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Lectures on Topological Field Theory and Four-Manifolds

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I give an introduction to topological field theory in four dimensions, focusing on the applications to the topology of four-manifolds. The main goal is to derive the relation between the Donaldson and the Seiberg-Witten invariants by using the u-plane integral of Moore and Witten. I present the relevant background, including classical and "quantum" four-manifold invariants, $\mathcal{N} = 2$ supersymmetry and its twisting, and the exact solution of Seiberg and Witten for $\mathcal{N} = 2$, $SU(2)$ Yang-Mills theory.

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1. Introduction

In 1995, Robbert Dijkgraaf started his lectures on four-manifolds and topological gauge theories [1] with the following question: "Why another set of lectures on four-manifolds and topological field theory?" Of course, this was a rather rethorical question, since in 1995 the Seiberg-Witten invariants [2] were only a few months old, and lecturing on this exciting discovery was one of the natural things to do at the time.

Why then yet another set of lectures on the same topic, this time in 2001? There are in fact two reasons to do it. The first reason is that, since 1995, there has been some further progress in the physical approach to four-manifold invariants. The study of the u-plane integral initiated by Moore and Witten in 1997 [3] made possible a complete understanding of Donaldson invariants for manifolds of $b_2^+ = 1$, a case which was not covered by Witten's seminal paper of 1994. Moreover, the techniques introduced in [3] made also possible a detailed derivation of Witten's magic formula [2] (in the words of Donaldson [4]) relating Donaldson to Seibeg-Witten invariants. The main purpose of these lectures is precisely to cover these more recent developments.

The second reason is mainly pedagogical. The background needed to fully understand this subject is considerable: it includes gauge-theoretic invariants of four-manifolds, twisted $\mathcal{N} = 2$ theories, and the exact solution of Seiberg and Witten for the low energy effective action of $\mathcal{N} = 2$ Yang-Mills theory [5]. Although these topics have been considered separately in excellent books and reviews, there is no -to my knowledge- a single reference which puts together all the relevant physical and mathematical information. This is precisely what I intend to do here.

Of course, providing a truly self-contained review would produce a rather large set of lectures. Therefore, I will be sketchy on many occasions and content myself with mentioning the relevant results rather than explaining them in detail. I should say from the very beginning that, in the case of four-manifold topology, detailed explanations are definitely beyond my competence.

The contents of these lectures are as follows: in sections 2, 3 and 4 I review the basics of four-manifold invariants. Section 2 covers the "classical" invariants, while sections 3 and 4 cover the Donaldson and the Seiberg-Witten invariants, respectively. I have chosen to introduce mathematically the SW invariants rather than discovering them through twisting. Next, in sections 5 and 6 I review $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetry rather quickly. The reader that does not know supersymmetry should be able to follow the lectures after being exposed to these sections (hopefully). Section 7 introduces topological field theories of the Witten or cohomological type, and also the twisting procedure. Sections 8 and 9 deal in some detail with the twisting of the $\mathcal{N}=2$ theories presented in section 6, and in particular section 8 describes Donaldson-Witten theory. At this point the reader knows what she wants to compute in field-theoretic terms, and section 10 gives a summary of the Seiberg-Witten solution, which is the main tool in the computation. The expert will find the exposition of this classic result too technical, but my exposition will hopefully provide the reader with some working knowledge which will be needed in the rest of the lectures. Sections 11 and 12 are devoted to the u-plane integral properly speaking. Here I have tried to be more systematic, since no reviews of [3] are available. Although I have skipped many technical results from [3], some of the key results of the u-plane integral are derived in detail, like the wall-crossing formulae and the relation between Donaldson and SW invariants. At this point, the interested reader should be ready to go to the original paper [3] to learn the rest.

2. Basics of four-manifolds

The purpose of this section is to collect a series of more or less elementary facts about the topology of four-manifolds that will be used in the rest of these lectures. We haven't made any attempt to be self-contained, and the reader should consult for example the excellent book [6] for a more complete survey. The first chapter of [7] gives also a very good summary. A general warning: in these lectures we will assume that the four-manifolds under consideration are closed, compact and orientable. We will also assume that they are endowed with a Riemannian metric.

2.1. Homology and cohomology

The most basic classical topological invariants of a four-manifold are the homology and cohomology groups $H_i(X, \mathbf{Z})$, $H^i(X, \mathbf{Z})$. These homology groups are abelian groups, and the rank of $H_i(X, \mathbf{Z})$ is called the i-th *Betti numbers* of X, denoted by b_i. Remember that by Poincaré duality one has

$$H^i(X, \mathbf{Z}) \simeq H_{n-i}(X, \mathbf{Z}). \tag{2.1}$$

and also $b_i = b_{n-i}$. We will also need the (co)homology groups with coefficients in other groups like $\mathbf{Z}_2$. To obtain these groups one uses the universal coefficient theorem, which states that

$$H_i(X, G) \simeq H_i(X, \mathbf{Z}) \otimes_{\mathbf{Z}} G \oplus \mathrm{Tor}(H_{i-1}(X, \mathbf{Z}), G). \tag{2.2}$$

Let's focus on the case $G = \mathbf{Z}_p$. Given an element x in $H_i(X, \mathbf{Z})$, one can always find an element in $H_i(X, \mathbf{Z}_p)$ by sending $x \to x \otimes 1$. This in fact gives a map:

$$H_i(X, \mathbf{Z}) \to H_i(X, \mathbf{Z}_p) \tag{2.3}$$

which is called the *reduction mod p* of the class x. Notice that, by construction, the image of (2.3) is in $H_i(X, \mathbf{Z}) \otimes \mathbf{Z}_p$. Therefore, if the torsion part in (2.2) is not zero, the map (2.3) is clearly not surjective. When the torsion product is zero, any element in $H_i(X, \mathbf{Z}_p)$ comes from the reduction mod p of an element in $H_i(X, \mathbf{Z})$. For the cohomology groups we have a similar result. Physicists are more familiar with the *de Rham cohomology groups*, $H^*_{DR}(X)$ which are defined in terms of differential forms. These groups are defined over $\mathbf{R}$, and therefore they are insensitive to the torsion part of the singular cohomology. Formally, one has $H^i_{DR}(X) \simeq (H^i(X, \mathbf{Z})/\mathrm{Tor}(H^i(X, \mathbf{Z}))) \otimes_{\mathbf{Z}} \mathbf{R}$.

Remember also that there is a nondegenerate pairing in cohomology, which in the de Rham case is the usual wedge product followed by integration. We will denote the pairing of the cohomology classes (or differential form representatives) α, β by (α, β).

Let's now focus on dimension four. Poincaré duality gives then an isomorphism between $H_2(X,\mathbf{Z})$ and $H^2(X,\mathbf{Z})$. It also follows that $b_1(X)=b_3(X)$. Recall that the Euler characteristic $\chi(X)$ of an n-dimensional manifold is defined as

$$\chi(X)=\sum_{i=0}^{n}(-1)^i b_i(X). \tag{2.4}$$

For a connected four-manifold X, we have then, using Poincaré duality, that

$$\chi(X)=2-2b_1(X)+b_2(X). \tag{2.5}$$

2.2. The intersection form

An important object in the geometry and topology of four-manifolds is the *intersection form*,

$$Q: H^2(X,\mathbf{Z})\times H^2(X,\mathbf{Z})\rightarrow \mathbf{Z} \tag{2.6}$$

which is just the pairing restricted to the two-classes. By Poincaré duality, it can be defined on $H_2(X,\mathbf{Z})\times H_2(X,\mathbf{Z})$ as well. Notice that Q is zero if any of the arguments is a torsion element, therefore one can define Q on the torsion free parts of homology and cohomology.

Another useful way to look at the intersection form is precisely in terms of intersection of submanifolds in X. One fundamental fact in this respect is that we can represent any two-homology class in a four-manifold by a closed, oriented surface S: given an embedding

$$i: S\hookrightarrow X, \tag{2.7}$$

we have a two-homology class $i_*([S])\in H_2(X,\mathbf{Z})$, where $[S]$ is the fundamental class of S. Conversely, any $a\in H_2(X,\mathbf{Z})$ can be represented in this way, and $a=[S_a]$ [6]. One can also prove that

$$Q(a,b)=S_a\cup S_b, \tag{2.8}$$

where the right hand side is the number of points in the intersection of the two surfaces, couted with signs which depend on the relative orientation of the surfaces. If, moreover, η_{S_a}, η_{S_b} denote the Poincaré duals of the submanifolds S_a, S_b (see [8]), one has

$$Q(a,b)=\int_X \eta_{S_a}\wedge\eta_{S_b}=Q([\eta_{S_a}],[\eta_{S_b}]). \tag{2.9}$$

If we choose a basis $\{a_i\}_{i=1,\dots,b_2(X)}$ for the torsion-free part of $H_2(X,\mathbf{Z})$, we can represent Q by a matrix with integer entries that we will also denote by Q. Under a change of basis, we obtain another matrix $Q \to C^T QC$, where C is the transformation matrix. This matrix is obviously symmetric, and it follows by Poincaré duality that it is unimodular, *i.e.* it has $\det(Q) = \pm 1$. If we consider the intersection form on the real vector space $H_2(X, \mathbf{R})$, we see that it is a symmetric, bilinear, nondegenerate form, and therefore it is classified by its rank and its signature. The rank of Q, $\mathrm{rk}(Q)$, is clearly given by $b_2(X)$, the second Betti number. The number of positive and negative eigenvalues of Q will be denoted by $b_2^+(X)$, $b_2^-(X)$, respectively, and the *signature* of the manifold X is then defined as

$$\sigma(X) = b_2^+(X) - b_2^-(X). \tag{2.10}$$

We will say that the intersection form is *even* if $Q(a,a) \equiv 0 \mod 2$. Otherwise, it is *odd.* An element x of $H_2(X, \mathbf{Z})/\mathrm{Tor}(H_2(X, \mathbf{Z}))$ is called *characteristic* if

$$Q(x,a) \equiv Q(a,a) \mod 2 \tag{2.11}$$

for any $a \in H_2(X, \mathbf{Z})/\mathrm{Tor}(H_2(X, \mathbf{Z}))$. An important property of characteristic elements is that

$$Q(x,x) \equiv \sigma(X) \mod 8. \tag{2.12}$$

In particular, if Q is even, then the signature of the manifold is divisible by 8.

Examples.

(1) The simplest intersection form is:

$$n(1) \oplus m(-1) = \mathrm{diag}(1, \dots, 1, -1, \dots, -1), \tag{2.13}$$

which is odd and has $b_2^+ = n$, $b_2^- = m$.

(2) Another important form is the hyperbolic lattice,

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.14}$$

which is even and has $b_2^+ = b_2^- = 1$.

(3) Finally, one has the even, positive definite form of rank 8

$$E_8 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix} \tag{2.15}$$

which is the Dynkin diagram of the exceptional Lie algebra E_8.

Fortunately, unimodular lattices have been classified. The result depends on whether the intersection form is even or odd and whether it is definite (positive or negative) or not, and can be read in the following table:

	odd	even
indef	$p\mathbf{1} \oplus q(-\mathbf{1})$	$pH \oplus qE_8$
def	$n\mathbf{1}$, exotic	exotic : $E_8, D_{16}, \ldots$

Table 1. Classification of unimodular lattices.

Clearly the intersection form is a homotopy invariant. It turns out that simply connected smooth four-manifolds are completely characterized topologically by the intersection form, *i.e.* two simply-connected, smooth four-manifolds are homeomorphic if their intersection forms are equivalent. This is a result due to Freedman. The classification of smooth four-manifolds up to diffeomorphism is another story, and this is the main reason to introduce new invariants which are sensitive to the differentiable structure. But before going into that, we have to give some more details about classical topology.

2.3. Self-dual and anti-self-dual forms

When working in de Rham cohomology, one can use the Hodge operator $*$ to obtain $b_2^\pm$. In four dimensions, the Hodge operator maps

$$* : H^2(X) \to H^2(X). \tag{2.16}$$

Notice that the Hodge operator depends on the Riemannian metric defined on the manifold. Given a differential form ψ, one can decompose it in its self-dual (SD) and anti-self-dual (ASD) parts, which are denoted respectively by $\psi^\pm$. Explicitly,

$$\psi^\pm = \frac{1}{2}(\psi \pm *\psi). \tag{2.17}$$

Since $*^2 = 1$, the Hodge operator has eigenvalues ± 1. The number of $+1$ eigenvalues is precisely b_2^+, and the number of -1 eigenvalues is b_2^-. This means that we can interpret b_2^+ as the number of self-dual harmonic forms on X. This interpretation will be useful in the context of gauge invariants.

2.4. Characteristic classes

An important set of topological invariants of X is given by the characteristic classes of its real tangent bundle. The most elementary ones are the Pontriagin class $p(X)$ and the Euler class $e(X)$, both in $H^4(X, \mathbf{Z}) \simeq \mathbf{Z}$. These classes are then completely determined by two integers, once a generator of $H^4(X, \mathbf{Z})$ is chosen. These integers will be also denoted by $p(X)$, $e(X)$, and they give the Prontriagin number and the Euler characteristic of the four-manifold X, so $e(X) = \chi$. The Pontriagin number is related to the signature of the manifold through the Hirzebruch theorem, which states that:

$$p(X) = 3\sigma(X). \tag{2.18}$$

If a manifold admits an almost-complex structure, one can define a holomorphic tangent bundle $T^{(1,0)}(X)$. This is a complex bundle of rank $r = \dim(X)$, therefore we can associate to it the Chern character $c(T^{(1,0)}(X))$ which is denoted by $c(X)$. For a four-dimensional manifold, one has $c(X) = 1 + c_1(X) + c_2(X)$. Since $c_1(X)$ is a two-form, its square can be paired with the fundamental class of the four manifold. The resulting number can be expressed in terms of the Euler characteristic and the signature as follows:

$$c_1^2(X) = 2\chi(X) + 3\sigma(X). \tag{2.19}$$

Finally, the second Chern class of X is just its Euler class: $c_2(X) = e(X)$. If the almost complex structure is integrable, then the manifold X is complex, and it is called a complex surface. Complex surfaces provide many examples in the theory of four-manifolds. Moreover, there is a very beautiful classification of complex surfaces due to Kodaira, using techniques of algebraic geometry. The interested reader can consult [9][10].

There is another set of characteristic classes which is perhaps less known in physics. These are the Stiefel-Whitney classes of real bundles F over X, denoted by $w_i(F)$. They take values in $H^i(X, \mathbf{Z}_2)$, and a precise definition can be found in [11][6], for example. The Stiefel-Whitney classes of a four-manifold X are defined as $w_i(X) = w_i(TX)$. The first Stiefel-Whitney class of a manifold measures its orientability, so we will always have $w_1(X) = 0$. The second Stiefel-Whitney class pays an important role in what follows. This is a two-cohomology class with coefficients in $\mathbf{Z}_2$, and it has three important properties. If the manifold admits an almost complex structure, then

$$c_1(X) \equiv w_2(X) \mod 2, \tag{2.20}$$

i.e. $w_2(X)$ is the reduction mod 2 of the first Chern class of the manifold. This is a general property of $w_2(X)$ for any almost-complex manifold. In four dimensions, $w_2(X)$ satisfies in addition two other properties: first, it always has a integer lift to an integer class [12] (for example, it the manifold is almost complex, then $c_1(X)$ is such a lift). The second property is the Wu formula, which states that

$$(w_2(X), \alpha) = (\alpha, \alpha) \mod 2, \tag{2.21}$$

for any $\alpha \in H^2(X, \mathbf{Z})$. The l.h.s can be interpreted as the pairing of α with the integer lift of $w_2(X)$. A corollary of the Wu formula is that an integer two-cohomology class is characteristic if and only if it is an integer lift of $w_2(X)$.

