymbol{\theta} \, \delta_1 \delta_2 \left(-\delta h \, \mathrm{e}^{-S(x,D,\boldsymbol{\theta})}\right), \end{aligned} \tag{2.46}$$

where we used that $\delta_j S = 0$. This is the integral of a total variation induced by a symmetry of the theory. These variations are typically zero, in the same way that the integrals of exact differentials vanish when integrating over a compact manifold. However, in most cases of interest the space where we integrate over is not compact, and the argument that integrals of total variations vanish is only formal or heuristic. Therefore, one has to exercise caution in using this argument.

Example 2.6. An example of this situation is the following integral:

$$\int_{\mathbb{R}^2} (x^2 - y^2) \mathrm{e}^{-x^2-y^2} \mathrm{d}x \mathrm{d}y. \tag{2.47}$$

It is not necessary to perform this integral to know that it vanishes. This is a consequence of rotational invariance, of course. Formally, this can be shown as follows. Let us consider the variation of x and y under an infinitesimal rotation with angle $\delta\theta$,

$$\delta x = -y\delta\theta, \qquad \delta y = x\delta\theta. \tag{2.48}$$

Then, it follows that

$$\delta(xy) = (x^2 - y^2)\delta\theta. \tag{2.49}$$

Therefore,

$$\delta\theta \int_{\mathbb{R}^2} (x^2 - y^2) \mathrm{e}^{-x^2-y^2} \mathrm{d}x \mathrm{d}y = \int_{\mathbb{R}^2} \delta(xy) \mathrm{e}^{-x^2-y^2} \mathrm{d}x \mathrm{d}y = \int_{\mathbb{R}^2} \delta\left(xy \mathrm{e}^{-x^2-y^2}\right) \mathrm{d}x \mathrm{d}y = 0. \tag{2.50}$$

□

In calculating Z, it is useful to integrate first over the variable D. This is a simple Gaussian integral

$$\begin{aligned} \int \mathrm{d}D \, \exp\left(-\frac{1}{2}D^2 - \mathrm{i}Dh'(x)\right) &= \int \mathrm{d}D \, \exp\left(-\frac{1}{2}(D + \mathrm{i}h'(x))^2 - \frac{1}{2}(h'(x))^2\right) \\ &= \sqrt{2\pi} \exp\left(-\frac{1}{2}(h'(x))^2\right). \end{aligned} \tag{2.51}$$

Note that the Gaussian is peaked around

$$D = -\mathrm{i}h'(x). \tag{2.52}$$

and we find

$$Z = \frac{1}{\sqrt{2\pi}} \int \mathrm{d}x \mathrm{d}\boldsymbol{\theta}\, \mathrm{e}^{-\frac{1}{2}(h'(x))^2 + \theta_1\theta_2 h''(x)}. \tag{2.53}$$

The variable D can be regarded as an auxiliary variable, and it is easy to see that it can be eliminated from the theory and replaced by the value

$$D \to -\mathrm{i}h'(x). \tag{2.54}$$

In field theory, such a field is usually called an auxiliary field, and the value (2.54) is sometimes called its "on-shell value". The reason is that this is the value that is obtained when we extremize the action w.r.t. D:

$$\frac{\delta S}{\delta D} = 0 \Rightarrow D = -\mathrm{i}h'(x). \tag{2.55}$$

We will assume that $(h'(x))^2$ is such that the integral

$$\int \mathrm{d}x\, \mathrm{e}^{-\frac{1}{2}(h'(x))^2} \tag{2.56}$$

converges. In other words, the exponential function has to decrease sufficiently fast at infinity (this happens e.g. if h is a polynomial).

One consequence of the above deformation argument is that the partition function is also invariant under rescalings of the function h:

$$h \to \gamma h, \qquad \gamma \neq 0. \tag{2.57}$$

More precisely, let us consider the partition function as a function of γ, after setting $h \to \gamma h$:

$$Z(\gamma) = \frac{1}{\sqrt{2\pi}} \int \mathrm{d}x \mathrm{d}\boldsymbol{\theta}\, \mathrm{e}^{-\frac{\gamma^2}{2}(h'(x))^2 + \gamma\theta_1\theta_2 h''(x)}. \tag{2.58}$$

Then, if we use the original representation of $Z(\gamma)$, we find

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} Z(\gamma) = \frac{1}{2\pi} \int \mathrm{d}x\, \mathrm{d}D\, \mathrm{d}\boldsymbol{\theta}\, \delta_1\delta_2\,(h)\; \mathrm{e}^{-S_\gamma(x,D,\boldsymbol{\theta})}, \tag{2.59}$$

where we have denoted by $S_\gamma(x, D, \boldsymbol{\theta})$ the action obtained after setting $h \to \gamma h$. This is again the integral of a total symmetry variation. We conclude that

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} Z(\gamma) = 0 \tag{2.60}$$

which means that $Z(\gamma)$, as a function of γ, is actually a constant. Note however that we can not set $\gamma = 0$ in the above invariance argument, since in that case the action vanishes and the integrand does not decrease sufficiently fast at infinity.

Since we can calculate $Z(\gamma)$ for any value of γ, we can choose a regime where the integral is easily done. Such a regime occurs when γ is very large. In this case, the Gaussian factor

$$\exp\left\{-\frac{1}{2}\gamma^2 (h'(x))^2\right\} \tag{2.61}$$

gives a vanishing result unless $h'(x) = 0$. More precisely, we remind the following identity:

$$\frac{\gamma}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\gamma^2 x^2\right\} \to \delta(x) \tag{2.62}$$

Therefore,

$$\frac{\gamma}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\gamma^2(h'(x))^2\right\}\to\delta(h'(x))=\sum_{x_k}\frac{1}{|h''(x_k)|}\delta(x-x_k), \tag{2.63}$$

as $\gamma\to\infty$, where x_k are the points such that

$$h'(x_k)=0. \tag{2.64}$$

In other words, the integral "localizes" onto the zeroes of the function $h'(x)$. We can therefore compute Z as a sum of the contributions around these points:

$$\begin{aligned} Z&=\frac{1}{\sqrt{2\pi}}\sum_{x_k}\frac{\sqrt{2\pi}}{\gamma}\frac{1}{|h''(x_k)|}\int \mathrm{d}x\mathrm{d}\boldsymbol{\theta}\,\delta(x-x_k)\exp\left\{\gamma\theta_1\theta_2h''(x)\right\}\\ &=\sum_{x_k}\frac{1}{\gamma|h''(x_k)|}\int \mathrm{d}\boldsymbol{\theta}\exp\left\{\gamma\theta_1\theta_2h''(x_k)\right\}\\ &=\sum_{x_k}\frac{h''(x_k)}{|h''(x_k)|}. \end{aligned} \tag{2.65}$$

This quantity is easy to understand: it counts the zeroes of $h'(x)$ with a sign ±1, depending on the sign of the slope of the tangent at the zeroes. If $h(x)$ is a polynomial, this is very easy to evaluate, and it depends only on the power of the term with the highest degree in $h'(x)$. If

$$h'(x)=a_{2N}x^{2N}+\cdots, \tag{2.66}$$

then we find that

$$Z=0. \tag{2.67}$$

If

$$h'(x)=a_{2N+1}x^{2N+1}+\cdots \tag{2.68}$$

then

$$Z=\mathrm{sign}(a_{2N+1}). \tag{2.69}$$

Note that in both cases Z is invariant under deformations of $h(x)$, as expected, provided that we do not change the behavior of $h'(x)$ at infinity. We illustrate this in Fig. 8, in the case $h'(x)$ is an even polynomial: the zeroes of $h'(x)$ come in pairs, and when we deform $h'(x)$ they appear or disappear in pairs, so that Z always vanishes.

The localization principle is a very powerful tool to compute partition functions and related quantities. It is interesting to note that the points where the integral localizes, namely the critical points $h'(x)=0$, are also the fixed points of the fermionic symmetry. Indeed, after replacing D by its on-shell value (2.54), we have

$$\delta\theta_1=h'(x)\epsilon_2,\qquad \delta\theta_2=-h'(x)\epsilon_1. \tag{2.70}$$

Requiring $\delta x=0$ selects $\theta_1=\theta_2=0$, and requiring

$$\delta\theta_1=\delta\theta_2=0 \tag{2.71}$$

selects $h'(x)=0$. Therefore, the fixed points of the fermionic symmetry are the points x_k where the integral localizes. This turns out to be a general principle which can be used to understand the localizing locus of theories with fermionic symmetries.

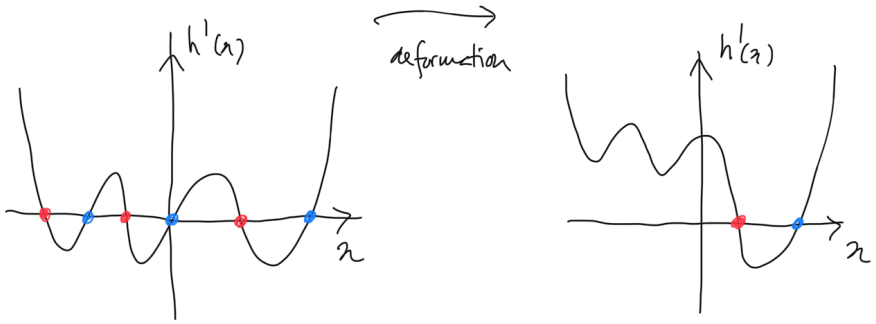

Figure 8. Evaluation of Z when $h'(x)$ is an even potential. Its zeroes come in pairs, and they appear or disappear in pairs when we deform $h(x)$.

2.3 Supersymmetry and geometry

We will now consider a more complicated example of a supersymmetric, zero-dimensional theory, with an important geometric meaning. Good references for this section are [3–5].

Let X be an orientable, compact n-dimensional manifold without boundary, and let us consider an oriented vector bundle $\mathcal{E} \to X$ of rank $2m \leq n$ over X. One of the most important topological invariants of a vector bundle is the so called *Euler class*

$$e(\mathcal{E}) \in H^{2m}(X) \tag{2.72}$$

which integrates to the Euler number, when $2m = n$,

$$\chi(\mathcal{E}) = \int_M e(\mathcal{E}). \tag{2.73}$$

If $\mathcal{E}$ is the tangent bundle to M, then $\chi(TM)$ is the Euler characteristic of M.

Very often, in geometry, we calculate topological invariants from quantities which are not manifestly invariant, and depend on a choice of an additional geometric structure. However, from this geometric structure we extract "invariant" information which makes it possible to obtain the wished-for topological invariant.

In the case of the Euler class of a bundle, there are two different routes to achieve this. The first one is based on Chern–Weil theory. This goes as follows. Let us first recall that a connection on $\mathcal{E}$ is a linear map

$$\nabla : \Gamma(\mathcal{E}) \to \Gamma\left(\mathcal{E} \otimes T^*M\right) \tag{2.74}$$

satisfying the Leibniz rule

$$\nabla(f\sigma) = (\nabla\sigma)\, f + \sigma \otimes \mathrm{d}f, \tag{2.75}$$

where f is a smooth function on M. We will also assume that $\mathcal{E}$ is equipped with a metric g compatible with the connection, i.e.

$$\mathrm{d}(g(\sigma,\tau)) = g(\nabla\sigma,\tau) + g(\sigma,\nabla\tau), \qquad \sigma,\tau \in \Gamma(\mathcal{E}). \tag{2.76}$$

We will denote by e_a, $a = 1, \cdots, n$ an orthonormal basis of sections of $\mathcal{E}$ in a local trivialization. With respect to this basis of sections, the connection is given by a one-form with values in the Lie algebra of $SO(2m)$ (i.e. an antisymmetric matrix):

$$\theta_{ab} = (\theta_{ab})_\mu \, \mathrm{d}x^\mu, \qquad a,b = 1, \cdots, 2m, \tag{2.77}$$

and

$$\nabla e_a = \theta_{ba} e_b, \tag{2.78}$$

where we used again Einstein's convention that repeated indices are summed over. In a local trivialization with orthonormal basis e_a, the section can be written as

$$\sigma = s_a e_a, \tag{2.79}$$

where the s_a are functions

$$s_a : U \subset M \to \mathbb{R} \tag{2.80}$$

and the covariant derivative is given by

$$\nabla s_a = \mathrm{d}s_a + \theta_{ab} s_b. \tag{2.81}$$

The curvature of the connection is given by

$$K = \mathrm{d}\theta + \theta \wedge \theta. \tag{2.82}$$

According to Chern-Weil theory, one can construct a representative $e_\nabla(\mathcal{E})$ of $e(\mathcal{E})$ associated to a connection ∇ in $\mathcal{E}$:

$$e_\nabla(\mathcal{E}) = (2\pi)^{-m} \mathrm{Pf}(K). \tag{2.83}$$

In constructing this representative, we have used a particular connection ∇. However, a key result of Chern–Weil theory (see e.g. [6]) is that different choices of connection give homologous representatives, i.e.

$$e_\nabla(\mathcal{E}) - e_{\widetilde{\nabla}}(\mathcal{E}) = \mathrm{d}(\cdot). \tag{2.84}$$

In other words, different choices of connection define the same cohomology class. In order to make a first contact with supersymmetric QFT, let us note that the Euler class (2.83) can be written as a Grassmann integral. Indeed, let us introduce a set of real Grassmann variables χ_a, $a = 1, \ldots, 2m$. Then, we can write:

$$e_\nabla(\mathcal{E}) = (2\pi)^{-m} \int \mathrm{d}\boldsymbol{\chi} \, \exp\left\{ \frac{1}{2} \chi_a K_{ab} \chi_b \right\}. \tag{2.85}$$

This example provides the prototype for topological field theories of the cohomological type: one uses a non-topological structure (a connection) to define an object which is topological. A change of the non-topological structure leads to a "trivial" change in the object, in the sense of cohomology.

There is another approach to computing the Euler class of $\mathcal{E}$, in terms of a generic section

$$s : X \to \mathcal{E}. \tag{2.86}$$

In that case, $e(\mathcal{E})$ is constructed as the Poincaré dual in X of the homology class of $X_s = s^{-1}(0)$, the zero locus of s. When $2m = n$, X_s is generically zero dimensional, i.e. it is a set of points:

$$X_s = \{x_k \in X : s(x_k) = 0\}. \tag{2.87}$$

We can integrate $e(\mathcal{E})$ over X to obtain the Euler number of $\mathcal{E}$ as

$$\chi(\mathcal{E}) = \sum_{s(x_k)=0} (\pm 1), \tag{2.88}$$

where the $\pm$ sign can be obtained by looking at the linearization of the section near the fixed point (we will be more precise about this later on). This is the famous Poincaré–Hopf theorem. Note that the result for the Euler class and the Euler characteristic are independent of the choice of section. We have again the same phenomenon: we use a geometric, auxiliary object (a section of the bundle) to extract topological information from it.

We would like to reformulate the above results in terms of a supersymmetric quantum field theory in zero dimensions, and shed light on three issues:

1. the invariance of the Euler class representative w.r.t. a change in the connection, in the Chern–Weil approach;

2. the invariance of the Euler class representative w.r.t. a change in the section, in the approach based on sections of the bundle;

3. the equivalence between both representatives.

To do this, we will use a physics reformulation of a beautiful result by Mathai and Quillen, who found an expression for the Euler class which interpolates between the expression obtained in the Chern–Weil approach and the Poincaré–Hopf approach based on sections.

Let us then consider the following Grassmann integral:

$$e_{s,\nabla}(\mathcal{E}) = (2\pi)^{-m} \int \mathrm{d}\boldsymbol{\chi} \exp\left\{-\frac{1}{2}|s|^2 + \frac{1}{2}\chi_a K_{ab}\chi_b + \mathrm{i}\chi_a \nabla s_a\right\}. \tag{2.89}$$

Here we work on a local trivialization for the bundle. Note that, when $s = 0$, we have

$$e_{s=0,\nabla}(\mathcal{E}) = e_{\nabla}(\mathcal{E}), \tag{2.90}$$

and we recover the expression (2.83). We will call this differential form the *Mathai–Quillen representative*.