2.5. Examples of four-manifolds

(1) A simple example is the four-sphere, $\mathbf{S}^4$. It has $b_1 = b_2 = 0$, and therefore $\chi = 2$, $\sigma = 0$, $Q = 0$.

(2) Next we have the complex projective space $\mathbf{CP}^2$. Recall that this is the complex manifold obtained from $\mathbf{C}^3 - \{(0,0)\}$ by indentifying $z_i \simeq \lambda z_i$, $i = 1, 2, 3$, with $\lambda \neq 0$. $\mathbf{CP}^2$ has $b_1 = 0$ and $b_2 = 1$. In fact, the basic two-homology class is the so-called class of the hyperplane h, which is given in projective coordinates by $z_1 = 0$. It is not difficult to prove that $h^2 = 1$, so $Q_{\mathbf{CP}^2} = (1)$. Notice that h is in fact a $\mathbf{CP}^1$, therefore it is an embedded sphere in $\mathbf{CP}^2$. The projective plane with the opposite orientation will be denoted by $\overline{\mathbf{CP}^2}$, and it has $Q = (-1)$.

(3) An easy way to obtain four-manifolds is by taking products of two Riemann surfaces. A simple example are the so-called product ruled surfaces $\mathbf{S}^2 \times \Sigma_g$, where Σ_g is a Riemann surface of genus g. This manifold has $b_1 = 2g$, $b_2 = 2$. The homology classes have the submanifold representatives $\mathbf{S}^2$ and Σ_g. They have self intersection zero and they intersect in one point, therefore $Q = H$, the hyperbolic lattice, with $b_2^+ = b_2^- = 1$. One then has $\chi = 4(1-g)$.

(4) Our last example is a hypersurface of degree d in $\mathbf{CP}^3$, described by a homogeneous polynomial $\sum_{i=1}^4 z_i^d = 0$. We will denote this surface by S_d. For $d = 4$, one obtains the so-called $K3$ surface.

Exercise. *Topology of S_d*

1) Compute $c_1(S_d)$ and $c_2(S_d)$. Deduce the values of χ and σ.

2) Use the classification of unimodular symmetric, bilinear forms to deduce Q_{K3} (for help, see [6]).

2.6. Spin *and* Spin_c *structures on four-manifolds*

The Spin_n group is a double covering of the orthonormal group $SO(n)$. If E is an oriented $SO(n)$ bundle on a manifold X, a natural question is wether one can lift it to a Spin_n-bundle $\mathrm{Sp}(E)$, producing a double covering $\mathrm{Sp}(E) \to E$. If this can be done, we say that E is endowed with a *Spin structure*. Locally, Spin structures can always be found, but globally there are topological obstructions which are encoded precisely in the second Stiefel-Whitney class of E, $w_2(E)$. We say that X admits a Spin structure if TX does, and X is then called a *Spin manifold*. The necessary and sufficent condition for X to be Spin is then $w_2(X) = 0$.

If a manifold is spin, then one can consistently construct the spinor bundle S, which is a vector bundle providing a representation of the Clifford algebra. A section of this bundle is nothing but a spinor field. In four dimensions, $\mathrm{Spin}_4 = SU(2)_+ \times SU(2)_-$, and the spinor bundle decomposes in two irreducible representations $S = S^+ \oplus S^-$, corresponding to positive and negative chirality spinors (our conventions for spinor algebra in Euclidean and Minkowskian signature are collected in the appendices).

Most of four-manifolds are however not Spin. In such a situation, one can still define something very similar to a spin structure: a Spin_c structure. The best way to think about Spin_c structures, at least for a physicist, is the following: as we said before, the second Stiefel-Whitney cohomology class of a manifold X, $w_2(X) \in H^2(X; \mathbf{Z}_2)$, is always the mod 2 reduction of an integral class w. Let L be the line bundle corresponding to w, *i.e*, $c_1(L) = w$. The square root of the line bundle L in principle does not exist, and the topological obstruction to define it globally is in this case $w_2(X)$, the reduction mod 2 of $c_1(L)$. In the same way, the spinor bundle S of X is not globally defined. However, a standard obstruction analysis in Čech cohomology shows that, although neither $L^{1/2}$ nor S are well defined separately, the product bundle

$$S_L = S \otimes L^{1/2} \tag{2.22}$$

is well defined. This is easy to understand intuitively: to construct a Spin structure, one has to choose a lifting of the rotation group to the spinor group in the different coordinate patches, and as it is well-known this lifting involves ± 1 ambiguities. When the choice of signs can not be made consistently for all patches, the Spin structure is obstructed globally. To define the square root of L we have the same kind of ambiguity in signs, but when tensoring boht bundles as in (2.22), these ambiguities compensate each other and the resulting object is globally well-defined.

We then see that, given an integer lifting of $w_2(X)$, we can construct a Spin_c structure and define a spinor bundle as in (2.22). Therefore, the set of Spin_c structures on a four-manifold is in one-to-one correspondence with the set of characteristic elements. The complex line bundle L is called the determinant line bundle of the Spin_c structure. Once we have found a Spin_c structure, we can generate all the rest of them by tensoring with other line bundles: if L_α is the line bundle associated to an element $\alpha \in H^2(M;\mathbf{Z})$, we can construct from (2.22) another spinor bundle

$$S_{L\otimes L_\alpha^2} = S \otimes \Big(L^{1/2} \otimes L_\alpha\Big). \tag{2.23}$$

Notice that, if $c_1(L) \equiv w_2(X) \mod 2$, the same is obviously true for $L \otimes L_\alpha^2$. This construction gives a map from $H^2(X,\mathbf{Z})$ to the set of Spin_c structures. If there is no two-torsion in $H^2(X,\mathbf{Z})$, then this map is a bijection, although a non-canonical one since it requires a choice of basepoint (the Spin_c structure associated to L).

In the case of an almost complex manifold X, there is however a canonical choice of the canonical Spin_c structure: this is for $L = K^{-1}$, where K denotes the canonical line bundle of X. One has in fact the isomorphism [13][14]

$$S_{K^{-1}} = \Omega^*_{\mathbf{C}} T^{(1,0)}(X), \tag{2.24}$$

where the right hand side denotes the complex exterior powers of the holomorphic tangent bundle of X. Positive chirality spinors correspond to the even powers, and negative chirality spinors to the odd ones.

Since a Spin_c structure can be simply understood as the tensor product (2.22), one can construct the Dirac operator D_L for the Spin_c-structure as the usual Dirac operator coupled to a $U(1)$ connection whose connection and curvature are formally given by 1/2 the connection and curvature of L.

Exercise. *A useful relation*

Using the index theorem for the Dirac operator, show that if L is the determinant line bundle of a Spin_c-structure, then

$$c_1^2(L) = \sigma(X) \bmod 8, \tag{2.25}$$

which is in fact a consequence of (2.12). Show also that if X is Spin, then $\sigma(X)$ is a multiple of eight. It turns out that in fact $\sigma(X)$ is a multiple of 16 for Spin manifolds (Rohlin's theorem).

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3. Basics of Donaldson invariants

Donaldson invariants can be mathematically motivated as follows: as we have mentioned, Freedman's results imply that two symply-connected smooth manifolds are homeomorphic if and only if they have the same intersection form. However, the classification of four-manifolds up to diffeomorphism turns out to be much more subtle: most of the techniques that one uses in dimension ≥ 5 to approach this problem (like cobordism theory) fail in four dimensions. For example, four dimensions is the only dimension in which a fixed homeomorphism type of closed four-manifolds is represented by infinitely many diffeomorphism types, and $n = 4$ is the only dimension where there are "exotic" $\mathbf{R}^n$'s, *i.e.* manifolds which are homeomorphic to $\mathbf{R}^n$ but not diffeomorphic to it. One has to look then for a new class of invariants of differentiable manifolds in order to solve the classification problem, and this was the great achievement of Donaldson. Remarkably, the new invariants introduced by Donaldson are defined by looking at instanton configurations of nonabelian gauge theories on the four-manifold. We will give here a sketch of the mathematical procedure to define Donaldson invariants, in a rather formal way and without entering into the difficult parts of the theory. The interested reader can consult the excellent book by Donaldson and Kronheimer [7]. Other useful resources include [15][16][17], on the mathematical side, and [18][19] on the physical side. The reference [20] gives a very nice review of the mathematical background.

3.1. Yang-Mills theory on a four-manifold

Donaldson theory defines differentiable invariants of smooth four-manifolds starting from Yang-Mills fields on a vector bundle over the manifold. The basic framework is then gauge theory on a four-manifold, and the moduli space of ASD connections. Here we review very quickly some basic notions of gauge connections on manifolds. A more detailed account can be found for example in [20][21].

Let G be a Lie group (usually we will take $G = SO(3)$ or $SU(2)$). Let $P \to M$ be a principal G-bundle over a manifold M with a connection A, taking values in the Lie algebra of G, $\mathbf{g}$. Given a vector space V and a representation ρ of G in $GL(V)$, we can form an associated vector bundle $E = P \times_G V$ in the standard way. G acts on V through the representation ρ. The connection A on P induces a connection on the vector bundle E (which we will also denote by A) and a covariant derivative d_A. Notice that, while the connection A on the principal bundle is an element in $\Omega^1(P, \mathbf{g})$, the induced connection on the vector bundle E is better understood in terms of a local trivialization U_α. On each U_α, the connection 1-form A_α is a $gl(V)$ valued one-form (where $gl(V)$ denotes the Lie algebra of $GL(V)$) and the transformation rule which glues together the different descriptions is given by:

$$A_\beta = g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta} + i g_{\alpha\beta}^{-1} d g_{\alpha\beta}, \tag{3.1}$$

where $g_{\alpha\beta}$ are the transition functions of E.

Recall that the representation ρ induces a representation of Lie algebras $\rho_* : \mathbf{g} \to gl(V)$. We will identify $\rho_*(\mathbf{g}) = \mathbf{g}$, and define the adjoint action of G on $\rho_*(\mathbf{g})$ through the representation ρ. On M one can consider the *adjoint bundle* $\mathbf{g}_E$, defined by:

$$\mathbf{g}_E = P \times_G \mathbf{g}, \tag{3.2}$$

which is a subbundle of $\mathrm{End}(E)$. For example, for $G = SU(2)$ and V corresponding to the fundamental representation, $\mathbf{g}_E$ consists of Hermitian, trace-free endomorphisms of E. If we look at (3.1), we see that the difference of two connections is an element in $\Omega^1(\mathbf{g}_E)$ (the one-forms on X with values in the bundle $\mathbf{g}_E$). Therefore, we can think about the space of all connections $\mathcal{A}$ as an affine space with tangent space at A given by $T_A\mathcal{A} = \Omega^1(\mathbf{g}_E)$.

The curvature F_A of the vector bundle E associated to the connection A can be also defined in terms of the local trivialization of E. On U_α, the curvature F_α is a $gl(V)$-valued two-form that behaves under a change of trivialization as:

$$F_\beta = g_{\alpha\beta}^{-1} F_\alpha g_{\alpha\beta}, \tag{3.3}$$

and this shows that the curvature can be considered as an element in $\Omega^2(\mathbf{g}_E)$.

The next geometrical objects we must introduce are gauge transformations, which are automorphisms of the vector bundle E, $u : E \to E$ preserving the fibre structure (*i.e.*, they map one fibre onto another) and descend to the identity on X. They can be described as sections of the bundle $\mathrm{Aut}(E)$. Gauge transformations form an infinite-dimensional Lie group $\mathcal{G}$, where the group structure is given by pointwise multiplication. The Lie algebra of $\mathcal{G} = \Gamma(\mathrm{Aut}(E))$ is given by $\mathrm{Lie}(\mathcal{G}) = \Omega^0(\mathbf{g}_E)$. This can be seen by looking at the local description, since on an open set U_α the gauge transformation is given by a map $u_\alpha : U_\alpha \to G$, where G acts through the representation ρ. As it is well-known, the gauge transformations act on the connections as

$$u^*(A_\alpha) = u_\alpha A_\alpha u_\alpha^{-1} + i d u_\alpha u_\alpha^{-1} = A_\alpha + i(\nabla_A u_\alpha) u_\alpha^{-1}, \tag{3.4}$$

where $\nabla_A u_\alpha = du_\alpha + i[A_\alpha, u_\alpha]$, and they act on the curvature as:

$$u^*(F_\alpha) = u_\alpha F_\alpha u_\alpha^{-1}. \tag{3.5}$$

3.2. $SU(2)$ and $SO(3)$ bundles

In these lectures we will restrict ourselves to the gauge groups $SU(2)$ and $SO(3)$, and E corresponding to the fundamental representation. Therefore, E will be a two-dimensional complex vector bundle or a three-dimensional real vector bundle, respectively. $SU(2)$ bundles over a compact four-manifold are completely classified by the second Chern class $c_2(E)$ (for a proof, see for example [15]).

In the case of a $SO(3)$ bundle V, the isomophism class is completely classified by the first Pontriagin class

$$p_1(V) = -c_2(V \otimes \mathbf{C}), \tag{3.6}$$

and the Stiefel-Whitney class $w_2(V) \in H^2(X, \mathbf{Z}_2)$. These characteristic classes are related by

$$w_2(V)^2 = p_1(V) \bmod 4. \tag{3.7}$$

$SU(2)$ bundles and $SO(3)$ bundles are of course related: given an $SU(2)$ bundle, we can form an $SO(3)$ bundle by taking the bundle $\mathbf{g}_E$ in (3.2). However, although an $SO(3)$ bundle can be always regarded locally as an $SU(2)$ bundle, there are global obstructions to lift the $SO(3)$ group to an $SU(2)$ group. The obstruction is measured precisely by the second Stiefel-Whitney class $w_2(V)$. Therefore, we can view $SU(2)$ bundles as a special case of $SO(3)$ bundle with zero Stiefel-Whitney class, and this is what we are going to do in these lectures. When the $SO(3)$ bundle can be lifted to an $SU(2)$ bundle, one has the relation:

$$p_1(V) = -4c_2(E). \tag{3.8}$$

Chern-Weil theory gives a representative of the characteristic class $p_1(V)/4$ in terms of the curvature of the connection:

$$\frac{1}{4}p_1(V) = \frac{1}{8\pi^2}\mathrm{Tr}F_A^2, \tag{3.9}$$

where F_A is a Hermitian, trace-free matrix valued two-form. Notice that Hermitian, trace-free matrices have the form:

$$\xi = \begin{pmatrix} a & -ib+c \\ ib+c & -a \end{pmatrix}, \quad a, b, c \in \mathbf{R}, \tag{3.10}$$

so the trace is a positive definite form:

$$\mathrm{Tr}\xi^2 = 2(a^2 + b^2 + c^2) = 2|\xi|^2, \qquad \xi \in \mathbf{su}(2). \tag{3.11}$$

We define the *instanton number* k as:

$$k = -\frac{1}{8\pi^2}\int_X \mathrm{Tr}F_A^2. \tag{3.12}$$

Notice that, if V has not a lifting to an $SU(2)$ bundle, the instanton number is not an integer. If V lifts to E, then $k = c_2(E)$.