The result of the integration in (2.83) and (2.89) is a differential form. We can use the correspondence between differential forms and Grassmann variables. We will denote

$$\mathrm{d}x^\mu \to \psi^\mu, \qquad \mu = 1, \cdots, n. \tag{2.91}$$

In this language, the Mathai-Quillen representative (2.89) can be rewritten as:

$$e_{s,\nabla}(\mathcal{E})(\psi) = (2\pi)^{-m} \int \mathrm{d}\boldsymbol{\chi} \exp\left\{-\frac{1}{2}|s|^2 + \frac{1}{2}\chi_a K_{ab}(\psi)\chi_b + \mathrm{i}\chi_a \nabla s_a(\psi)\right\}, \tag{2.92}$$

and, in the case $n = 2m$, one has the following expression for the Euler number of $\mathcal{E}$:

$$\chi(\mathcal{E}) = (2\pi)^{-m} \int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\psi}\mathrm{d}\chi \exp\left\{-\frac{1}{2}|s|^2 + \frac{1}{2}\chi_a K_{ab}(\psi)\chi_b + \mathrm{i}\chi_a \nabla s_a(\psi)\right\}. \tag{2.93}$$

We now note that (2.93) looks like the partition function of a supersymmetric field theory in zero dimensions, whose "action" is:

$$S(x, \psi, \chi) = \frac{1}{2}|s|^2 - \frac{1}{2}\chi_a K_{ab}(\psi)\chi_b - \mathrm{i}\chi_a \nabla s_a(\psi). \tag{2.94}$$

In order to fully exploit the supersymmetric structure of the theory, we introduce a set of variables B_a, $a = 1, \cdots, 2m$, which has the meaning of a basis of differential forms for the fiber. This is done through the following Gaussian trick:

$$\mathrm{e}^{-\frac{1}{2}|s|^2} = \frac{1}{(2\pi)^m} \int \mathrm{d}B\, \mathrm{e}^{-\frac{1}{2}|B|^2 - \mathrm{i}B_a s_a} \tag{2.95}$$

In this way we obtain two pairs of fields (x, ψ) and (χ, B), where the first pair is associated to the base manifold X, while the second pair is associated to the fiber of $\mathcal{E}$. Notice that these pairs have opposite Grassmann character. We now consider the following supersymmetry transformation of the different fields, follows:

$$\begin{aligned} \delta x^\mu &= \psi^\mu, \qquad \delta\psi^\mu = 0, \\ \delta\chi_a &= B_a, \qquad \delta B_a = 0. \end{aligned} \tag{2.96}$$

The first thing to notice is that this transformation is *nilpotent* $\delta^2 = 0$. Indeed, δ is simply the action of the de Rham differential for the base manifold and for the fiber. The resulting action is

$$S(x, \psi, \chi, B) = \frac{1}{2}|B|^2 + \mathrm{i}B_a s_a - \frac{1}{2}\chi_a K_{ab}(\psi)\chi_b - \mathrm{i}\chi_a \nabla s_a(\psi). \tag{2.97}$$

A crucial property of this action is that it is not only δ-invariant, but also δ-exact:

$$S(x, \psi, \chi, \hat{B}) = \delta\Psi_{\mathrm{loc}}(x, \psi, \chi, B), \tag{2.98}$$

where $\Psi_{\mathrm{loc}}(x, \psi, \chi, B)$ is given by

$$\Psi_{\mathrm{loc}} = \chi_a \left(\mathrm{i}s_a + \frac{1}{2}\theta_{ab}(\psi)\chi_b + \frac{1}{2}B_a \right) \tag{2.99}$$

and

$$\hat{B}_a = B_a + \theta_{ab}\chi_b. \tag{2.100}$$

A linear shift in the auxiliary field is of no consequence in integrated quantities. We conclude that the Mathai–Quillen representative can be written as an integral over the supermanifold associated to the fiber of $\mathcal{E}$, with a δ-exact integrand:

$$\begin{aligned} e_{s,\nabla}(\mathcal{E}) &= \frac{1}{(2\pi)^{2m}} \int \mathrm{d}\boldsymbol{\chi}\, \mathrm{d}\boldsymbol{B}\, \mathrm{e}^{-S(x,\psi,\chi,B)} = \frac{1}{(2\pi)^{2m}} \int \mathrm{d}\boldsymbol{\chi}\, \mathrm{d}\boldsymbol{B}\, \mathrm{e}^{-S(x,\psi,\chi,\hat{B})} \\ &= \frac{1}{(2\pi)^{2m}} \int \mathrm{d}\boldsymbol{\chi}\, \mathrm{d}\boldsymbol{B}\, \mathrm{e}^{-\delta\Psi_{\mathrm{loc}}}, \end{aligned} \tag{2.101}$$

and the Euler characteristic of $\mathcal{E}$ is given by

$$\chi(\mathcal{E}) = \frac{1}{(2\pi)^{2m}} \int \mathrm{d}\boldsymbol{\chi}\, \mathrm{d}\boldsymbol{B}\, \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\psi}\mathrm{e}^{-\delta\Psi_{\mathrm{loc}}}. \tag{2.102}$$

The δ-exactness of the action shows immediately that, under an infinitesimal variation of the section, we have

$$\delta_s S(x, \psi, \chi, B) = \delta\left(\mathrm{i}\chi_a \delta s_a\right). \tag{2.103}$$

Therefore,

$$\delta_s \chi(\mathcal{E}) = -\frac{1}{(2\pi)^{2m}} \int \mathrm{d}\boldsymbol{\chi}\, \mathrm{d}\boldsymbol{B}\, \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\psi}\delta\left(\mathrm{i}\chi_a \delta s_a\right) \mathrm{e}^{-S(x,\psi,\chi,B)}. \tag{2.104}$$

This is the integral of a total variation, so it is zero provided the integrand decays sufficiently rapid at infinity. Similarly, under an infinitesimal variation of the section, we have

$$\delta_\theta S(x,\psi,\chi,B) = \delta\left(\frac{1}{2}\theta_{ab}(\psi)\chi_b\right), \tag{2.105}$$

and again the Euler characteristic is formally invariant under this transformation.

The invariance under deformations of the section s leads to the following result. Let us scale the section as

$$s \to \gamma s, \tag{2.106}$$

where γ is a real parameter. Then, one obtains the same value for the Euler characteristic independently of the value of γ. In particular, there are two important limiting cases. When $\gamma = 0$, we recover (2.85) and the Chern–Weil expression for the Euler characteristic. We can also evaluate the $\chi(\mathcal{E})$ when $\gamma \to \infty$. We can do it directly in the expression (2.93), which after rescaling reads

$$\chi(\mathcal{E}) = (2\pi)^{-m}\int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\psi}\mathrm{d}\chi\,\exp\left\{-\frac{1}{2}\gamma^2|s|^2 + \frac{1}{2}\chi_a K_{ab}(\psi)\chi_b + \mathrm{i}\gamma\chi_a\nabla s_a(\psi)\right\}. \tag{2.107}$$

What happens when $\gamma \to \infty$? The Gaussian factor

$$\exp\left\{-\frac{1}{2}\gamma^2|s|^2\right\} \tag{2.108}$$

makes every contribution to the integral go to zero, unless $s = 0$. Therefore, the integral localizes onto the zeroes of the section s, i.e. the points $\boldsymbol{x}_k$ such that

$$s(\boldsymbol{x}_k) = 0. \tag{2.109}$$

We can therefore compute $\chi(\mathcal{E})$ as a sum of the contributions around these points. Each contribution can be in addition computed as follows. First of all, we use the multi-dimensional generalization of (2.63),

$$\frac{\gamma^{2m}}{(2\pi)^m}\exp\left\{-\frac{1}{2}\gamma^2|s|^2\right\} \to \delta(s(\boldsymbol{x})) = \sum_{s(\boldsymbol{x}_k)=0}\frac{1}{|\det\, H^{(k)}|}\delta(\boldsymbol{x}-\boldsymbol{x}_k) \tag{2.110}$$

as $\gamma \to \infty$, where

$$H^{(k)}_{\mu a} = \partial_\mu s_a\Big|_{\boldsymbol{x}_k}. \tag{2.111}$$

At a zero of the section, we have

$$\nabla s_a = H^{(k)}_{\mu a}\psi^\mu. \tag{2.112}$$

We can use the delta function to integrate over the bosonic coordinates $\boldsymbol{x}$. We obtain in this way:

$$\chi(\mathcal{E}) = \frac{1}{\gamma^{2m}}\sum_{\boldsymbol{x}_k}\frac{1}{|\det\, H^{(k)}|}\int \mathrm{d}\boldsymbol{\psi}\mathrm{d}\boldsymbol{\chi}\,\exp\left\{\mathrm{i}\gamma\chi_a H^{(k)}_{\mu a}\psi^\mu + \frac{1}{2}\chi_a K^{(k)}_{ab}(\psi)\chi_b\right\}, \tag{2.113}$$

where $K^{(k)}_{ab}(\psi)$ denotes the curvature evaluated at $\boldsymbol{x} = \boldsymbol{x}_k$. To perform the integration over the Grassmann variables, it is useful to do the rescaling

$$\hat{\chi}_a = \gamma^{1/2}\chi_a, \qquad \hat{\psi}^\mu = \gamma^{1/2}\psi^\mu. \tag{2.114}$$

We now recall that the measure for Grassmann variables transform with the inverse of the Jacobian, i.e.

$$\mathrm{d}\boldsymbol{\psi}\mathrm{d}\boldsymbol{\chi} = \gamma^{2m}\mathrm{d}\hat{\boldsymbol{\psi}}\mathrm{d}\hat{\boldsymbol{\chi}}. \tag{2.115}$$

We finally obtain

$$\chi(\mathcal{E}) = \sum_{\boldsymbol{x}_k} \frac{1}{|\det\, H^{(k)}|} \int \mathrm{d}\hat{\boldsymbol{\psi}}\mathrm{d}\hat{\boldsymbol{\chi}} \exp\left\{ \mathrm{i}\hat{\chi}_a H^{(k)}_{\mu a}\hat{\psi}^\mu + \frac{1}{2\gamma}\hat{\chi}_a K^{(k)}_{ab}(\hat{\psi})\hat{\chi}_b \right\}. \tag{2.116}$$

In the limit $\gamma \to \infty$, the second term in the exponent does not contribute, and we obtain

$$\chi(\mathcal{E}) = \sum_{\boldsymbol{x}_k} \frac{\det\, H^{(k)}}{|\det\, H^{(k)}|}. \tag{2.117}$$

This gives back the Poincaré-Hopf result (2.88) for the Euler characteristic of $\mathcal{E}$. The ± 1 appearing in this formulae is given by the quotient of determinants.

Remark 2.7. One particular case of the above construction happens when $\mathcal{E} = TM$ is the tangent bundle to a Riemannian manifold M, a manifold of even dimension. In this case the metric on the bundle is the Riemannian metric. The connection θ_{ab} is the so-called spin or Levi-Civita connection. It can be expressed in terms of the vierbein e^a_μ and inverse vierbein e^μ_a as

$$(\theta_{ab})_\mu = e^a_\nu \left(\partial_\mu e^\nu_b + \Gamma^\nu_{\mu\lambda} e^\lambda_b \right), \tag{2.118}$$

see e.g. [6] for more details.

3 Supersymmetric QFT in one dimension

3.1 SUSY quantum mechanics

In order to introduce supersymmetric QM, we have to add fermionic coordinates to the standard bosonic coordinate q. In terms of operators, this means that on top of the usual bosonic operators q, p, we have to introduce Grassmann operators $\Psi_{1,2}$, obeying the anticommutation relations,

$$\{\Psi_\alpha, \Psi_\beta\} = \delta_{\alpha\beta}. \tag{3.1}$$

These operators are formally Hermitian,

$$\Psi^\dagger_\alpha = \Psi_\alpha, \qquad \alpha = 1, 2. \tag{3.2}$$

It is also useful to consider the operators

$$\Psi_\pm = \frac{1}{\sqrt{2}} \left(\Psi_1 \pm \mathrm{i}\Psi_2 \right), \tag{3.3}$$

which satisfy

$$\{\Psi_+, \Psi_-\} = 1, \qquad \Psi^2_\pm = 0, \tag{3.4}$$

and

$$\Psi^\dagger_\pm = \Psi_\mp. \tag{3.5}$$

We can regard $\Psi_\pm$ as fermionic destruction/creation operators, respectively. The algebra of these two operators can be represented by a two-dimensional vector space $\mathbb{C}^2$ generated by the two states

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.6}$$

Physically, $|0\rangle$ is a fermionic vacuum, defined by

$$\Psi_+|0\rangle = 0, \tag{3.7}$$

while $|1\rangle$ is a state occupied by a single fermion defined as

$$\Psi_-|0\rangle = |1\rangle. \tag{3.8}$$

In addition,

$$\Psi_+|1\rangle = |0\rangle. \tag{3.9}$$

The operators $\Psi_\pm$ are represented by the following matrices:

$$\Psi_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \Psi_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{3.10}$$

Their commutator is given by

$$[\Psi_+, \Psi_-] = \sigma_3. \tag{3.11}$$

The Hilbert space for supersymmetric quantum mechanics is the tensor product

$$\mathcal{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2. \tag{3.12}$$

It can be decomposed as the direct sums of two Hilbert spaces:

$$\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F, \tag{3.13}$$

where

$$\mathcal{H}_B = L^2(\mathbb{R})|0\rangle, \qquad \mathcal{H}_F = L^2(\mathbb{R})|1\rangle. \tag{3.14}$$

Elements of this Hilbert space, or wave functions, can be represented as vector-valued objects,

$$\Psi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}. \tag{3.15}$$

An important operator in the theory is the *fermion number operator*

$$\mathsf{F} = \Psi_-\Psi_+, \tag{3.16}$$

which is represented by the matrix

$$\frac{\mathbf{1} - \sigma_3}{2}. \tag{3.17}$$

Therefore, eigenstates of σ_3 with eigenvalue ± 1 have fermion number $F = 0$ or $F = 1$, respectively (we denote by F the eigenvalue of the operator F, and we will refer to it as the fermion number). In other words, the elements of $\mathcal{H}_B$ have $F = 0$, and elements of $\mathcal{H}_F$ have $F = 1$. By acting with a

fermionic operator on one of these eigenstates one changes the fermion number. Mathematically, the Hilbert space of this theory has a $\mathbb{Z}_2$ grading, defined by the operator

$$(-1)^{\mathsf{F}}, \tag{3.18}$$

whose eigenvalues are ± 1.

We will define the fermionic version of QM by the Hamiltonian,

$$\mathsf{H} = \frac{1}{2}\mathsf{p}^2 + W(\mathsf{q}) + \frac{1}{2}Y(\mathsf{q})[\Psi_+, \Psi_-]. \tag{3.19}$$

When acting on wave functions of the form (3.15), the Hamiltonian is given by the operator

$$\mathsf{H} = -\frac{1}{2}\frac{\partial^2}{\partial q^2} + W(q) + \frac{1}{2}Y(q)\sigma_3. \tag{3.20}$$

In matrix notation, we have

$$\mathsf{H} = \begin{pmatrix} -\partial_q^2 + W(q) + Y(q)/2 & 0 \\ 0 & -\partial_q^2 + W(q) - Y(q)/2 \end{pmatrix}. \tag{3.21}$$

Since σ_3 commutes with the Hamiltonian, we can diagonalize them simultaneously. Therefore, we can study the spectrum by considering wavefunctions of the form

$$\begin{pmatrix} \phi_1(q) \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \phi_2(q) \end{pmatrix}. \tag{3.22}$$

Let us assume that the functions $W(q)$, $Y(q)$ appearing in (3.20) satisfy

$$W(q) = \frac{1}{2}\omega^2(q), \qquad Y(q) = \omega'(q), \tag{3.23}$$

where $\omega(q)$ is called the *superpotential*. In this case, the above quantum-mechanical system has an additional fermionic symmetry i.e. a *supersymmetry* (SUSY). There are two equivalent ways to see this. In the Hamiltonian picture, we note that there are two conserved fermionic charges, or *supercharges*, defined by

$$\begin{aligned} \mathsf{Q}_+ &= (\mathsf{p} - \mathrm{i}\omega(\mathsf{q}))\,\Psi_+, \\ \mathsf{Q}_- &= (\mathsf{p} + \mathrm{i}\omega(\mathsf{q}))\,\Psi_-, \end{aligned} \tag{3.24}$$

and satisfying

$$\mathsf{Q}_\pm^\dagger = \mathsf{Q}_\mp. \tag{3.25}$$

In matrix notation, they can be written as

$$\mathsf{Q}_+ = \begin{pmatrix} 0 & -\mathrm{i}\,(\partial_q + \omega(q)) \\ 0 & 0 \end{pmatrix}, \qquad \mathsf{Q}_- = \begin{pmatrix} 0 & 0 \\ -\mathrm{i}\,(\partial_q - \omega(q)) & 0 \end{pmatrix}. \tag{3.26}$$