The topological invariant $w_2(V)$ for $SO(3)$ bundles may be less familiar to physicists, but it has been used by 't Hooft [22] when $X = \mathbf{T}^4$, the four-torus, to construct gauge configurations called *torons*. To construct torons, one considers $SU(N)$ gauge fields on a four-torus of lengths a_μ, $\mu = 1, \cdots, 4$. To find configurations which are topologically nontrivial, we require of the gauge fields to be periodic up to a gauge transformation in two directions:

$$\begin{aligned} A_\mu(a_1, x_2) =& \Omega_1(x_2)A_\mu(0, x_2), \\ A_\mu(x_1, a_2) =& \Omega_2(x_1)A_\mu(x_1, 0), \end{aligned} \tag{3.13}$$

where we have denoted by $\Omega\, A$ the action of the gauge transformation Ω on the connection A. Looking at the corners, we find the compatibility condition

$$\Omega_1(a_2)\Omega_2(0) = \Omega(a_1)\Omega_1(0)Z, \tag{3.14}$$

where $Z \in C(SU(N)) = \mathbf{Z}_N$ is an element in the center of the gauge group. We can allow a nontrivial Z since a gauge transformation which is in the center of $SU(N)$ does not act on the $SU(N)$ gauge fields. This means that when we allow torons we are effectively dealing with an $SU(N)/\mathbf{Z}_N$ gauge theory. For $SU(2)$, this means that we are dealing with an $SO(3)$ theory, and the toron configurations are in fact topologically nontrivial $SO(3)$ gauge fields with nonzero Stiefel-Whitney class.

3.3. ASD connections

We suppose now that X is an oriented, compact, Riemannian four-manifold. The Riemannian structure allows us to define the Hodge star operator $*$, which is related to the induced metric on the forms by:

$$\psi \wedge *\theta = (\psi, \theta)d\mu, \tag{3.15}$$

where $d\mu$ is the Riemannian volume element. $*$ gives a splitting of the two-forms $\Omega^2(M)$ in self-dual (SD) and anti-self-dual (ASD) forms, defined as the ± 1 eigenspaces of $*$ and denoted by $\Omega^{2,+}(X)$ and $\Omega^{2,-}(X)$, respectively. This splitting extends in a natural way to bundle-valued forms, in particular to the curvature associated to the connection A, $F_A \in \Omega^2(\mathbf{g}_E)$. We call a connection ASD if

$$F_A^+ = 0. \tag{3.16}$$

It is instructive to consider this condition in the case of $X = \mathbf{R}^4$ with the Euclidean metric. If $\{dx_1, dx_2, dx_3, dx_4\}$ is an oriented orthonormal frame, a basis for SD (ASD) forms is given by:

$$\{dx_1 \wedge dx_2 \pm dx_3 \wedge dx_4, dx_1 \wedge dx_4 \pm dx_2 \wedge dx_3, dx_1 \wedge dx_3 \pm dx_4 \wedge dx_1\}. \tag{3.17}$$

If we write $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$, then the ASD condition reads:

$$\begin{aligned} F_{12} + F_{34} =& 0 \\ F_{14} + F_{23} =& 0 \\ F_{13} + F_{42} =& 0. \end{aligned} \tag{3.18}$$

Notice that the second Chern class density can be written as

$$\mathrm{Tr}(F_A^2) = \{|F_A^+|^2 - |F_A^-|^2\} d\mu, \tag{3.19}$$

where $d\mu$ is the volume element and the norm is defined as:

$$|\psi|^2 = \frac{1}{2} \mathrm{Tr}(\psi \wedge *\psi). \tag{3.20}$$

We then see that, with our conventions, if A is an ASD connection the instanton number k is positive. This gives a topological constraint on the existence of ASD connections.

One of the most important properties of ASD connections is that they minimize the Yang-Mills action

$$S_{\mathrm{YM}} = \frac{1}{2} \int_X F \wedge *F = \frac{1}{4} \int_X d^4x \sqrt{g} F_{\mu\nu} F^{\mu\nu} \tag{3.21}$$

in a given topological sector. This is so because the integrand of (3.21) can be written as $|F_A^+|^2 + |F_A^-|^2$, therefore

$$S_{\mathrm{YM}} = \frac{1}{2} \int_X |F_A^+|^2 d\mu + 8\pi^2 k, \tag{3.22}$$

which is bounded from below by $8\pi^2 k$. The minima is attained precisely when (3.16) holds.

The ASD condition is a nonlinear differential equation for non-abelian gauge connections, and it defines a subspace of the (infinite dimensional) configuration space of connections $\mathcal{A}$. This subspace can be regarded as the zero locus of the section

$$s : \mathcal{A} \longrightarrow \Omega^{2,+}(\mathbf{g}_E) \tag{3.23}$$

given by

$$s(A) = F_A^+. \tag{3.24}$$

Our main goal is to define a finite-dimensional moduli space starting from $s^{-1}(0)$. The key fact to take into account is that the section (3.23) is equivariant with respect to the action of the gauge group: $s(u^*(A)) = u^*(s(A))$. Therefore, if a gauge connection A satisfies the ASD condition, then any gauge-transformed connection $u^*(A)$ will also be ASD. To get rid of the gauge redundancy in order to obtain a finite dimensional moduli space, one must "divide by $\mathcal{G}$" *i.e.* one has to quotient $s^{-1}(0)$ by the action of the gauge group. We are thus led to define the moduli space of ASD connections, $\mathcal{M}_{\mathrm{ASD}}$, as follows:

$$\mathcal{M}_{\mathrm{ASD}} = \big\{[A] \in \mathcal{A}/\mathcal{G} \,|\, s(A) = 0\big\}, \tag{3.25}$$

where $[A]$ denotes the gauge-equivalence class of the connection A. Notice that, since s is gauge-equivariant, the above space is well-defined. The fact that the ASD connections form a moduli space is well-known in field theory. For example, on $\mathbf{R}^4$ $SU(2)$ instantons are parameterized by a finite number of data (which include, for example, the position of the instanton), giving $8k-3$ parameters for instanton number k [23]. The moduli space $\mathcal{M}_{\mathrm{ASD}}$ is in general a complicated object, and in the next subsections we will analyze some of its aspects in order to provide a local model for it.

3.4. Reducible connections

In order to analyze $\mathcal{M}_{\mathrm{ASD}}$, we will first look at the map

$$\mathcal{G} \times \mathcal{A} \to \mathcal{A} \tag{3.26}$$

and the associated quotient space $\mathcal{A}/\mathcal{G}$. The first problem we find when we quotient by $\mathcal{G}$ is that, if the action of the group is not free, one has singularities in the resulting quotient space. If we want a smooth moduli space of ASD connections, we have to exclude the points of $\mathcal{A}$ which are fixed under the action of $\mathcal{G}$. To characterize these points, we define the isotropy group of a connection A, Γ_A, as

$$\Gamma_A = \{u \in \mathcal{G} | u(A) = A\}, \tag{3.27}$$

which measures the extent at which the action of $\mathcal{G}$ on a connection A is not free. If the isotropy group is the center of the group $C(G)$, then the action is free and we say that the connection A is *irreducible*. Otherwise, we say that the connection A is *reducible*. Reducible connections are well-known in field theory, since they correspond to gauge configurations where the gauge symmetry is broken to a smaller subgroup. For example, the $SU(2)$ connection

$$A = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \tag{3.28}$$

should be regarded in fact as a $U(1)$ connection in disguise. It is clear that a constant gauge transformation of the form $u\sigma_3$ leaves (3.28) invariant, therefore the isotropy group of A is bigger than the center of $SU(2)$. We will denote the space of irreducible connections by $\mathcal{A}^*$. It follows from the definition that the reduced group of gauge transformations $\hat{\mathcal{G}} = \mathcal{G}/C(G)$ acts freely on $\mathcal{A}^*$.

By using the description of u as a section of $\mathrm{Aut}(E)$ and the action on A given in (3.4), we see that

$$\Gamma_A = \{u \in \Gamma(\mathrm{Aut}(E)) | d_A u = 0\}, \tag{3.29}$$

i.e. the isotropy group at A is given by the covariantly constant sections of the bundle $\mathrm{Aut}(E)$. It follows that Γ_A is a Lie group, and its Lie algebra is given by

$$\mathrm{Lie}(\Gamma_A) = \{f \in \Omega^0(\mathbf{g}_E) | d_A f = 0\}. \tag{3.30}$$

Therefore, a useful way to detect if Γ_A is bigger than $C(G)$ (and has positive dimension) is to study the kernel of d_A in $\Omega^0(\mathbf{g}_E)$. Reducible connections correspond then to a non-zero kernel of

$$-d_A : \Omega^0(\mathbf{g}_E) \to \Omega^1(\mathbf{g}_E). \tag{3.31}$$

In the case of $SU(2)$ and $SO(3)$, a reducible connection has precisely the form (3.28), with isotropy group $\Gamma_A/C(G) = U(1)$. This means, topologically, that the $SU(2)$ bundle E splits as:

$$E = L \oplus L^{-1}, \tag{3.32}$$

with L a complex line bundle, while a reducible $SO(3)$ bundle splits as

$$V = \mathbf{R} \oplus T, \tag{3.33}$$

where $\mathbf{R}$ denotes the trivial rank-one real bundle over X. The above structure for V is easily derived by considering the real part of $\mathrm{Sym}^2(E)$. Notice that, if V admits a $SU(2)$ lifting E, then $T = L^2$. There are topological constraint to have these splittings, because (3.32) implies that $c_2(E) = -c_1(L)^2$, and (3.33) that

$$p_1(V) = c_1(T)^2. \tag{3.34}$$

When E exists, the first Chern class $\lambda = c_1(L)$ is an integral cohomology class. However, when $w_2(E) \neq 0$, then it follows from (3.7) that L does not exist as a line bundle, since its first Chern class is not an integral class but lives in the lattice

$$H^2(X, \mathbf{Z}) + \frac{1}{2} w_2(V). \tag{3.35}$$

In particular, one has that

$$c_1(T) \equiv w_2(V) \bmod 2. \tag{3.36}$$

Therefore, reductions of V are in one-to-one correspondence with cohomology classes $\alpha \in H^2(M; \mathbf{Z})$ such that $\alpha^2 = p_1(V)$. In the following, when we study the local model of $\mathcal{M}_{\mathrm{ASD}}$, we will restrict ourselves to irreducible connections.

3.5. A local model for the moduli space

To construct a local model for the moduli space means essentially to give a characterization of its tangent space at a given point. The way to do that is to consider the tangent space at an ASD connection A in $\mathcal{A}$, which is isomorphic to $\Omega^1(\mathbf{g}_E)$, and look for the directions in this vector space which preserve the ASD condition and which are not gauge orbits (since we are quotienting by $\mathcal{G}$). The local model for $\mathcal{M}_{\mathrm{ASD}}$ was first obtained by Atiyah, Hitchin and Singer in [24].

Let us first address the second condition. In order to find which directions in the tangent space at a connection A are pure gauge, *i.e.* we want to find slices of the action of the gauge group $\hat{\mathcal{G}}$. The procedure is simply to consider the derivative of the map (3.26) in the $\mathcal{G}$ variable at a point $A \in \mathcal{A}^*$ to obtain

$$C : \mathrm{Lie}(\mathcal{G}) \longrightarrow T_A\mathcal{A}, \tag{3.37}$$

which is nothing but (3.31) (notice the minus sign in d_A, which comes from the definition of the action in (3.4)). Since there is a natural metric in the space $\Omega^*(\mathbf{g}_E)$, we can define a formal adjoint operator:

$$C^\dagger : \Omega^1(\mathbf{g}_E) \longrightarrow \Omega^0(\mathbf{g}_E) \tag{3.38}$$

given by $C^\dagger = d_A^*$. We can then orthogonally decompose the tangent space at A into the gauge orbit Im C and its complement:

$$\Omega^1(\mathbf{g}_E) = \text{Im } C \oplus \text{Ker } C^\dagger. \tag{3.39}$$

This is precisely the slice of the action we were looking for. Locally, this means that the neighbourhood of $[A]$ in $\mathcal{A}^*/\mathcal{G}$ can be modelled by the subspace of $T_A\mathcal{A}$ given by Ker d_A^*. Furthermore, the isotropy group Γ_A has a natural action on $\Omega^1(\mathbf{g}_E)$ given by the adjoint multiplication, as in (3.5). If the connection is reducible, the moduli space is locally modelled on (Ker $d_A^*)/\Gamma_A$ (see [7][15]).

We have obtained a local model for the orbit space $\mathcal{A}^*/\mathcal{G}$, and now we need to enforce the ASD condition. Let A be an irreducible ASD connection, verifying $F_A^+ = 0$, and let $A + a$ be another ASD connection, where $a \in \Omega^1(\mathbf{g}_E)$. The condition we get on a starting from $F_{A+a}^+ = 0$ is $p^+(d_A a + a \wedge a) = 0$, where p^+ is the projector on the SD part of a two-form. At linear order we find:

$$p^+ d_A a = 0. \tag{3.40}$$

Notice that the map $p^+ d_A$ is nothing but the linearization of the section s, ds:

$$ds : T_A\mathcal{A} \longrightarrow \Omega^{2,+}(\mathbf{g}_E) \tag{3.41}$$

The kernel of ds corresponds to tangent vectors that satisfy the ASD condition at linear order (3.40). We can now give a precise descriptionof the tangent space of $\mathcal{M}_{\text{ASD}}$ at $[A]$: we want directions which are in $\text{Ker}\, ds$ but which are not in $\text{Im}\, d_A$. First notice that, since s is gauge-equivariant, $\text{Im}\, d_A \subset \text{Ker}\, ds$. This can be checked by direct computation:

$$p^+ d_A d_A \phi = [F_A^+, \phi] = 0, \qquad \phi \in \Omega^0(\mathbf{g}_E), \tag{3.42}$$

since A is ASD. Taking now into account (3.39), we finally find:

$$T_{[A]}\mathcal{M}_{\text{ASD}} \simeq (\text{Ker}\, ds) \cap (\text{Ker}\, d_A^\dagger). \tag{3.43}$$

This space can be regarded as the kernel of the operator $D = p^+ d_A \oplus d_a^\dagger$:

$$D : \Omega^1(\mathbf{g}_E) \longrightarrow \Omega^0(\mathbf{g}_E) \oplus \Omega^{2,+}(\mathbf{g}_E). \tag{3.44}$$

Since $\mathrm{Im}\, d_A \subset \mathrm{Ker}\, ds$ there is a short exact sequence:

$$0 \longrightarrow \Omega^0(\mathbf{g}_E) \xrightarrow{d_A} \Omega^1(\mathbf{g}_E) \xrightarrow{p^+ d_A} \Omega^{2,+}(\mathbf{g}_E) \longrightarrow 0. \tag{3.45}$$

This complez is called the *instanton deformation complex* or *Atiyah-Hitchin-Singer (AHS) complex* [24], and gives a very elegant local model for the moduli space of ASD connections. In particular, one has that

$$T_{[A]}\mathcal{M}_{\mathrm{ASD}} = H_A^1, \tag{3.46}$$

where H_A^1 is the middle cohomology group of the complex (3.45):

$$H_A^1 = \frac{\ker \ \ p^+ d_A}{\mathrm{Im} \ \ d_A}. \tag{3.47}$$

The index of the AHS complex (3.45) is given by

$$\mathrm{ind} = \dim H_A^1 - \dim H_A^0 - \dim H_A^2, \tag{3.48}$$

where $H_A^0 = \mathrm{Ker}\, d_A$ and $H_A^2 = \mathrm{Coker}\ \ p^+ d_A$. This index is usually called the *virtual* dimension of the moduli space. When A is an irreducible connection (in particular, $\mathrm{Ker}\, d_A = 0$) and in addition it satisfies $H_A^2 = 0$, it is called a *regular* connection [7]. For these connections, the dimension of $T_{[A]}\mathcal{M}_{\mathrm{ASD}}$ is given by the virtual dimension. This index can be computed for any gauge group G using the Atiyah-Singer index theorem. The computation is done in [24], and the result for $SO(3)$ is:

$$\dim \ \mathcal{M}_{\mathrm{ASD}} = -2p_1(V) - \frac{3}{2}(\chi + \sigma). \tag{3.49}$$

Exercise. *Dimension of $\mathcal{M}_{\mathrm{ASD}}$*

Compute $\mathcal{M}_{\mathrm{ASD}}$ using the index theorem for the twisted Dirac operator. *Hint*: use that $\Omega^1(X) \simeq S^+ \otimes S^-$, and $\Omega^{2,+} \simeq S^+ \otimes S^+$.

The conclusion of this analysis is that, if A is an irreducible ASD connection, the moduli space in a neighbourhood of this point is smooth and can be modelled by the cohomology (3.47). If the connection is also regular, the index of the instanton deformation complex gives minus the dimension of moduli space. Of course, the most difficult part of Donaldson theory is to find the global structure of $\mathcal{M}_{\mathrm{ASD}}$. In particular, in order to define the invariants one has to compactify the moduli space. We are not going to deal with these subtle issues here, and refer the reader to the references mentioned at the beginning of this section.

3.6. Donaldson invariants

Donaldson invariants are roughly defined in terms of integrals of differential forms in the moduli space of irreducible ASD connections. These differential forms come from the rational cohomology ring of $\mathcal{A}^*/\mathcal{G} = \mathcal{B}^*$, and it is necessary to have an explicit description of this ring. The construction involves the *universal bundle* or *universal instanton* associated to this moduli problem. The construction goes as follows: if the gauge group is $SU(2)$, we consider the $SO(3)$ bundle $\mathbf{g}_E$ associated to E, and if the gauge group is $SO(3)$ we consider the vector bundle V. We will denote both of them by $\mathbf{g}_E$, since the construction is the same in both cases. We then consider the space $\mathcal{A}^* \times \mathbf{g}_E$. This can be regarded as a bundle:

$$\mathcal{A}^* \times \mathbf{g}_E \to \mathcal{A}^* \times X \tag{3.50}$$

which is the pullback from the bundle $\pi : \mathbf{g}_E \to X$. The space $\mathcal{A}^* \times \mathbf{g}_E$ is called a *family of tautological connections*, since the natural conection on $\mathcal{A}^* \times \mathbf{g}_E$ is tautological in the $\mathbf{g}_E$ direction and trivial in the $\mathcal{A}^*$ direction: at the point (A, p), the connection is given by $A_\alpha(\pi(p))$ (where we have chosen a trivialization of $\mathbf{g}_E$ as in 3.1, and $\pi(p) \in U_\alpha$). Since the group of reduced gauge transformations $\hat{\mathcal{G}}$ acts on both factors, $\mathcal{A}^*$ and $\mathbf{g}_E$, the quotient

$$\mathbf{P} = \mathcal{A}^* \times_{\hat{\mathcal{G}}} \mathbf{g}_E \tag{3.51}$$

is a $G/C(G)$-bundle over $\mathcal{B}^* \times X$. This is the *universal bundle* associated to E (or V). In the case of $G = SU(2)$ or $SO(3)$, the universal bundle is an $SO(3)$ bundle (since $SU(2)/\mathbf{Z}_2 = SO(3)$ and $SO(3)$ has no center). Its Pontriagin class $p_1(\mathbf{P})$ can be computed using Chern-Weil theory in terms of the curvature of a connection on $\mathbf{P}$. One can construct a natural connection on $\mathbf{P}$, called the *universal connection*, by considering the quotient of the tautological connection (see [7][18] for details). The curvature of the universal connection will be denoted by $K_{\mathbf{P}}$. It is a form in $\Omega^2(\mathcal{B}^* \times X, \mathbf{g}_E)$, and splits according to the bigrading of $\Omega^*(\mathcal{B}^* \times X)$ into three pieces: a two-form with respect to $\mathcal{B}^*$, a two-form with respect to X, and a mixed form (one-form on $\mathcal{B}^*$ and one-form on X), all with values in $\mathbf{g}_P$. The Pontriagin class is:

$$\frac{p_1(\mathbf{P})}{4} = \frac{1}{8\pi^2}\mathrm{Tr}(K_{\mathbf{P}} \wedge K_{\mathbf{P}}) \tag{3.52}$$

and defines a cohomology class in $H^4(\mathcal{B}^* \times X)$ By decomposing according to the bigrading, we obtain an element in $H^*(\mathcal{B}^*) \otimes H^*(X)$. To get differential forms on $\mathcal{B}^*$, we just take the slant product with *homology* classes in X (*i.e.* we simply pair the forms on X with cyles on X). In this way we obtain the *Donaldson map*:

$$\mu : H_i(X) \longrightarrow H^{4-i}(\mathcal{B}^*). \tag{3.53}$$

One can prove [7] that the differential forms obtained in this way actually generate the cohomology ring of $\mathcal{B}^*$. Finally, after restriction to $\mathcal{M}_{\mathrm{ASD}}$ we obtain the following differential forms on the moduli space of ASD connections:

$$\begin{aligned} x \in H_0(X) &\to \mathcal{O}(x) \in H^4(\mathcal{M}_{\mathrm{ASD}}), \\ \delta \in H_1(X) &\to I_1(\delta) \in H^3(\mathcal{M}_{\mathrm{ASD}}), \\ S \in H_2(X) &\to I_2(S) \in H^2(\mathcal{M}_{\mathrm{ASD}}). \end{aligned} \tag{3.54}$$

There are also cohomology classes associated to three-cycles in X, but we will not consider them in these lectures (see [25] for a detailed treatment in the context of the u-plane integral). In the next lecture we will see that the Donaldson map arises very naturally in the context of topological field theory in what is called the *descent procedure*. In any case, we can now formally define the Donaldson invariants as follows. Consider the space

$$\mathbf{A}(X) = \mathrm{Sym}(H_0(X) \oplus H_2(X)) \otimes \wedge^* H_1(X), \tag{3.55}$$

with a typical element written as $x^\ell S_{i_1} \cdots S_{i_p} \delta_{j_1} \cdots \delta_{j_q}$. The *Donaldson invariant* corresponding to this element of $\mathbf{A}(X)$ is the following intersection number:

$$\begin{aligned} &\mathcal{D}_X^{w_2(V),k}(x^\ell S_{i_1} \cdots S_{i_p} \delta_{j_1} \cdots \delta_{j_q}) = \\ &\int_{\mathcal{M}_{\mathrm{ASD}}(w_2(V),k)} \mathcal{O}^\ell \wedge I_2(S_{i_1}) \wedge \cdots \wedge I_2(S_{i_p}) \wedge I_1(\delta_{j_1}) \wedge \cdots \wedge \ I_1(\delta_{j_q}), \end{aligned} \tag{3.56}$$

where we denoted by $\mathcal{M}_{\mathrm{ASD}}(w_2(V), k)$ the moduli space of ASD connections specified by the second Stiefel-Whitney class $w_2(V)$ and the instanton number k. Notice that, since the integrals of differential forms are different from zero only when the dimension of the space equals the total degree of the form, it is clear that the integral in (3.56) will be different from zero only if the degrees of the forms add up to $\dim(\mathcal{M}_{\mathrm{ASD}}(w_2(V), k))$. It follows from (3.56) that Donaldson invariants can be understood as functionals:

$$\mathcal{D}_X^{w_2(V),k} : \mathbf{A}(X) \to \mathbf{Q}. \tag{3.57}$$

The reason that the values of the invariants are rational rather than integer is subtle and has to do with the fact that they are rigorously defined as intersection numbers only in certain situations (the so-called *stable range*). Outside this range, there is a natural way to extend the definition which involves dividing by 2 (for more details, see [16]).

It is very convenient to pack all Donaldson invariants in a generating function. Let $\{\delta_i\}_{i=1,\dots,b_1}$ be a basis of one-cycles, and $\{S_i\}_{i=1,\dots,b_2}$ a basis of two-cycles. We introduce the formal sums

$$\delta = \sum_{i=1}^{b_1} \zeta_i \, \delta_i, \qquad\qquad S = \sum_{i=1}^{b_2} v_i \, S_i, \tag{3.58}$$

where v_i are complex numbers, and ζ_i are Grassmann variables. We then define the *Donaldson-Witten generating function* as:

$$Z_{DW}^{w_2(V)}(p, \zeta_i, v_i) = \sum_{k=0}^{\infty} \mathcal{D}_X^{w_2(V),k}(\mathrm{e}^{px+\delta+S}), \tag{3.59}$$

where in the right hand side we are summing over all instanton numbers, *i.e.* we are summing over all topological configurations of the $SO(3)$ gauge field with a *fixed* $w_2(V)$. This gives a formal power series in p, ζ_i and v_i. The Donaldson invariants are the coefficients of this formal series. If we assign degree 4 to p, 2 to v_i and 3 to ζ_i, and we fix the total degree (*i.e.* we fix k), we get a finite polynomial which encodes all the Donaldson invariants for a fixed instanton number. Therefore, Donaldson invariants at fixed instanton number can be also regarded as polynomials in the (dual of the) cohomology of the manifold. Sometimes we will also write (3.59) as a functional $Z_{DW}^{w_2(V)}(p, S, \delta)$. It should be mentioned that in the math literature the most common object is the so-called *Donaldson series* [26], which is defined when $\delta = 0$ as follows:

$$\mathbf{D}^{w_2(V)}(S) = Z_{DW}^{w_2(V)}(p, S)\big|_{p=0} + \frac{1}{2}\frac{\partial}{\partial p} Z_{DW}^{w_2(V)}(p, S)\big|_{p=0}. \tag{3.60}$$

The Donaldson series can then be regarded as a map:

$$\mathbf{D}^{w_2(V)} : \mathrm{Sym}(H_2(X)) \to \mathbf{Q}. \tag{3.61}$$

The basic goal of Donalson theory is the computation of the generating functional (3.59) (or, in the simply-connected case, of the Donaldson series (3.60)). Many results have been obtained along the years for different four-manifolds (a good review is [27]). The major breakthrough in this sense was the structure theorem of Kronheimer and Mrowka [26] (see also [28]) for the Donaldson series of simply-connected four-manifolds with $b_2^+ > 1$ and of the so-called *Donaldson simple type*. A four-manifold is said to be of *Donaldson simple type* if

$$\left(\frac{\partial^2}{\partial p^2} - 4\right) Z_{DW}^{w_2(V)}(p, S) = 0, \tag{3.62}$$

for all choices of $w_2(V)$. When this holds, then, according to the results of Kronheimer and Mrowka, the Donaldson series has the following structure:

$$\mathbf{D}^{w_2(V)}(S) = \exp(S^2/2) \sum_{s=1}^{p} a_s \mathrm{e}^{(\kappa_s, S)}, \tag{3.63}$$

for finitely many homology classes $\kappa_1, \cdots, \kappa_p \in H_2(X, \mathbf{Z})$ and nonzero rational numbers $a_1, \cdots, a_p$. Furthermore, each of the classes κ_i is characteristic. The classes κ_i are called *Donaldson basic classes.*

A simple example of this situation is the $K3$ surface. In this case, the Donaldson-Witten generating functional is given by

$$Z_{DW}^{w_2(V)} = \frac{1}{2} \mathrm{e}^{2\pi i \lambda_0^2} \left(\mathrm{e}^{\frac{S^2}{2} + 2p} - i^{-w_2(V)^2} \mathrm{e}^{-\frac{S^2}{2} - 2p} \right). \tag{3.64}$$

In this expression, $2\lambda_0$ is a choice of an integer lifting of $w_2(E)$. The overall factor $\mathrm{e}^{2\pi i \lambda_0^2}$ gives a dependence on the choice a such a lifting, and this is due to the fact that the orientation of instanton moduli space depends on such a choice [7]. From the above expression one can deduce that for example for $w_2(V) = 0$, one has

$$\int I_2(S)^2 = (S, S), \quad \int I_2(S)^6 = \frac{1}{8}(S, S)^3, \tag{3.65}$$

and so on. Notice that in the first integral in (3.65) we integrate over the moduli space $\mathcal{M}_{\mathrm{ASD}}$ with instanton number $k = 2$, and in the second one we have $k = 6$. Notice that, according to (3.64), $K3$ is of simple type, and the Donalson series is simply given by:

$$\mathbf{D}^{w_2(V)} = \mathrm{e}^{S^2/2}, \tag{3.66}$$

which satisfies indeed the structure theorem of Kronheimer and Mrowka and shows that $K3$ has only one Donaldson basic class, namely $\kappa = 0$.

The main goal of these lectures is to give an expression for the Donaldson-Witten generating function (3.59) as explicit as possible. In particular, we want to derive the structure theorem (3.63) and be more precise about the unknown data κ_i, a_i. As Witten discovered in [2], these data involve some simpler invariants of four-manifolds, the Seiberg-Witten invariants, which will be the subject of the next section.

3.7. Metric dependence

Donaldson invariants turn out to be independent of the metric when $b_2^+ > 1$. This is closely related to the existence of reducible connections. Remember that a reducible connection gives a splitting of the $SU(2)$ bundle according to (3.32). In order for the reducible connection to solve the ASD equations, we have to solve the equation for an abelian instanton $F^+ = 0$. This means, in particular, that $c_1(L)$ can be represented by an ASD harmonic two-form, and in particular it belongs to the intersection

$$H^{2,-}(X, \mathbf{R}) \cap H^2(X, \mathbf{Z}). \tag{3.67}$$

For a generic metric, however, this intersection it just the zero element (unless $b_2^+ = 0$), since it is the intersection of an integer lattice in $H^2(X, \mathbf{R})$ with a proper subspace. Therefore, for a generic metric there are no reducible ASD connections if $b_2^+ > 0$.