Indeed, one finds that

$$[\mathsf{H}, \mathsf{Q}_\pm] = 0 \tag{3.27}$$

if and only if

$$Y(q) = \omega'(q), \qquad W'(q) = Y(q)\omega(q), \tag{3.28}$$

which lead to the conditions (3.23). In addition, one has

$$\mathsf{H} = \frac{1}{2}\{\mathsf{Q}_+, \mathsf{Q}_-\}. \tag{3.29}$$

The charges $\mathsf{Q}_\pm$ are nilpotent, i.e.

$$\mathsf{Q}_\pm^2 = 0. \tag{3.30}$$

Since they are fermionic, they exchange the bosonic and fermionic sectors of the Hilbert space:

$$\mathsf{Q}_\pm : \mathcal{H}_B \to \mathcal{H}_F, \qquad \mathsf{Q}_\pm : \mathcal{H}_F \to \mathcal{H}_B. \tag{3.31}$$

Equivalently, they anticommute with the $\mathbb{Z}_2$ grading:

$$\{\mathsf{Q}_\pm, (-1)^{\mathsf{F}}\} = 0. \tag{3.32}$$

Remark 3.1. We can deduce the existence of conserved fermionic charges from the existence of fermionic symmetries in the Lagrangian describing the theory. This will be done in the exercise section.

Note that in the supersymmetric theory, with Hamiltonian

$$\mathsf{H} = \frac{1}{2}\mathsf{p}^2 + \frac{1}{2}\omega^2(\mathsf{q}) + \frac{1}{2}\omega'(\mathsf{q})\sigma_3, \tag{3.33}$$

the fermionic sectors with σ_3 eigenvalues ± 1 have different potentials,

$$W_\pm(q) = \frac{1}{2}\omega^2(q) \pm \frac{1}{2}\omega'(q). \tag{3.34}$$

3.2 Properties of SUSY quantum mechanics

Quantum-mechanical theories with supersymmetry have peculiar properties, from the point of view of spectral theory.

First of all, the energy of an eigenstate can only be *positive or zero.* Moreover, *a state can have zero energy if and only if it is annihilated by all supercharges.*

To see this, let $|\psi\rangle$ be an eigenstate of the Hamiltonian:

$$\mathsf{H}|\psi\rangle = E|\psi\rangle, \tag{3.35}$$

then

$$E = \langle\psi|\mathsf{H}|\psi\rangle = \frac{1}{2}\langle\psi|\left(\mathsf{Q}_+\mathsf{Q}_- + \mathsf{Q}_+\mathsf{Q}_-\right)|\psi\rangle = \frac{1}{2}\left(\|\mathsf{Q}_-|0\rangle\|^2 + \|\mathsf{Q}_+|0\rangle\|^2\right), \tag{3.36}$$

where we used (3.25) and (3.29). Since the last term is a sum of positive definite quantities, we must have $E \geq 0$. Moreover, $E = 0$ if and only if

$$\mathsf{Q}_+|\psi\rangle = \mathsf{Q}_-|\psi\rangle = 0. \tag{3.37}$$

This means that a state with zero energy must be annihilated by the supercharges. Conversely, a state which is not annihilated by both supercharges must have a non-zero energy.

States annihilated by the supercharges are called supersymmetric states. Therefore, *in SUSY QM, a zero energy ground state is a supersymmetric state, and viceversa.*

The second property is that *states with non-zero energy appear necessarily in pairs, related by the supersymmetric charges.*

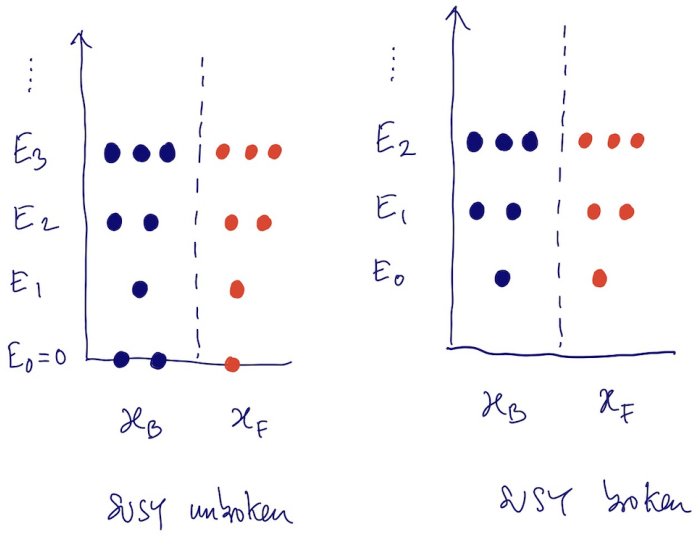

Figure 9. The spectra in a theory with unbroken SUSY (left) and with broken SUSY (right).

To see this, let us denote by $\mathcal{H}_{(n)}$ the subspace of $\mathcal{H}$ spanned by eigenstates of H with energy E_n. Since $\mathsf{Q}_\pm$, $(-1)^{\mathsf{F}}$ commute with H, they map $\mathcal{H}_{(n)}$ to itself:

$$\mathsf{Q}_\pm,\ (-1)^{\mathsf{F}} : \mathcal{H}_{(n)} \to \mathcal{H}_{(n)}. \tag{3.38}$$

We can decompose $\mathcal{H}_{(n)}$ according to the $(-1)^{\mathsf{F}}$ grading as

$$\mathcal{H}_{(n)} = \mathcal{H}^B_{(n)} \oplus \mathcal{H}^F_{(n)}, \tag{3.39}$$

and these spaces are exchanged by $\mathsf{Q}_\pm$. Let us now show that, if $E_n > 0$, we have an isomorphism

$$\mathcal{H}^B_{(n)} \simeq \mathcal{H}^F_{(n)}. \tag{3.40}$$

To see this, let us consider the operator

$$\mathsf{Q}_1 = \mathsf{Q}_+ + \mathsf{Q}_-. \tag{3.41}$$

This operator is self-adjoint. It maps

$$\mathsf{Q}_1 : \mathcal{H}^B_{(n)} \to \mathcal{H}^F_{(n)} \tag{3.42}$$

and it satisfies

$$\mathsf{Q}_1^2 = 2\mathsf{H}, \tag{3.43}$$

which acts on $\mathcal{H}_{(n)}$ as $E_n > 0$. This means that Q_1 is invertible when acting on $\mathcal{H}_{(n)}$. This proves the isomorphism (3.40), and shows that states with non-zero energy come in pairs.

The above argument is *not* true for zero-energy ground states. In such a case, we can have supersymmetric ground states (both fermionic and bosonic) which are not paired.

In general, given a SUSY QM, there are two types of situation:

1. There are supersymmetric ground states, with zero energy, which are not necessarily paired. In this case, we say that supersymmetry is *unbroken*, since the ground states are invariant under the supersymmetric charges.

2. There are *no* supersymmetric ground states. The ground states have positive energy $E_0 > 0$, and they are paired. In this case, we say that supersymmetry is *spontaneously broken*.

We illustrate these two possibilities in Fig. 9.

One important quantity that we can associate to a supersymmetric theory with the properties just mentioned is the *Witten index*, which is defined as

$$\mathcal{I} = \mathrm{Tr}_{\mathcal{H}}\,(-1)^{\mathsf{F}} \mathrm{e}^{-\beta \mathsf{H}}. \tag{3.44}$$

Here, β is a parameter. If the spectrum is discrete, the Witten index does not depend on the value of β, and we can simply set

$$\mathcal{I} = \mathrm{Tr}_{\mathcal{H}}(-1)^{\mathsf{F}}. \tag{3.45}$$

When the spectrum is continuous, the Witten index can depend on β. We will assume, as we have done so far, that the spectrum is discrete. It is easy to see to express the Witten index in terms of the structure of the supersymmetric theory, and to prove the independence w.r.t. β . After decomposing the Hilbert space as

$$\mathcal{H} = \oplus_{n=0}^{\infty} \mathcal{H}_{(n)}, \tag{3.46}$$

we immediately find the following result. If there are *not* supersymmetric ground states, there are as many fermionic and bosonic states in each space $\mathcal{H}_{(n)}$, and

$$\mathcal{I} = 0. \tag{3.47}$$

If there are supersymmetric ground states, then

$$\mathcal{I} = \mathrm{Tr}_{\mathcal{H}_{(0)}}(-1)^{\mathsf{F}} = n_B - n_F, \tag{3.48}$$

where $n_{B,F}$ is the number of bosonic (respectively, fermionic) supersymmetric ground states.