In order to test metric dependence, however, we have to see what happens to the moduli space of ASD connections as we move along a generic one-dimensional family of metrics (more precisely, along a one-dimensional family of conformal classes of metrics, since the instanton equation is conformally invariant). If $b_2^+ = 1$, then on a generic one-dimensional family we can in fact find a reducible ASD connection [7]. This will provoke a singularity in the moduli space, and as consequence the Donaldson invariants will "jump." If, on the contrary, $b_2^+ > 1$, then we don't find reducible connections along generic one-dimensional families, and the Donaldson invariants will be truly metric-independent.

For $b_2^+ = 1$, the metric dependence can be described in more detail as follows [29]. Let X be a four-manifold of $b_2^+ = 1$. The dependence on the metric is through the so-called *period point* ω. The period point is defined as the harmonic two-form satisfying

$$*\omega = \omega, \qquad \omega^2 = 1. \tag{3.68}$$

Clearly, ω depends on the metric through the Hodge dual $*$. More precisely, it only depends on the conformal class of the metric (*i.e.* rescaled metrics $g \to tg$ give the same ω). When we vary the conformal class of the metric, we vary at the same time the period point ω in the space $H^2(X, \mathbf{R})$. As we vary the metric, the period point will describe a hypersurface in the cone

$$V_+ = \{\omega \in H^2(X, \mathbf{R}) : \omega^2 > 0\}. \tag{3.69}$$

In some cases, the period point ω can be described in a fairly concrete way. For example, if we take $X = \mathbf{S}^2 \times \Sigma_g$, then the general period point can be written as an element in the Kähler cone:

$$\omega = \frac{1}{\sqrt{2}}(\mathrm{e}^{\theta}[\mathbf{S}^2] + \mathrm{e}^{-\theta}[\Sigma_g]). \tag{3.70}$$

This period point ω describes a hyperbola in $H^2(X, \mathbf{R})$ parametrized by $-\infty < \theta < \infty$, and each point in this hyperbola corresponds to a choice of Kähler metric in $\mathbf{S}^2 \times \Sigma_g$. The limits $\theta \to \pm\infty$, correspond to limiting metrics which give a very small volume to $\mathbf{S}^2$ and Σ_g, respectively.

Imagine now that, on a manifold of $b_2^+ = 1$, we start varying the period point ω in such a way that, at a certain value, there exists a cohomology class $\zeta \in H^2(X)$ which satisfies $\zeta \equiv w_2(V)$ (mod 2) and

$$\zeta^2 < 0, \qquad (\zeta, \omega) = 0. \tag{3.71}$$

We then say that the element ζ defines a *wall* in V_+:

$$W_\zeta = \{\omega : (\zeta, \omega) = 0\}. \tag{3.72}$$

The complements of these walls are called *chambers*, and the cone V_+ is then divided in chambers separated by walls.

What is the meaning of these walls? If $\zeta \in H^2(X, \mathbf{Z})$ satisfies (3.71), then it is the first Chern class of a line bundle T which admits an ASD connection, since $\zeta^+ = (\zeta, \omega)\omega = 0$. The condition that ζ is congruent to $w_2(V)$ mod 2 is precisely the condition (3.36). Therefore, ζ is the first Chern class of a reducible solution which causes a singularity in moduli space: the Donaldson invariants jump when we pass through such a wall.

In summary, when $b_2^+ = 1$, the Donaldson invariants depend on the metric because they jump at walls, but they are metric independent *in each chamber*. We will represent by $Z_\pm^\zeta(p, S, \delta)$ the Donaldson-Witten generating function after and before passing the wall defined by ζ, respectively. One of the basic problems in Donaldson theory for manifolds of $b_2^+ = 1$ is to determine the jump in the generating function,

$$Z_+^\zeta(p, S, \delta) - Z_-^\zeta(p, S, \delta) = WC_\zeta(p, S, \delta), \tag{3.73}$$

which is usually called the *wall-crossing* term. It was conjectured by Kotschick and Morgan [30] that the wall-crossing only depends on the classical homology ring of the four-manifold.

Assuming the validity of this conjecture, Göttsche [31] was able to find a universal formula for $WC(\zeta)$ in the simply-connected case. As we will see, the u-plane integral of Moore and Witten [3] allows an explicit evaluation of the wall-crossing term that confirms the Kotschik-Morgan conjecture and reproduces Göttsche's formula. In the nonsimply connected case, only partial results are known mathematically [32]. Using the u-plane integral one can find a universal formula in the nonsimply connected case as well [25].

Exercise. *Walls for the Hirzebruch surface*

Consider the Hirzebruch surface $\mathbf{F}_1$, which can be regarded as the blowup of $\mathbf{P}^2$ at one point. Compute $H_2(\mathbf{F}_1)$, and write the period point in the Kähler cone. Find the walls.

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4. Basics of Seiberg-Witten invariants

One of the major breakthroughs in the study of four-manifolds came in 1994, when Witten showed that the mathematical problem that motivated the introduction of Donaldson invariants -the classification of four-manifolds up to diffeomorphism- can be studied successfully by looking at a much simpler set of equations, the so-called Seiberg-Witten monopole equations, which were introduced in [2]. Witten also showed, based on physical arguments, that Seiberg-Witten invariants contain all the information of Donaldson invariants, and that they provide the missing ingredient in the structure theorem of Kronheimer and Mrowka. In this section we will give a brief review of Seiberg-Witten invariants and their basic properties. A more detailed mathematical treatment can be found in [2][14][6][4]. The book [33] is primarily mathematical, but it also covers part of the physics. From a physicist' point of view, one should look at [1][34][35].

4.1. The Seiberg-Witten equations

Let X be an oriented, closed, Riemannian four-manifold, and let L be the determinant line bundle of a Spin_c structure. Positive chirality spinors, *i.e.* sections of $S^+ \otimes L^{1/2}$, will be denoted by M. The SW monopole equations are moduli equations for a pair (A, M) given by a $U(1)$ connection on L and a spinor M. Moduli equations for pairs consisting of a gauge connection and a section of some bundle have been considered in the past. One of the best known examples are the Hitchin equations [36]. Although the rationale for introducing the SW equations for pairs (A, M) comes from the analysis of a twisted $\mathcal{N} = 2$

supersymmetric theory, one can just write down the equations and explore their properties and the invariants that they define. This is what we are going to do in this section.

A section of the spinor bundle $M_{\dot\alpha}$ transforms in the **2** of $SU(2)_+$. On the other hand, F^+ is a SD form and it transforms in the **3**. In order to couple them, we form a symmetric tensor in the **3** out of M_α by considering

$$\overline{M}_{(\dot\alpha} M_{\dot\beta)} = \begin{pmatrix} -M_1 M_2^* & \frac{1}{2}(|M_1|^2 - |M_2|^2) \\ \frac{1}{2}(|M_1|^2 - |M_2|^2) & M_1^* M_2 \end{pmatrix}. \tag{4.1}$$

The Seiberg-Witten equations are:

$$\begin{aligned} F^+_{\dot\alpha\dot\beta} + 4i\overline{M}_{(\dot\alpha} M_{\dot\beta)} =& 0, \\ \nabla_L^{\alpha\dot\alpha} M_{\dot\alpha} =& 0, \end{aligned} \tag{4.2}$$

where $F^+_{\dot\alpha\dot\beta} = \overline{\sigma}^{\mu\nu}_{\dot\alpha\dot\beta} F^+_{\mu\nu}$, F is the curvature of the $U(1)$ connection on L, and ∇_L is the Dirac operator for the bundle $S^+ \otimes L^{1/2}$. Using (4.1) and the explicit form of $\overline{\sigma}^{\mu\nu}_{\dot\alpha\dot\beta}$ given in Appendix A, we can write the first equation as:

$$\begin{aligned} \frac{1}{2}(F_{12} + F_{34}) =& |M_1|^2 - |M_2|^2, \\ \frac{1}{2}(F_{13} + F_{42}) =& i(M_1 M_2^* - M_1^* M_2), \\ \frac{1}{2}(F_{14} + F_{23}) =& M_1 M_2^* + M_1^* M_2. \end{aligned} \tag{4.3}$$

4.2. The Seiberg-Witten invariants

The procedure to define the Seiberg-Witten invariants is very similar to the procedure followed in Donaldson theory. First, one has to construct the moduli space of solutions to the equations (4.2). We will be content with a local analysis, as we did in the case of $\mathcal{M}_{\rm ASD}$.

In the case of the Seiberg-Witten equations, the configuration space is $\mathcal{C} = \mathcal{A} \times \Gamma(X, S^+ \otimes L^{1/2})$, where $\mathcal{A}$ is the moduli space of $U(1)$-connections on L. The group $\mathcal{G}$ of gauge transformations of the bundle L acts on this configuration space according to (3.4):

$$\begin{aligned} g^*(A_\mu) &= A_\mu + i\partial_\mu \log g, \\ g^*(M) &= g^{1/2} M, \end{aligned} \tag{4.4}$$

where $M \in \Gamma(X, S^+ \otimes L^{1/2})$ and g takes values in $U(1)$. The infinitesimal form of these transformations becomes, after putting $g = \exp(i\phi)$:

$$\begin{aligned} \delta A &= -d\phi, \\ \delta M &= \frac{i}{2}\phi M. \end{aligned} \tag{4.5}$$

The moduli space of solutions to the monopole equations, modulo gauge transformations, will be denoted by $\mathcal{M}_{\rm SW}$. The tangent space to the configuration space at the point (A,M) is just $T_{(A,M)}\mathcal{C} = T_A\mathcal{A} \oplus T_M\Gamma(X, S^+ \otimes L^{1/2}) = \Omega^1(X) \oplus \Gamma(X, S^+ \otimes L^{1/2})$, since $\Gamma(X, S^+ \otimes L^{1/2})$ is a vector space.

A first step to understand the structure of the moduli space of solutions to the monopole equations modulo gauge transformations is to construct a slice of the gauge action, as we did in Donaldson theory. For this we need an explicit construction of the gauge orbits, given by the map $\mathcal{G} \times \mathcal{C} \to \mathcal{C}$. The tangent space to these orbits is the map

$$C : {\rm Lie}(\mathcal{G}) \longrightarrow T\mathcal{C}. \tag{4.6}$$

whose explicit expression in local coordinates can be obtained from (4.5):

$$C(\phi) = (-d\phi, \frac{i}{2}\phi M) \in \Omega^1(X) \oplus \Gamma(X, S^+ \otimes L^{1/2}), \quad \phi \in \Omega^0(X). \tag{4.7}$$

The local model of the moduli space is then given by the zero locus in ker $C^\dagger$ of the following map $s : \mathcal{C} \to \mathcal{F}$, where $\mathcal{F} = \Omega^{2,+}(X) \oplus \Gamma(X, S^- \otimes L^{1/2})$:

$$s(A,M) = \Big(F^+_{\dot\alpha\dot\beta} + 4i\overline{M}_{(\dot\alpha} M_{\dot\beta)}, \nabla_L^{\alpha\dot\alpha} M_{\dot\alpha} \Big). \tag{4.8}$$

As in Donaldson theory, we study the linearization of this map, $ds : T_{(A,M)}\mathcal{C} \to \mathcal{F}$. The explicit expression is:

$$ds(\psi,\mu) = \Big((p^+(d\psi))_{\dot\alpha\dot\beta} + 4i\big(\overline{M}_{(\dot\alpha}\mu_{\dot\beta)} + \bar\mu_{(\dot\alpha}M_{\dot\beta)}\big), \nabla_L^{\alpha\dot\alpha}\mu_{\dot\alpha} + \frac{i}{2}\psi^{\alpha\dot\alpha}M_{\dot\alpha} \Big), \tag{4.9}$$

where p^+ is the projector on SD forms, and we have written the connection and spinor as $A+\psi$, $M+\mu$. Instead of studying the restriction of this map to ker $C^\dagger$, we can consider the instanton deformation complex:

$$0 \to \Omega^0(X) \stackrel{C}{\to} \Omega^1(X) \oplus \Gamma(X, S^+ \otimes L^{1/2}) \stackrel{ds}{\to} \Omega^{2,+}(X) \oplus \Gamma(X, S^- \otimes L^{1/2}) \to 0. \tag{4.10}$$

which encodes the local model of the Seiberg-Witten moduli space and its virtual dimension.

Exercise. *Dimension of Seiberg-Witten moduli space*

1) Verify that (4.10) is in fact a complex.

2) Using the index theorem, compute the virtual dimension of Seiberg-Witten moduli space. You should find:

$$\dim \mathcal{M}_{\rm SW} = \frac{1}{4}\big(c_1(L)^2 - 2\chi - 3\sigma\big). \tag{4.11}$$

It is convenient to denote

$$\lambda = \frac{1}{2} c_1(L), \tag{4.12}$$

which is in general not an integer class. Since $c_1(L) \equiv w_2(X) \bmod 2$ (it is characteristic), λ is an element in the lattice

$$H_2(X, \mathbf{Z}) + \frac{1}{2} w_2(X). \tag{4.13}$$

From now on, we will use λ to specify the topological class of the determinant line bundle involved in the SW monopole equations. In particular, the dimension of the moduli space (4.11) will be denoted by

$$d_\lambda = \lambda^2 - \frac{2\chi + 3\sigma}{4}. \tag{4.14}$$

Notice that the index of the Dirac operator for the Spin_c structure specified by λ is $-\sigma/4 + \lambda^2$, and has to be an integer. Then, d_λ will be an integer if an only if

$$\Delta = \frac{\chi + \sigma}{4} \tag{4.15}$$

is an integer number. Therefore, on manifolds where Δ is not an integer, one can not define the Seiberg-Witten invariants. There are many classes of manifolds for which Δ is always an integer. For example, on almost complex manifolds one has:

$$\Delta = 1 - h^{1,0} + h^{2,0}, \tag{4.16}$$

where $h^{p,q}$ is the dimension of $H^{p,q}(X)$.

Exercise. *More practice with deformation complexes*

Consider the following moduli equations:

$$\begin{aligned} &F^+_{\mu\nu} - \frac{i}{2}[B^+_{\mu\tau}, B^{+\tau}_{\nu}] - \frac{i}{\sqrt{2}}[B^+_{\mu\nu}, C] = 0, \\ &D_\mu C + \sqrt{2} D^\nu B^+_{\nu\mu} = 0, \end{aligned} \tag{4.17}$$

where C is a scalar field and $B^+_{\mu\nu}$ is a self-dual two form, and both of them are valued in the Lie algebra of $SU(N)$. These are the so-called *Vafa-Witten equations* [37] (see also [38][39]). Work out the deformation complex and compute the virtual dimension of moduli space.