It is interesting to point out that the Witten index is a "robust" characterization of the theory, and in particular it is invariant under deformations of the theory. Indeed, let us suppose that we perturb smoothly the Hamiltonian, so that the states change their energies. The states which were formerly supersymmetric ground states might get a non-zero energy after the perturbation, and become excited states. However, excited states always occur in pairs, so that if, say, m bosonic supersymmetric ground states become excited states, they will have to be accompanied by m fermionic ground states which also get excited, to complete the pairs. Therefore, after the perturbation we have

$$n_B \to n_B - m, \qquad n_F \to n_F - m, \tag{3.49}$$

so that $\mathcal{I}$ remains unchanged. In the same way, if formerly excited states become supersymmetric ground states, they will do so in pairs, and $\mathcal{I}$ will not change. One can think about $\mathcal{I}$ as a "topological invariant" of the supersymmetric model. One could suspect that, in some cases, the Witten index might be related to *actual* topological invariants of the space where the theory is defined. We will see that this is the case.

There is a very simple criterium to know whether SUSY is broken, in supersymmetric QM, for a given superpotential $\omega(q)$. A SUSY ground state must be annilihated by both supercharges, i.e. it has to satisfy the equations

$$\left(\frac{\partial}{\partial q} - \omega(q)\right)\phi_1(q) = 0, \qquad \left(\frac{\partial}{\partial q} + \omega(q)\right)\phi_2(q) = 0, \tag{3.50}$$

with the immediate solution

$$\phi_1(q) = \phi_1(q_0)\exp\left(\int_{q_0}^{q}\omega(q')\mathrm{d}q'\right), \qquad \phi_2(q) = \phi_2(q_0)\exp\left(-\int_{q_0}^{q}\omega(q')\mathrm{d}q'\right). \tag{3.51}$$

Let us assume that $\omega(q)$ is a polynomial. Then, there are two cases to consider:

1. If the highest power of $\omega(q)$ is odd, i.e.

   $$\omega(q) = gq^{2k+1} + \cdots, \tag{3.52}$$

   the highest power of $\int\omega(q')\mathrm{d}q'$ will be even, and equal to

   $$\frac{g}{2k+2}q^{2k+2} + \cdots \tag{3.53}$$

   and we will have one normalizable ground state: when $g > 0$, ϕ_2 is square integrable and we have to set $\phi_1 = 0$. Therefore, there is a fermionic ground state, and the Witten index is

   $$\mathrm{Tr}\,(-1)^{\mathsf{F}} = -1. \tag{3.54}$$

   When $g < 0$, ϕ_1 is square integrable and we have to set $\phi_2 = 0$. Therefore, there is a bosonic ground state, and the Witten index is

   $$\mathrm{Tr}\,(-1)^{\mathsf{F}} = 1. \tag{3.55}$$

   In this case supersymmetry is clearly unbroken.

2. If the highest power of $\omega(q)$ is even, the highest power of $\int\omega(q')\mathrm{d}q'$ will be odd, and none of the above functions is normalizable. In this case SUSY is broken, and we expect a degenerate ground state of nonzero energy. The Witten index is

   $$\mathrm{Tr}(-1)^{\mathsf{F}} = 0. \tag{3.56}$$

Example 3.2. Let us consider the case in which

$$\omega(q) = \omega q, \tag{3.57}$$

where ω is a constant. In this case, the Hamiltonian is given by

$$\mathsf{H} = \frac{\mathsf{p}^2}{2} + \frac{\omega^2\mathsf{q}^2}{2} + \frac{\omega}{2}\sigma_3. \tag{3.58}$$

In this case, the Hamiltonian can be diagonalized exactly. The bosonic part of the Hamiltonian is the standard harmonic oscillator, leading to energy levels

$$|\omega|\left(n + \frac{1}{2}\right), \qquad n = 0, 1, 2, \cdots \tag{3.59}$$

Bosonic states have then energies

$$E_n^B = |\omega| n + \frac{1}{2}\left(|\omega| + \omega\right), \qquad n = 0, 1, 2, \cdots, \tag{3.60}$$

while the fermionic states have energies

$$E_n^F = |\omega| n + \frac{1}{2}\left(|\omega| - \omega\right), \qquad n = 0, 1, 2, \cdots \tag{3.61}$$

If $\omega > 0$, the fermionic state with $n = 0$ is a ground state with zero energy, and we have the spectrum:

$$\begin{aligned} B: &\quad |\omega|,\, 2|\omega|,\, 3|\omega|, \cdots \\ F: &\quad 0,\, |\omega|,\, 2|\omega|,\, 3|\omega|, \cdots \end{aligned} \tag{3.62}$$

Note that each energy level is doubly degenerate. When $\omega < 0$ it is the bosonic state with $n = 0$ which is a ground state with zero energy, and we have the spectrum:

$$\begin{aligned} B: &\quad 0,\, |\omega|,\, 2|\omega|,\, 3|\omega|, \cdots, \\ F: &\quad |\omega|,\, 2|\omega|,\, 3|\omega|, \cdots \end{aligned} \tag{3.63}$$

This is in agreement with the general discussion above.

□

The above result on the ground state structure of the theory, depending on the behavior of $\omega(q)$, is an exact result. But often in quantum theory we have to rely on approximation techniques. What would we obtain by using e.g. a semiclassical analysis of the ground states? In order to see this, it is useful to restore $\hbar$ by using dimensional analysis. We note that the superpotential must have dimensions of momentum, since it is added to p in the expression for the supercharges. Therefore, there must be a relative factor of $\hbar$ in the expression for the potentials $W_\pm(q)$ in (3.34),

$$W_\pm(q) = \frac{1}{2}\omega^2(q) \pm \frac{\hbar}{2}\omega'(q). \tag{3.64}$$

Classically, when $\hbar \to 0$, the number of supersymmetric ground states corresponds to the number of zeroes of $\omega(q)^2/2$, i.e. to the zeroes of $\omega(q)$, since each of them leads to a state of zero energy. Each of these approximate ground states corresponds to the ground states we found above in the analysis of the harmonic oscillator. Indeed, the superpotential behaves near the zero at $q = q_i$ as

$$\omega(q) = \omega_i(q - q_i) + \mathcal{O}\left((q - q_i)^2\right), \qquad \omega_i = \omega'(q_i), \tag{3.65}$$

and the Hamiltonian near the minimum $q = q_i$ is given, at leading order, by

$$\mathsf{H} \approx \frac{\mathsf{p}^2}{2} + \frac{\omega_i^2}{2}\mathsf{x}^2 + \frac{\hbar\omega_i}{2}\sigma_3, \qquad \mathsf{x} = \mathsf{q} - q_i. \tag{3.66}$$

As we know from the previous example, this approximate Hamiltonian has a supersymmetric ground state, which is fermionic or bosonic depending on the sign of ω_i.

It is important to note that this semiclassical analysis is at odds in general with the exact result found before: when the superpotential is *even*, the perturbative analysis predicts that the number of supersymmetric vacua is equal to the number of zeroes of $\omega(q)$, while we have just seen that in an exact analysis there are none. For example, when $\omega(q)$ is given by

$$\omega(q) = \lambda q^2 - \mu^2, \tag{3.67}$$

the Hamiltonian is

$$\mathsf{H} = \frac{1}{2}\mathsf{p}^2 + \frac{\lambda^2}{2}\left(\mathsf{q}^2 - \frac{\mu^2}{\lambda}\right)^2 + \lambda \mathsf{q}\sigma_3. \tag{3.68}$$

One finds two approximate ground states, located at

$$q_\pm = \pm\frac{\mu}{\sqrt{\lambda}}. \tag{3.69}$$

However, the exact analysis shows that there are *no* true ground states for this Hamiltonian. Therefore, these approximate ground states get a non-zero energy in an exact treatment.

Remark 3.3. We might think that higher orders of perturbation theory will give a non-zero energy to these states, but this is not the case. The reason is that, in perturbation theory, we expand around a given zero of the superpotential. This procedure is not sensitive to the parity of the total number of zeroes of $\omega(q)$, which is the crucial ingredient to determine whether supersymmetry is unbroken or not. Therefore, the ground state energy will vanish at all orders in perturbation theory if it vanishes at leading order.

Interestingly, although an approximate perturbative analysis does not give the right ground state structure, it *does give* the right value of the Witten index. Indeed, each approximate ground state contributes to this index as

$$-\operatorname{sgn}(\omega_i). \tag{3.70}$$

Perturbatively, the Witten index is then given by a sum over the zeroes of $\omega(q)$:

$$\operatorname{Tr}(-1)^{\mathsf{F}} = -\sum_i \operatorname{sgn}(\omega_i). \tag{3.71}$$

This clearly reproduces the right result, due to the analysis we made after the calculation (2.65): it vanishes if $\omega(q)$ is an even polynomial, and it gives ∓ 1 depending on the $\pm$ sign of the highest power of $\omega(q)$.

This result was expected due to the invariance of the Witten index under deformations of the theory: if we rescale

$$\omega(q) \to \lambda\omega(q), \tag{3.72}$$

and take $\lambda \to 0$, the zeroes of $\omega(q)$ become the ground states of the theory, and the above analysis is correct. Since the index should be invariant under these deformations, we can calculate it in this extreme regime and obtain the right result.

3.3 Supersymmetric quantum mechanics and Morse theory

Let us first generalize the QM system introduced before by including various operators q_I, p_J, $I, J = 1, \cdots n$ (which correspond to a quantum particle in $\mathbb{R}^n$), satisfying the standard Heisenberg algebra,

$$[\mathsf{x}^I, \mathsf{p}_J] = \mathrm{i}\delta^I_J, \tag{3.73}$$

as well as fermionic operators $\Psi^I_\pm$, $I = 1, \cdots, n$, satisfying

$$\{\Psi^I_\pm, \Psi^J_\pm\} = 0, \qquad \{\Psi^I_+, \Psi^J_-\} = \delta^{IJ}. \tag{3.74}$$

The representation of the fermionic algebra has now dimension 2^n. We define the vacuum $|0\rangle$ as

$$\Psi^I_+|0\rangle = 0, \qquad I = 1, \cdots, n, \tag{3.75}$$

and we can construct $2^n - 1$ additional states by acting with the n creation operators Ψ_-^I:

$$\begin{aligned} &\Psi_-^I|0\rangle, \qquad I = 1, \cdots, n, \\ &\Psi_-^I \Psi_-^J|0\rangle, \qquad 1 \le I < J \le n, \\ &\cdots\cdots \\ &\Psi_-^1 \cdots \Psi_-^n|0\rangle. \end{aligned} \tag{3.76}$$

Therefore, the Hilbert space will now be

$$\mathcal{H} = L_2(\mathbb{R}^n) \otimes \mathbb{C}^{2^n}. \tag{3.77}$$

The states obtained in (3.76) are very similar to the basis of a Grassmann algebra Gr_n. In particular, we can put these states in one-to-one correspondence with a basis of differential forms in $\mathbb{R}^n$, by the correspondence

$$\Psi_-^I \leftrightarrow \mathrm{d}x^I. \tag{3.78}$$

This will be important in what follows.

The Hamiltonian is now

$$\mathsf{H} = \frac{1}{2}\sum_{I=1}^n \mathsf{p}_I^2 + \frac{1}{2}\sum_{I=1}^n (\partial_I h(\mathsf{q}))^2 + \frac{1}{2}\sum_{I,J=1}^n \partial_I\partial_J h(\mathsf{q})[\Psi_-^I, \Psi_+^J], \tag{3.79}$$

We have introduced the function $h(\mathsf{q})$. In the case of a single variable, we have

$$\omega(q) = -h'(q). \tag{3.80}$$

The supercharges are

$$\begin{aligned} \mathsf{Q}_+ &= \Psi_+^I \left(\mathsf{p}_I + \mathrm{i}\partial_I h(\mathsf{q})\right), \\ \mathsf{Q}_- &= \Psi_-^I \left(\mathsf{p}_I - \mathrm{i}\partial_I h(\mathsf{q})\right). \end{aligned} \tag{3.81}$$

It is easy to verify that

$$\mathsf{H} = \frac{1}{2}\{\mathsf{Q}_+, \mathsf{Q}_-\}. \tag{3.82}$$

The supercharges are nilpotent and commute with the Hamiltonian, so we have again a supersymmetric QM system.

Let us now do a perturbative analysis of ground states, as we did before. These are located, perturbatively, at

$$\partial_I h = 0, \tag{3.83}$$

which are the critical points of h, and will be denoted by x_i^I, $i = 1, \cdots, r$. We will assume that h is non-degenerate, i.e. that

$$\det H_{IJ}^{(i)} \neq 0, \tag{3.84}$$

where

$$H_{IJ}^{(i)} = \partial_I\partial_J h(x_i) \tag{3.85}$$

is the Hessian of h at the critical point. If we expand $h(x)$ around the critical point, we find

$$h(x) = h(x_i) + \frac{1}{2}H_{IJ}^{(i)}(x_I - x_i^I)(x^J - x_i^J) + \cdots \tag{3.86}$$

As usual in these problems, we want to diagonalize the quadratic fluctuations. Since $H^{(i)}_{IJ}$ is symmetric, we can diagonalize it as

$$H^{(i)} = A^{-1} C^{(i)} A, \qquad C^{(i)} = \text{diag}\left(c^{(i)}_1, \cdots, c^{(i)}_n\right), \tag{3.87}$$

where A is an orthogonal matrix: $A^{-1} = A^t$. Let us denote

$$\xi^I_{(i)} = A^I{}_J (x^J - x^J_i). \tag{3.88}$$

Then,

$$h(x) = h(x_i) + \frac{1}{2} \sum_{I=1}^{n} c^{(i)}_I \left(\xi^I_{(i)}\right)^2 + \cdots \tag{3.89}$$

If we also define new momentum operators as

$$\widetilde{\mathsf{p}}_I = A^I_J \mathsf{p}_J, \tag{3.90}$$

it is easy to check that the operator version of $\xi^I_{(i)}$ satisfies the commutation relations

$$[\boldsymbol{\xi}^I_{(i)}, \widetilde{\mathsf{p}}_J] = \mathrm{i}\delta^I_J. \tag{3.91}$$

In addition, the new fermionic operators as

$$\widehat{\Psi}^I_\pm = A^I_J \Psi^J_\pm, \qquad I = 1, \cdots, n, \tag{3.92}$$

satisfy the same anticommutation relations:

$$\{\widehat{\Psi}^I_+, \widehat{\Psi}^J_-\} = \delta^{IJ} \tag{3.93}$$

We will need

$$H^{(i)}_{IJ}[\Psi^I_-, \Psi^J_+] = \left(A^{-1}\right)^I{}_K H^{(i)}_{IJ} \left(A^{-1}\right)^J{}_L [\widehat{\Psi}^K_-, \widehat{\Psi}^L_+] = \delta_{KL} c^{(i)}_K [\widehat{\Psi}^K_-, \widehat{\Psi}^L_+] = \sum_{I=1}^{n} c^{(i)}_I [\widehat{\Psi}^I_-, \widehat{\Psi}^I_+]. \tag{3.94}$$

In terms of these new variables, the Hamiltonian near a critical point is given approximately by

$$\mathsf{H} \sim \frac{1}{2} \sum_{I=1}^{n} \left(\widetilde{\mathsf{p}}^2_I + c^{(i)}_I \left(\boldsymbol{\xi}^I_{(i)}\right)^2 + \frac{1}{2} c^{(i)}_I [\widehat{\Psi}^I_-, \widehat{\Psi}^I_+] \right). \tag{3.95}$$