We then have a local model for the moduli space of solutions to the SW moduli equations, which has virtual dimension (4.11). A more detailed analysis shows that this moduli space, in contrast to the ASD moduli space, is *compact* [2]. This makes life much easier!

The second step in defining the invariants is to construct a universal bundle, as in Donaldson theory. The procedure is very similar. Let $P_{U(1)}$ be the principal $U(1)$ bundle associated to the determinant line bundle $L \to X$ of a Spin_c structure on X. We will denote by $\mathcal{M}^* \subset \mathcal{M} = \mathcal{A} \times \Gamma(X, S^+ \otimes L^{1/2})$ the subspace of irreducible pairs in the configuration space. On the space

$$\mathcal{M}^* \times P_{U(1)}, \tag{4.18}$$

there is an action of the gauge group $\mathcal{G} = \mathrm{Map}(X, U(1))$. The quotient

$$\mathbf{P}_{U(1)} = \mathcal{M}^* \times_{\mathcal{G}} P_{U(1)} \tag{4.19}$$

is a $U(1)$-bundle over $(\mathcal{M}^*/\mathcal{G}) \times X$. This is the universal bundle associated to $P_{U(1)}$ for the SW moduli problem. The first Chern class of this bundle is a closed two-form on the base $(\mathcal{M}^*/\mathcal{G}) \times X$,

$$c_1(\mathbf{P}_{U(1)}) = \frac{1}{2\pi}\mathcal{F} \tag{4.20}$$

and therefore gives an element in $H^*(\mathcal{M}^*/\mathcal{G}) \otimes H^*(X)$. The analog of the Donaldson map is now,

$$\mu : H_i(X) \longrightarrow H^{2-i}(\mathcal{M}^*/\mathcal{G}). \tag{4.21}$$

The image of a point in X gives, after restriction to $\mathcal{M}_{\mathrm{SW}}$,

$$\phi \in H^2(\mathcal{M}_{\mathrm{SW}}). \tag{4.22}$$

Given a basis of one-cycles $\delta_1, \cdots, \delta_r \in H_1(X, \mathbf{Z})$ with duals $\beta_1, \cdots, \beta_r \in H^1(X, \mathbf{Z})$, we define the following one-forms:

$$\nu_i = \mu(\delta_i), \quad i = 1, \cdots, b_1, \tag{4.23}$$

restricted again to $\mathcal{M}_{SW}$. We can now define the Seiberg-Witten invariant associated to a Spin^c-structure specified by $\lambda \in H^2(X, \mathbf{Z}) + \frac{1}{2}w_2(X)$,

$$SW(\lambda, \beta_{i_1} \wedge \cdots \wedge \beta_{i_r}) = \int_{\mathcal{M}_\lambda} \nu_{i_1} \wedge \cdots \wedge \nu_{i_r} \wedge \phi^{\frac{1}{2}(d_\lambda - r)}. \tag{4.24}$$

Clearly, $d_\lambda - r$ has to be even, otherwise the invariant is zero. Also, nontriviality of the invariants requires Δ (defined in (4.15)) to be an integer. Usually the SW invariants are defined with no insertion of one-classes. The extension that we are considering here has been studied for example in [40].

The class λ is called a *Seiberg-Witten basic class* if the map $SW(\lambda) : \wedge^* H^1(X, \mathbf{Z}) \to \mathbf{Z}$ defined by (4.24) is not identically zero. In the simply-connected case, a basic class is just a class λ such that $SW(\lambda) \neq 0$. We will also say that a manifold X is of *Seiberg-Witten simple type* if all basic classes have $d_\lambda = 0$. This means, in particular, that $SW(\lambda, \beta_1 \wedge \cdots \wedge \beta_r) = 0$ for $r > 0$. All known simply-connected manifolds with $b_2^+ > 1$ are of SW simple type. Finally, we note the following property of the SW invariants [2]:

$$SW(-\lambda, \beta_{i_1} \wedge \cdots \beta_{i_r}) = (-1)^\Delta SW(\lambda, \beta_{i_1} \wedge \cdots \wedge \beta_{i_r}). \tag{4.25}$$

4.3. Metric dependence

Like Donaldson invariants, Seiberg-Witten invariants exhibit metric dependence when $b_2^+ = 1$. The reason is also very similar. If we look at (4.4), we see that reducible pairs (*i.e.* pairs (A, M) with a nontrivial isotropy group) must have $M = 0$. Therefore, reducible pairs that satisfy the SW monopole equations are just abelian instantons $F^+ = 0$. This means that, if $b_2^+ = 1$, as we move along a one-dimensional family we will find a reducible solution to the SW moduli equations, and therefore the value of the SW invariants will change. Like in Donaldson theory, the metric dependence has again a structure of chambers and walls in the cone V_+. However, the walls are defined in this case by the conditions:

$$2\lambda \equiv w_2(X) \bmod 2, \quad \lambda^2 < 0, \quad (\lambda, \omega) = 0, \quad d_\lambda \geq 0, \tag{4.26}$$

where 2λ is the first Chern class of the determinant line bundle of a Spin_c structure. The third condition in (4.26) tells us that this line bundle provides a reducible solution to the Seiberg-Witten equations: $M = 0$, $F^+ = 0$, and this causes a jump in the value of the SW invariant, so we have a wall-crossing term:

$$SW_+(\lambda, \beta_1 \wedge \cdots \beta_r) - SW_-(\lambda, \beta_1 \wedge \cdots \beta_r) = WC_\lambda(\beta_1 \wedge \cdots \beta_r). \tag{4.27}$$

In this case, life is much simpler than in Donaldson theory. It is not very difficult to show that, when X is simply-connected [2][41]:

$$SW_+(\lambda) - SW_-(\lambda) = 1. \tag{4.28}$$

The case in which X is not simply-connected has also been worked out [42][40]. As we will see at the end of these lectures, their general wall-crossing formula for the SW invariants in the nonsimpy connected case can also be obtained by using the u-plane integral [25].

5. $\mathcal{N}=1$ supersymmetry

In this section, we give some useful background on supersymmetry. Since our motivation is the construction of topological field theories, our presentation will be rather sketchy. The standard reference is [43]. A very useful and compact presentation can be found in the excellent review by Álvarez-Gaumé and Hassan [44], which is the main source for this very quick review. We follow strictly the conventions of [44], which are essentially those in [43], although there are some important differences. Some of these conventions can be found in Appendix A.

5.1. The supersymmetry algebra

Supersymmetry is the only nontrivial extension of Poincaré symmetry which is compatible with the general principles of relativistic quantum field theory. In $\mathbf{R}^{1,3}$ one introduces $\mathcal{N}$ fermionic generators

$$Q_u = \begin{pmatrix} Q_{\alpha u} \\ \overline{Q}^{\dot\alpha u} \end{pmatrix} \tag{5.1}$$

where $u = 1, \cdots, \mathcal{N}$. The superPoincaré algebra extends the usual Poincaré algebra, and the (anti)commutators of the fermionic generators are:

$$\begin{aligned}
&\{Q_{\alpha u}, \overline{Q}_{\dot\beta v}\} = 2\epsilon_{uv}\sigma^{\mu}_{\alpha\dot\beta}P_\mu && \{Q_{\alpha u}, Q_{\beta v}\} = 2\sqrt{2}\epsilon_{\alpha\beta}Z_{uv} \\
&[P_\mu, Q_{\alpha v}] = 0 && [P_\mu, \overline{Q}_{\dot\alpha u}] = 0 \\
&[M_{\mu\nu}, Q_{\alpha u}] = -(\sigma_{\mu\nu})_\alpha{}^\beta Q_{\beta u} && [M_{\mu\nu}, \overline{Q}^{\dot\alpha u}] = -(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\overline{Q}^{\dot\beta u}
\end{aligned} \tag{5.2}$$

where $u, v = 1, \cdots, \mathcal{N}$, and $M_{\mu\nu}$ are the generators of the Lorentz group $SO(4) \simeq SU(2)_+ \times SU(2)_-$. The terms Z_{uv} are the so-called central charges. They satisfy

$$Z_{uv} = -Z_{vu} \tag{5.3}$$

and they commute with all the generators of the algebra.

When the central charges vanish, the theory has an internal $U(\mathcal{N})_R$ symmetry:

$$Q_{\alpha v} \to U_v{}^w Q_{\alpha w} \qquad \bar{Q}^{\,v}_{\dot\alpha} \to U^{*w}_{v}\bar{Q}_{\dot\alpha}{}^{w}. \tag{5.4}$$

The generators of this symmetry will be denoted by B_a, and their commutation relations with the fermionic supercharges are:

$$[Q_{\alpha v}, B_a] = (b_a)_v{}^w Q_{\alpha w} \qquad [\overline{Q}_{\dot\alpha}{}^{w}, B_a] = -\overline{Q}_{\dot\alpha}{}^{v}(b_a)_v{}^w \tag{5.5}$$

where $b_a = b_a^+$. The central charges are linear combinations of the $U(N)$ generators

$$Z_{uv} = d_{uv}{}^a B_a. \tag{5.6}$$

If the central charges are not zero, the internal symmetry gets reduced to USp($\mathcal{N}$), formed by the unitary transformations that leave invariant the 2-form (5.6) in $\mathcal{N}$ dimensions. The $U(1)_R$ of the internal symmetry (5.4), with generator R, gives a chiral symmetry of the theory,

$$[Q_{\alpha v}, R] = Q_{\alpha v}, \qquad [\overline{Q}^{\,w}_{\dot\alpha}, R] = -\overline{Q}^{\,v}_{\dot\alpha}. \tag{5.7}$$

This symmetry is typically anomalous, quantum-mechanically, and the quantum effects break it down to a discrete subgroup. We will see a explicit realization of this in $\mathcal{N} = 2$ super Yang-Mills theory.

5.2. $\mathcal{N} = 1$ superspace and superfields

In order to find a local realization of supersymmetry, one has to extend the usual Minkowski space to the so-called *superspace*. In this section we are going to develop the basics of $\mathcal{N} = 1$ superspace, which is extremely useful to formulate supersymmetric field multiplets and supersymmetric Lagrangians. Therefore, we are going to construct a local realization of the supersymmetry algebra (5.2) when we have two supercharges Q_α, $\bar{Q}^{\dot\alpha}$.

The superspace is obtained by adding four spinor coordinates $\theta^\alpha, \bar\theta_{\dot\alpha}$ to the four space-time coordinates x^μ. The generator of supersymmetric transformations in superspace is

$$-i\xi^\alpha Q_\alpha - i\bar\xi_{\dot\alpha}\bar{Q}^{\dot\alpha} \tag{5.8}$$

where ξ^α, $\bar\xi_{\dot\alpha}$ are (fermionic) transformation parameters[1]. Under this generator, the superspace coordinates transform as

$$\begin{aligned} x^\mu \to & x'^\mu = x^\mu + i\theta\sigma^\mu\bar\xi - i\xi\sigma^\mu\bar\theta \\ \theta \to & \theta' = \theta + \xi\,, \\ \bar\theta \to & \bar\theta' = \bar\theta + \bar\xi\,. \end{aligned} \tag{5.9}$$

The representation of the supercharges acting on the superspace is then given by

$$Q_\alpha = i\left(\frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\,\partial_\mu\right), \qquad \overline{Q}_{\dot\alpha} = -i\left(\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\,\partial_\mu\right) \tag{5.10}$$

[1] This is the only case in which we do not follow the conventions of [44]: their susy charges are $-i$ times ours.

and they satisfy $\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = -2i\sigma^\mu_{\alpha\dot\alpha}\,\partial_\mu$. Since $P_\mu = -i\partial_\mu$, this gives a representation of the supersymmetry algebra. It is also convenient to introduce the super-covariant derivatives

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot\alpha}\,\partial_\mu\,, \qquad \bar{D}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\sigma^\mu_{\alpha\dot\alpha}\theta^\alpha\,\partial_\mu\,, \tag{5.11}$$

which satisfy $\{D_\alpha, \bar{D}_{\dot\alpha}\} = -2i\sigma^\mu_{\alpha\dot\alpha}\,\partial_\mu$ and commute with Q and $\bar{Q}$.

A *superfield* is just a function on the superspace $F(x,\theta,\bar\theta)$. Since the θ-coordinates are anti-commuting, the Taylor expansion in the fermionic coordinates truncates after a finite number of terms. Therefore, the most general $\mathcal{N}=1$ superfield can always be expanded as

$$\begin{aligned} F(x,\theta,\bar\theta) =& f(x) + \theta\phi(x) + \bar\theta\bar\chi(x) + \theta\theta m(x) + \bar\theta\bar\theta n(x) + \theta\sigma^\mu\bar\theta v_\mu(x) \\ &+ \theta\theta\bar\theta\bar\lambda(x) + \bar\theta\bar\theta\theta\psi(x) + \theta\theta\bar\theta\bar\theta d(x)\,. \end{aligned} \tag{5.12}$$

Under a supersymmetry transformation (5.8), the superfield transforms as $\delta F = (\xi Q + \bar\xi\bar{Q})F$, and from this expression one can obtain the transformation of the components.

The generic superfield gives a reducible representation of the supersymmetry algebra. Therefore, in order to obtain irreducible representations one must impose constraints. There are two different $\mathcal{N}=1$ irreducible supermultiplets:

a) Chiral multiplet: The $\mathcal{N}=1$ scalar multiplet is a superfield which satisfies the following constraint:

$$\bar{D}_{\dot\alpha}\Phi = 0 \tag{5.13}$$

and it is called the chiral superfield. The constraint can be easily solved by noting that, if $y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta$, then

$$\bar{D}_{\dot\alpha}y^\mu = 0, \quad \bar{D}_{\dot\alpha}\theta^\beta = 0\,. \tag{5.14}$$

Therefore, any function of (y,θ) is a chiral superfield. We can then write

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta^\alpha\psi_\alpha(y) + \theta^2 F(y)\,, \tag{5.15}$$

and we see that a chiral superfield contains two complex scalar fields, ϕ and F, and a Weyl spinor ψ_α. In a similar way we can define an anti-chiral superfield by $D_\alpha\Phi^\dagger = 0$, which can be expanded as

$$\Phi^\dagger(y^\dagger,\bar\theta) = \phi^\dagger(y^\dagger) + \sqrt{2}\bar\theta\bar\psi(y^\dagger) + \bar\theta^2 F^\dagger(y^\dagger)\,, \tag{5.16}$$

where, $y^{\mu\dagger} = x^\mu - i\theta\sigma^\mu\bar\theta$.

Exercise.

Show that, in terms of the original variables, Φ and $\Phi^\dagger$ take the form

$$\begin{aligned}\Phi(x,\theta,\bar\theta) &= A(x) + i\theta\sigma^\mu\bar\theta\partial_\mu A - \frac{1}{4}\theta^2\bar\theta^2\nabla^2 A + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi\sigma^\mu\bar\theta + \theta\theta F(x)\,,\\ \Phi^\dagger(x,\theta,\bar\theta) &= A^\dagger(x) - i\theta\sigma^\mu\bar\theta\partial_\mu A^\dagger - \frac{1}{4}\theta^2\bar\theta^2\nabla^2 A^\dagger + \sqrt{2}\bar\theta\bar\psi(x) + \frac{i}{\sqrt{2}}\bar\theta\bar\theta\,\theta\sigma^\mu\,\partial_\mu\bar\psi + \bar\theta\bar\theta F^\dagger(x)\,.\end{aligned} \tag{5.17}$$

b) Vector Multiplet: this is a real superfield satifying $V = V^\dagger$. In components, it takes the form

$$\begin{aligned}V(x,\theta,\bar\theta) &= C + i\theta\chi - i\bar\theta\bar\chi + \frac{i}{2}\theta^2(M+iN) - \frac{i}{2}\bar\theta^2(M-iN) - \theta\sigma^\mu\bar\theta A_\mu\\ &+ i\theta^2\bar\theta(\bar\lambda + \frac{i}{2}\bar\sigma^\mu\partial_\mu\chi) - i\bar\theta^2\theta(\lambda + \frac{i}{2}\sigma^\mu\partial_\mu\bar\chi) + \frac{1}{2}\theta^2\bar\theta^2(D - \frac{1}{2}\nabla^2 C)\,.\end{aligned} \tag{5.18}$$

By performing an abelian gauge transformation $V \to V + \Lambda + \Lambda^\dagger$, where Λ ($\Lambda^\dagger$) are chiral (antichiral) superfields, one can set $C = M = N = \chi = 0$. This is the so called Wess-Zumino gauge, where

$$V = -\theta\sigma^\mu\bar\theta A_\mu + i\theta^2\bar\theta\bar\lambda - i\bar\theta^2\theta\lambda + \frac{1}{2}\theta^2\bar\theta^2 D\,. \tag{5.19}$$

In this gauge, $V^2 = \frac{1}{2}A_\mu A^\mu\theta^2\bar\theta^2$ and $V^3 = 0$. The Wess-Zumino gauge breaks supersymmetry, but not the gauge symmetry of the abelian gauge field A_μ. The Abelian field strength is defined by

$$W_\alpha = -\frac{1}{4}\bar D^2 D_\alpha V\,, \qquad \bar W_{\dot\alpha} = -\frac{1}{4}D^2\bar D_{\dot\alpha}V\,,$$

and W_α is a chiral superfield. In the Wess-Zumino gauge it takes the form

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D - \frac{i}{2}(\sigma^\mu\bar\sigma^\nu\theta)_\alpha\, F_{\mu\nu} + \theta^2(\sigma^\mu\partial_\mu\bar\lambda)_\alpha\,. \tag{5.20}$$

The non-Abelian case is similar: V is in the adjoint representation of the gauge group, $V = V_A T^A$, and the gauge transformations are

$$e^{-2V} \to e^{-i\Lambda^\dagger}e^{-2V}e^{i\Lambda}$$

where $\Lambda = \Lambda_A T^A$. The non-Abelian gauge field strength is defined by

$$W_\alpha = \frac{1}{8}\bar D^2 e^{2V} D_\alpha e^{-2V}$$

and transforms as

$$W_\alpha \to W'_\alpha = e^{-i\Lambda} W_\alpha e^{i\Lambda} \,.$$

In components, it takes the form

$$W_\alpha = T^a \left(-i\lambda^a_\alpha + \theta_\alpha D^a - \frac{i}{2}(\sigma^\mu \bar{\sigma}^\nu \theta)_\alpha F^a_{\mu\nu} + \theta^2 \sigma^\mu D_\mu \bar{\lambda}^a \right) \tag{5.21}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu \,, \qquad D_\mu \bar{\lambda}^a = \partial_\mu \bar{\lambda}^a + f^{abc} A^b_\mu \bar{\lambda}^c \,.$$

5.3. Construction of $\mathcal{N} = 1$ Lagrangians

In the previous subsection we have constructed supermultiplets of $\mathcal{N} = 1$ supersymmetry. The next step is to construct manifestly supersymmetric Lagrangians. Again, this is easily done in superspace.

The most general $\mathcal{N} = 1$ supersymmetric Lagrangian for the scalar multiplet (including the interaction terms) is given by

$$\mathcal{L} = \int d^4\theta \, K(\Phi, \Phi^\dagger) + \int d^2\theta \mathcal{W}(\Phi) + \int d^2\bar{\theta} \bar{\mathcal{W}}(\Phi^\dagger) \,. \tag{5.22}$$

We are following here the usual rules of Grasmmannian integration, and the θ-integrals pick up the highest component of the superfield. In our conventions, $\int d^2\theta\theta^2 = 1$ and $\int d^2\bar{\theta}\bar{\theta}^2 = 1$. The kinetic term for the scalar fields A_i has the form

$$g^{ij} \partial_\mu A_i \partial^\mu A^\dagger_j \tag{5.23}$$

where

$$g^{ij} = \frac{\partial^2 K}{\partial A_i \partial A^\dagger_j} \tag{5.24}$$

is in general a nontrivial metric for the space of fields Φ. This has the form of a Kähler metric derived from a Kähler potential $K(A_i, A^\dagger_j)$. For this reason, the function $K(\Phi, \Phi^\dagger)$ is referred to as the *Kähler potential.* The simplest Kähler potential, corresponding to the flat metric, is

$$K(\Phi, \Phi^\dagger) = \sum_{i=1} \Phi^\dagger_i \Phi_i$$

which gives the free Lagrangian for a massless scalar and a massless fermion with an auxiliary field which can be eliminated by its equation of motion:

$$\mathcal{L} = \sum_i \Phi^\dagger_i \Phi_i \,|_{\theta^2\bar{\theta}^2} = \partial_\mu A^\dagger_i \partial^\mu A_i + F^\dagger_i F_i - i\bar{\psi}_i \bar{\sigma}^\mu \partial_\mu \psi_i \,. \tag{5.25}$$

Exercise.

Show that

$$\begin{aligned}\Phi_i^\dagger \Phi_j \mid_{\theta^2\bar{\theta}^2} =& -\frac{1}{4} A_i^\dagger \nabla^2 A_j - \frac{1}{4}\nabla^2 A_i^\dagger A_j + F_i^\dagger F_j + \frac{1}{2}\partial_\mu A_i^\dagger \partial^\mu A_j \\ & - \frac{i}{2}\psi_j \sigma^\mu \partial_\mu \bar{\psi}_i + \frac{i}{2}\partial_\mu \psi_j \sigma^\mu \bar{\psi}_i \,.\end{aligned} \tag{5.26}$$

and from this derive (5.25).

The function $W(\Phi)$ in (5.22) is an arbitrary holomorphic function of chiral superfields, and it is called the *superpotential*. It can be expanded as,

$$\begin{aligned}\mathcal{W}(\Phi_i) =& \mathcal{W}(A_i + \sqrt{2}\theta\psi_i + \theta\theta F_i) \\ =& \mathcal{W}(A_i) + \frac{\partial \mathcal{W}}{\partial A_i}\sqrt{2}\theta\psi_i + \theta\theta\left(\frac{\partial \mathcal{W}}{\partial A_i}F_i - \frac{1}{2}\frac{\partial^2 \mathcal{W}}{\partial A_i A_j}\psi_i\psi_j\right) .\end{aligned} \tag{5.27}$$

Supersymmetric interaction terms can be constructed in terms of the superpotential and its conjugate. Finally, we have to mention that there is $U(1)_R$ symmetry that acts as follows:

$$\begin{aligned}R\,\Phi(x,\theta) = \Phi'(x,\theta) =& e^{2in\alpha}\Phi(x, e^{-i\alpha}\theta)\,, \\ R\,\Phi^\dagger(x,\bar{\theta}) = \Phi'^\dagger(x,\theta) =& e^{-2in\alpha}\Phi^\dagger(x, e^{i\alpha}\bar{\theta})\,.\end{aligned} \tag{5.28}$$

Under this, the component fields transform as

$$\begin{aligned}& A \to e^{2in\alpha} A\,, \\ & \psi \to e^{2i(n-1/2)\alpha}\psi\,, \\ & F \to e^{2i(n-1)\alpha} F\,.\end{aligned} \tag{5.29}$$

Let us now present the Lagrangian for vector superfields. The super Yang-Mills Lagrangian with a θ-term can be written as

$$\begin{aligned}\mathcal{L} =& \frac{1}{8\pi}\mathrm{Im}\left(\tau\,\mathrm{Tr}\int d^2\theta\, W^\alpha W_\alpha\right) \\ =& -\frac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} + \frac{\theta}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{a\mu\nu} + \frac{1}{g^2}(\frac{1}{2}D^a D^a - i\lambda^a\sigma^\mu D_\mu \bar{\lambda}^a)\,,\end{aligned} \tag{5.30}$$

where $\tau = \theta/2\pi + 4\pi i/g^2$.

Exercise.

Using the normalization $\mathrm{Tr}T^aT^b = \delta^{ab}$, show that

$$\mathrm{Tr}(W^\alpha W_\alpha \mid_{\theta\theta}) = -2i\lambda^a\sigma^\mu D_\mu\bar{\lambda}^a + D^aD^a - \frac{1}{2}F^{a\mu\nu}F^a_{\mu\nu} + \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}\,, \tag{5.31}$$

and from here derive (5.30).

Now we can present the general Lagrangian that describes chiral multiplets coupled to a gauge field. Let the chiral superfields Φ_i belong to a given representation of the gauge group in which the generators are the matrices T^a_{ij}. The kinetic energy term $\Phi_i^\dagger \Phi_i$ is invariant under global gauge transformations $\Phi' = e^{-i\Lambda}\Phi$. In the local case, to insure that Φ' remains a chiral superfield, Λ has to be a chiral superfield. The supersymmetric gauge invariant kinetic energy term is then given by $\Phi^\dagger e^{-2V}\Phi$. The full $\mathcal{N}=1$ supersymmetric Lagrangian is

$$\mathcal{L} = \frac{1}{8\pi}\,\mathrm{Im}\left(\tau \mathrm{Tr}\int d^\theta\, W^\alpha W_\alpha\right) + \int d^2\theta d^2\bar\theta\, \Phi^\dagger e^{-2V}\Phi + \int d^2\theta\, \mathcal{W} + \int d^2\bar\theta\, \bar{\mathcal{W}}\,. \tag{5.32}$$

Exercise.

Expand (5.32) in components to obtain

$$\begin{aligned}\mathcal{L} =& -\frac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} + \frac{\theta}{32\pi^2}F^a_{\mu\nu}\widetilde{F}^{a\mu\nu} - \frac{i}{g^2}\,\lambda^a\sigma^\mu D_\mu\bar\lambda^a + \frac{1}{2g^2}D^aD^a\\ &+(\partial_\mu A - iA^a_\mu T^a A)^\dagger(\partial^\mu A - iA^{a\mu}T^a A) - i\,\bar\psi\bar\sigma^\mu(\partial_\mu\psi - iA^a_\mu T^a\psi)\\ &- D^a A^\dagger T^a A - i\sqrt{2}\,A^\dagger T^a\lambda^a\psi + i\sqrt{2}\,\bar\psi T^a A\bar\lambda^a + F_i^\dagger F_i\\ &+\frac{\partial\mathcal{W}}{\partial A_i}F_i + \frac{\partial\bar{\mathcal{W}}}{\partial A_i^\dagger}F_i^\dagger - \frac{1}{2}\frac{\partial^2\mathcal{W}}{\partial A_i\partial A_j}\psi_i\psi_j - \frac{1}{2}\frac{\partial^2\bar{\mathcal{W}}}{\partial A_i^\dagger\partial A_j^\dagger}\bar\psi_i\bar\psi_j\,.\end{aligned} \tag{5.33}$$

In (5.33), the auxiliary fields F and D^a can be eliminated by using their equations of motion. The terms involving these fields, thus, give rise to the scalar potential

$$V = \sum_i\left|\frac{\partial\mathcal{W}}{\partial A_i}\right|^2 - \frac{1}{2}g^2(A^\dagger T^a A)^2\,. \tag{5.34}$$

6. $\mathcal{N}=2$ supersymmetry

To construct topological field theories in four dimensions, we are actually interested in models with two supersymmetries (*i.e.* with eight supercharges). In this section we will present $\mathcal{N}=2$ supersymmetric gauge theories (with and without matter) in some detail, following the conventions in [3][45]. We will use the $\mathcal{N}=1$ superspace formalism of the previous section as our starting point, and then we will write the supersymmetric transformations in a manifest $\mathcal{N}=2$ supersymmetric way.

6.1. $\mathcal{N}=2$ Yang-Mills theory

$\mathcal{N}=2$ Yang-Mills theory contains a real superfields V, and a massless chiral superfield Φ, both in the adjoint representation of the gauge group G. The action, written in $\mathcal{N}=1$ superspace, is a particular case of the general action (5.32) with $W=0$, and it reads:

$$\int d^4xd^2\theta\mathrm{Tr}(W^2)+\int d^4xd^2\bar{\theta}\mathrm{Tr}(W^{\dagger 2})+\int d^4xd^2\theta d^2\bar{\theta}\mathrm{Tr}(\Phi^{\dagger k}\mathrm{e}^{2V_{kl}}\Phi^l). \tag{6.1}$$

In this equation, $V_{kl}=T^a{}_{kl}V^a$, where T^a is a Hermitian basis for the Lie algebra in the adjoint representation, and the real superfield is in the WZ gauge, with components A_μ, $\lambda^1{}_\alpha=\lambda_{2\alpha}$ and D (all in the adjoint representation of the gauge group):

$$V=-\theta\sigma^\mu\bar{\theta}A_\mu-i\bar{\theta}^2\theta\lambda^1-i\theta^2\bar{\theta}\bar{\lambda}^2+\frac{1}{2}\theta^2\bar{\theta}^2D. \tag{6.2}$$

Notice that the conjugate of $\lambda^1{}_\alpha$ is $\bar{\lambda}_{1\dot{\alpha}}=-\bar{\lambda}^2{}_{\dot{\alpha}}$. The chiral superfield Φ, also in the adjoint representation, has components ϕ, $\lambda^2{}_\alpha=-\lambda_{1\alpha}$ and F:

$$\begin{aligned}\Phi&=\phi^a+\sqrt{2}\theta\lambda^2+\theta^2F,\\ \Phi^\dagger&=\phi^\dagger+\sqrt{2}\bar{\theta}\bar{\lambda}^1+\bar{\theta}^2\overline{F},\end{aligned} \tag{6.3}$$

and the conjugate of $\lambda^2{}_\alpha$ is $\bar{\lambda}_{2\dot{\alpha}}=\bar{\lambda}^1{}_{\dot{\alpha}}$. We can now write the action in components. First, we redefine the auxiliary field D as $D\to D+[\phi,\phi^\dagger]$. The action then reads:

$$\begin{aligned}S=\int d^4x\mathrm{Tr}\{&\nabla_\mu\phi^\dagger\nabla^\mu\phi-i\lambda^1\sigma^\mu\nabla_\mu\bar{\lambda}_1-i\lambda^2\sigma^\mu\nabla_\mu\bar{\lambda}_2-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\\ &+\tfrac{1}{2}D^2+|F|^2-\tfrac{1}{2}[\phi,\phi^\dagger]^2-i\sqrt{2}\lambda^{1\alpha}[\phi^\dagger,\lambda^2_\alpha]+i\sqrt{2}\bar{\lambda}_{1\dot{\alpha}}[\bar{\lambda}^{\dot{\alpha}}_2,\phi]\}.\end{aligned} \tag{6.4}$$