We have now completely decoupled the n degrees of freedom at the quadratic level. We note that, in each sector, we can represent

$$[\widehat{\Psi}^I_-, \widehat{\Psi}^I_+] \to -\sigma_3, \tag{3.96}$$

therefore we have

$$\mathsf{H} \sim \frac{1}{2} \sum_{I=1}^{n} \left(\widetilde{\mathsf{p}}^2_I + c^{(i)}_I \left(\boldsymbol{\xi}^I_{(i)}\right)^2 - \frac{1}{2} c^{(i)}_I \sigma_3 \right), \tag{3.97}$$

so we have n harmonic oscillators (3.58) with frequencies $-c^{(i)}_I$. The structure of the ground state in each sector I depends on the sign of these frequencies. If $c^{(i)}_I > 0$, the ground state is the bosonic one, $|0\rangle$. If $c^{(i)}_I < 0$, the ground state is the fermionic one

$$\widehat{\Psi}^I_- |0\rangle. \tag{3.98}$$

When we put all the sectors together, we find that the ground state is a Gaussian peaked around x_i with the fermionic structure

$$\prod_{I:c_I^{(i)}<0} \widehat{\Psi}_-^I|0\rangle. \tag{3.99}$$

Let us denote by

$$\mu_i = \text{number of negative eigenvalues of } H_{IJ}^{(i)}. \tag{3.100}$$

This is called the *Morse index of the critical point* x_i. It follows that the approximate ground state we have found has an eigenvalue of $(-1)^{\mathsf{F}}$ given by

$$(-1)^{\mu_i}. \tag{3.101}$$

In particular, the Witten index is

$$\mathrm{Tr}\,(-1)^{\mathsf{F}} = \sum_{i=1}^{r}(-1)^{\mu_i}. \tag{3.102}$$

The most important application of these results occurs when we put this supersymmetric system on a compact Riemannian manifold M, with metric g_{IJ}. In a local coordinate system we have the commutation relations (3.73), while the anticommutation relations are generalized to

$$\{\Psi_+^I, \Psi_-^J\} = g^{IJ}. \tag{3.103}$$

Let us consider the simple case in which $h = 0$. We consider the supercharges

$$\mathsf{Q}_\pm = \Psi_\pm^I \mathsf{p}_I. \tag{3.104}$$

We now recall that, due to the equivalence between the representation of the fermionic algebra and differential forms, we can regard the Hilbert space as the space of differential forms

$$\mathcal{H} = \Omega(M) \otimes \mathbb{C}. \tag{3.105}$$

Then, the fermionic creation operators act as elementary differentials

$$\Psi_-^I \to \mathrm{d}x^I \wedge, \tag{3.106}$$

while the annihilation operators can be represented as

$$\Psi_+^I \to g^{IJ} \iota_{\partial/\partial x^J}, \tag{3.107}$$

where ι_X is the operator which acts on forms by contraction with a vector field X. Since $p_I = -\mathrm{i}\nabla_I$, it follows that, under this correspondence,

$$\mathsf{Q}_- \to -\mathrm{i}\,\mathrm{d}, \tag{3.108}$$

where d is the de Rham exterior differential. Since Q_+ is the adjoint of this, we find

$$\mathsf{Q}_- \to \mathrm{i}\,\mathrm{d}^\dagger, \tag{3.109}$$

where $\mathrm{d}^\dagger = \star\mathrm{d}\star$ and $\star$ is the Hodge dual. It follows also that

$$\mathsf{H} \to \frac{1}{2}(\mathrm{d}\,\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d}) = \frac{1}{2}\Delta, \tag{3.110}$$

where Δ is the Laplace–Beltrami operator. Therefore, in this supersymmetric quantum mechanics, *supersymmetric ground states are harmonic forms*, providing a representation of the cohomology of M. The fermion number gives the degree of the form, and the Witten index

$$\mathrm{Tr}\,(-1)^{\mathsf{F}} = \text{number of even harmonic forms} - \text{number of odd harmonic forms} = \chi(M) \quad (3.111)$$

is the Euler characteristic of M.

We can now include a non-zero function $h : M \to \mathbb{R}$. We will assume that h is non-degenerate The supersymmetric charge is now

$$\mathsf{Q}_- = \Psi_-^I(p_I - \mathrm{i}dh) \to -\mathrm{i}\,(\mathrm{d} + \mathrm{d}h\wedge) = -\mathrm{i}\mathrm{e}^{-h}\mathrm{d}\mathrm{e}^{h}. \quad (3.112)$$

By taking the adjoint, we deduce

$$\mathsf{Q}_+ \to \mathrm{i}\,\mathrm{e}^{h}\mathrm{d}^{\dagger}\mathrm{e}^{-h}. \quad (3.113)$$

We can think of these operators as defining a new cohomology theory, with exterior differential

$$\mathrm{d}_h = \mathrm{e}^{-h}\,\mathrm{d}\,\mathrm{e}^{h}. \quad (3.114)$$

However, since multiplying by e^h is an invertible operation, it is clear that if a differential form ψ defines a cohomology class in the standard de Rham cohomology, the form

$$\psi_h = \mathrm{e}^{-h}\psi \quad (3.115)$$

defines a cohomology class in the d_h cohomology. Note in particular that small deformations of the function h will not change the cohomology. In particular, rescalings of h leave the cohomology unchanged.

We can now compute the number of harmonic forms by computing the number of ground states of the Hamiltonian H, which is given by the extension of (3.79) to a Riemannian manifold:

$$\mathsf{H} = \frac{1}{2}\sum_{I=1}^{n} g^{IJ}\mathsf{p}_I\mathsf{p}_J + \frac{1}{2}\sum_{I=1}^{n} g^{IJ}\,(\nabla_I h(\mathsf{q}))\,(\nabla_J h(\mathsf{q})) + \frac{1}{2}\sum_{I,J=1}^{n} \nabla_I\nabla_J h(\mathsf{q})[\Psi_-^I, \Psi_+^J]. \quad (3.116)$$

Of course, when $h = 0$, we recover the Laplace–Beltrami operator. Now, in order to compute the ground states for this Hamiltonian, we can take h to be very large. The approximate ground states are in that case concentrated around the critical points of h. A critical point with Morse index μ_i corresponds to an approximately harmonic form of degree μ_i. As we know from our simple analysis, not all approximate ground states obtained in this way are true ground states, since the analysis is only perturbative. However, this analysis clearly overcounts harmonic forms: some approximate ground states might get a non-zero energy from non-perturbative effects, but if a state has a non-zero energy already in perturbation theory, it cannot be a true ground state of the theory. We deduce then that

$$m_p \geq b_p, \quad (3.117)$$

where m_p are the critical points with Morse index p, and b_p is the p-th Betti number of M. This is known as Morse's inequality, which we have then "derived" by a physical argument.

On the other hand, although a precise determination of ground states using perturbation theory is not possible, we can still compute the Witten index in perturbation theory. We then conclude that

$$\chi(M) = \sum_{i=1}^{r}(-1)^{\mu_i}, \quad (3.118)$$

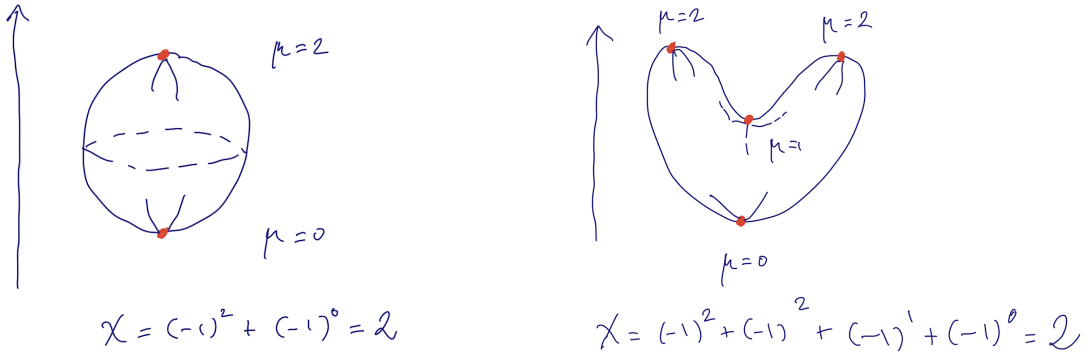

Figure 10. Calculating the Euler characteristic of a sphere with Morse theory. When we deform the sphere, new critical points can appear, but they do it in pairs and cancel in the total result.

i.e. the Euler characteristic of M can be computed by analysis of the critical points. This is another basic result of Morse theory. In Fig. 10, we showed an example of how one would calculate the Euler characteristic of

It turns out that the physics of QM does not only provide a derivation of these results: it provides an improvement. The reason is that it is possible to determine the *exact* ground states of these quantum-mechanical systems by taking into account tunneling effects. This leads to the so-called *Morse–Witten theory*, which makes it possible to determine the cohomology of M by analyzing "tunneling" between critical points.

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