This action is not manifestly $\mathcal{N}=2$ supersymmetric, since the $\mathcal{N}=2$ supersymmetric algebra has an internal $SU(2)_R$ symmetry, which is not manifest in (6.4). $SU(2)_R$ invariance is easily achieved: the scalars ϕ and the gluons A_μ are singlets, while the gluinos λ_v, $v=1,2$ form a doublet. The auxiliary fields form a real triplet:

$$D^{vw}=\begin{pmatrix}\sqrt{2}F & iD\\ iD & \sqrt{2}\overline{F}\end{pmatrix}, \tag{6.5}$$

and the $SU(2)_R$ indices are raised and lowered with the matrices ϵ_{vw}, ϵ^{vw}. Notice that $D^{vw}=D^*_{vw}$. Finally, by covariantizing the $\mathcal{N}=1$ transformations, one finds the $\mathcal{N}=2$

transformations:

$$
\begin{aligned}
&[Q_{v\alpha}, \phi] = i\sqrt{2}\lambda_{v\alpha}, \\
&[Q_{v\alpha}, \phi^\dagger] = 0, \\
&[Q_{v\alpha}, A_\mu] = (\sigma_\mu)_{\alpha\dot{\beta}}\bar{\lambda}_v{}^{\dot{\beta}} \\
&[Q_{v\alpha}, \lambda_{w\beta}] = -i\epsilon_{\alpha\beta}D_{vw} - [\phi, \phi^\dagger]\epsilon_{\alpha\beta}\epsilon_{vw} - i(\sigma^{\mu\nu})_{\alpha\beta}\epsilon_{vw}F_{\mu\nu} \\
&[Q_{v\alpha}, \bar{\lambda}_{w\dot{\beta}}] = \sqrt{2}(\bar{\sigma}^\mu)_{\dot{\beta}\alpha}\nabla_\mu\phi^\dagger\epsilon_{vw}, \\
&[Q_{u\alpha}, D_{vw}] = (\sigma^\mu)_{\alpha\dot{\beta}}\big(\epsilon_{uv}(\nabla_\mu\bar{\lambda})_w{}^{\dot{\beta}} + \epsilon_{uw}(\nabla_\mu\bar{\lambda})_v{}^{\dot{\beta}}\big) + \sqrt{2}\big(\epsilon_{uv}[\lambda_{w\alpha}, \phi^\dagger] + \epsilon_{uw}[\lambda_{v\alpha}, \phi^\dagger]\big), \\
&[\overline{Q}_{v\dot{\alpha}}, \phi] = 0, \\
&[\overline{Q}_{v\dot{\alpha}}, \phi^\dagger] = i\sqrt{2}\bar{\lambda}_{v\dot{\alpha}}, \\
&[\overline{Q}_{v\dot{\alpha}}, A_\mu] = (\bar{\sigma}_\mu)_{\dot{\alpha}\beta}\lambda_v{}^{\beta}, \\
&[Q_{v\alpha}, \lambda_{w\beta}] = -\sqrt{2}(\sigma^\mu)_{\beta\dot{\alpha}}\nabla_\mu\phi\epsilon_{vw}, \\
&[\overline{Q}_{v\dot{\alpha}}, \bar{\lambda}_{w\dot{\beta}}] = i\epsilon_{\dot{\alpha}\dot{\beta}}D_{vw} + [\phi, \phi^\dagger]\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{vw} + i(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}\epsilon_{vw}F_{\mu\nu} \\
&[\overline{Q}_{u\dot{\alpha}}, D_{vw}] = -(\bar{\sigma}^\mu)_{\dot{\alpha}\beta}\big(\epsilon_{uv}(\nabla_\mu\lambda)_w{}^{\beta} + \epsilon_{uw}(\nabla_\mu\lambda)_v{}^{\beta}\big) - \sqrt{2}\big(\epsilon_{uv}[\bar{\lambda}_{w\dot{\alpha}}, \phi] + \epsilon_{uw}[\bar{\lambda}_{v\dot{\alpha}}, \phi]\big).
\end{aligned}
\tag{6.6}
$$

The action reads, once the $SU(2)_R$ symmetry is manifest,:

$$
\begin{aligned}
S = \int d^4x \mathrm{Tr}\big\{&\nabla_\mu\phi^\dagger\nabla^\mu\phi - i\lambda^v\sigma^\mu\nabla_\mu\bar{\lambda}_v - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}D_{vw}D^{vw} \\
&- \tfrac{1}{2}[\phi, \phi^\dagger]^2 - \frac{i}{\sqrt{2}}\epsilon^{vw}\lambda_v{}^\alpha[\phi^\dagger, \lambda_{w\alpha}] - \frac{i}{\sqrt{2}}\epsilon_{vw}\bar{\lambda}^v{}_{\dot{\alpha}}[\bar{\lambda}^{w\dot{\alpha}}, \phi]\big\}.
\end{aligned}
\tag{6.7}
$$

The above action also has a classical $U(1)_R$ symmetry:

$$
\begin{aligned}
&A_\mu \to A_\mu, && D_{vw} \to D_{vw}, \\
&\lambda_{v\alpha} \to \mathrm{e}^{i\varphi}\lambda_{v\alpha}, && \phi \to \mathrm{e}^{2i\varphi}\phi, \\
&\bar{\lambda}^v{}_{\dot{\alpha}} \to \mathrm{e}^{-i\varphi}\bar{\lambda}^v{}_{\dot{\alpha}}, && \phi^\dagger \to \mathrm{e}^{-2i\varphi}\phi^\dagger.
\end{aligned}
\tag{6.8}
$$

6.2. $\mathcal{N} = 2$ hypermultiplets

The $\mathcal{N} = 2$ generalization of the chiral multiplet is the hypermultiplet. Its on-shell spectrum consists of a $SU(2)_R$ doublet of complex scalar fields and a Dirac spinor. There is also a doublet of auxiliary fields off-shell. From the point of view of $\mathcal{N} = 1$ superspace, the hypermultiplet consists of two chiral multiplets, Q and $\widetilde{Q}$, with components:

$$
\begin{aligned}
Q =& q_2 + \sqrt{2}\theta\psi + \theta^2 F_1, \\
\widetilde{Q} =& i(q^{\dagger 1} + \sqrt{2}\theta\chi + \theta^2 F^{\dagger 2}).
\end{aligned}
\tag{6.9}
$$

The action is

$$S = \int d^4x d^2\theta d^2\bar{\theta} \{ Q^\dagger Q + \widetilde{Q}^\dagger \widetilde{Q} \} - im \int d^4x d^2\theta \, Q\widetilde{Q} + im \int d^4x d^2\bar{\theta} \, Q^\dagger \widetilde{Q}^\dagger. \tag{6.10}$$

Exercise. *The action for the free hypermultiplet*

Write (6.10) in components to find

$$\begin{aligned} \mathcal{A} = \int d^4x \, (&\partial_\mu q^{\dagger 1} \partial^\mu q_1 + \partial_\mu q^{\dagger 2} \partial^\mu q_2 - i\psi\sigma^\mu \partial_\mu \bar{\psi} - i\chi\sigma^\mu \partial_\mu \bar{\chi} \\ &+ F^{\dagger 1} F_1 + F^{\dagger 2} F_2 + mF^{\dagger 1} q_1 + mF^{\dagger 2} q_2 + mF_1 q^{\dagger 1} + mF_2 q^{\dagger 2} + \\ &- m\bar{\psi}\bar{\chi} - m\psi\chi \,). \end{aligned} \tag{6.11}$$

The action (6.11) has a manifest $SU(2)_R$ invariance, with two doublets q_w and F_w, and can be written as

$$\begin{aligned} \int d^4x \, (&\partial_\mu q^{\dagger w} \partial^\mu q_w - i\psi\sigma^\mu \partial_\mu \bar{\psi} - i\chi\sigma^\mu \partial_\mu \bar{\chi} - m\bar{\psi}\bar{\chi} - m\psi\chi \\ &+ F^{\dagger w} F_w + mF^{\dagger w} q_w + mF_w q^{\dagger w} \,). \end{aligned} \tag{6.12}$$

The usual $\mathcal{N} = 1$ supersymmetry can be extended to an $\mathcal{N} = 2$ supersymmetry with transformations:

$$\begin{aligned} \delta q_v &= \sqrt{2}\xi_v \psi + \sqrt{2}\epsilon_{vw}\bar{\xi}^w \bar{\chi}, \\ \delta\psi &= \sqrt{2}\epsilon^{vw}\xi_v F_w + i\sqrt{2}\sigma^\mu \bar{\xi}^w \partial_\mu q_w, \\ \delta\chi &= \sqrt{2}\xi_w F^{\dagger w} - i\sqrt{2}\epsilon_{vw}\sigma^\mu \bar{\xi}^v \partial_\mu q^{\dagger w}, \\ \delta F_v &= -i\sqrt{2}\partial_\mu \psi \sigma^\mu \epsilon_{vw} \bar{\xi}^w + i\sqrt{2}\xi_v \sigma^\mu \partial_\mu \bar{\chi}. \end{aligned} \tag{6.13}$$

Notice that this algebra has now a central charge:

$$\{Q_{u\alpha}, Q_{v\beta}\} = 2\sqrt{2}\epsilon_{\alpha\beta} Z_{uv}. \tag{6.14}$$

Exercise. *The central charge for the free hypermultiplet*

Compute Z_{uv} using the explicit transformations (6.13). Notice that, after taking into account that $F_v = -mq_v$ on-shell, the central charge is proportional to the mass of the hypermultiplet.

6.3. $\mathcal{N}=2$ gauge theories with matter

Now we combine the $\mathcal{N}=2$ matter hypermultiplet with the $\mathcal{N}=2$ vector multiplet. From the $\mathcal{N}=1$ point of view, the theory consists of two chiral superfields Q and $\widetilde{Q}$, in conjugate representations of the gauge group, and a real superfield V, together with a chiral superfield Φ in the adjoint. The action is, again, a particular case of (5.32), but now it has a nontrivial superpotential

$$W=\sqrt{2}\widetilde{Q}\Phi Q-im\widetilde{Q}Q. \tag{6.15}$$

The full action reads, then,

$$\begin{aligned}S=&\int d^4xd^2\theta\mathrm{Tr}(W^2)+\int d^4xd^2\bar{\theta}\mathrm{Tr}(W^{\dagger 2})+\int d^4xd^2\theta d^2\bar{\theta}\,\mathrm{Tr}(\Phi^\dagger \mathrm{e}^{2V}\Phi)+\\&+\int d^4xd^2\theta d^2\bar{\theta}\{\,Q^{\dagger k}\mathrm{e}^{2V_{kl}}Q^l+\widetilde{Q}^k\mathrm{e}^{-2V_{kl}}\widetilde{Q}^{\dagger l}\,\}+\sqrt{2}\int d^4xd^2\theta\,\widetilde{Q}^k\Phi_{kl}Q^l+\\&+\sqrt{2}\int d^4xd^2\bar{\theta}\,Q^{\dagger k}\Phi^\dagger_{kl}\widetilde{Q}^{\dagger l}-im\int d^4xd^2\theta\,\widetilde{Q}^lQ^l+im\int d^4xd^2\bar{\theta}\,Q^{\dagger l}\widetilde{Q}^{\dagger l}.\end{aligned} \tag{6.16}$$

In order to find an $SU(2)_R$ invariant expression, it is convenient to redefine the auxiliary fields in the following way:

$$\begin{aligned}\tilde{F}_1&=F_1+mq_1+i\sqrt{2}\phi^\dagger q_1,\\\tilde{F}_2&=F_2+mq_2-i\sqrt{2}\phi q_2,\\\tilde{D}^a&=D^a+[\phi,\phi^\dagger]^a-q^{\dagger 1}T^aq_1+q^{\dagger 2}T^aq_2,\\\widetilde{F}^a&=F^a-i\sqrt{2}q^{\dagger 2}T^aq_1,\\\widetilde{\overline{F}}^a&=\overline{F}^a+i\sqrt{2}q^{\dagger 1}T^aq_2.\end{aligned} \tag{6.17}$$

Notice that the algebraic equation of motion for the triplet of auxiliaries (6.5) can be written in a manifestly $SU(2)_R$ covariant form as follows:

$$D^a_{vw}=-2iq^\dagger_{(v}T^aq_{w)}. \tag{6.18}$$

The action reads, in components, and after using the redefinition (6.17):

$$\begin{aligned}S=&\int d^4x\mathrm{Tr}\{\nabla_\mu\phi^\dagger\nabla^\mu\phi-i\lambda_w\sigma^\mu\nabla_\mu\bar{\lambda}^w-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{4}D_{vw}D^{vw}\\&-\tfrac{1}{2}[\phi,\phi^\dagger]^2-\frac{i}{\sqrt{2}}\epsilon^{vw}\lambda^\alpha_v[\phi^\dagger,\lambda_{w\alpha}]-\frac{i}{\sqrt{2}}\epsilon_{vw}\bar{\lambda}^v{}_{\dot\alpha}[\bar{\lambda}^{w\dot\alpha},\phi]\}\\&+\int d^4x\,\{\nabla_\mu q^{\dagger w}\nabla^\mu q_w-i\psi\sigma^\mu\partial_\mu\bar{\psi}-i\chi\sigma^\mu\partial_\mu\bar{\chi}-m\bar{\psi}\bar{\chi}-m\psi\chi+m^2q^{\dagger w}q_w\\&+F^{\dagger v}F_v+i\sqrt{2}q^{\dagger v}\lambda_v\psi+i\sqrt{2}\epsilon_{vw}q^{\dagger v}\bar{\lambda}^w\bar{\chi}-i\sqrt{2}\epsilon^{vw}\chi\lambda_vq_w-i\sqrt{2}\bar{\psi}\bar{\lambda}^vq_v\\&+i\sqrt{2}\,\chi^\alpha\phi\psi_\alpha-i\sqrt{2}\,\bar{\psi}_{\dot\alpha}\phi^\dagger\bar{\chi}^{\dot\alpha}-q^{\dagger w}\{\phi,\phi^\dagger\}q_w+im\sqrt{2}q^{\dagger w}\phi^\dagger q_w-im\sqrt{2}q^{\dagger w}\phi q_w\\&+q^{\dagger (v}T^aq^{w)}q^\dagger{}_{(v}T^aq_{w)}\,\}\end{aligned} \tag{6.19}$$

where we have supressed the tildes over the auxiliaries. The $\mathcal{N}=2$ transformations turn out to be,

$$
\begin{aligned}
\delta\phi &= \sqrt{2}\epsilon^{vw}\xi_v\lambda_w, \\
\delta A_\mu &= i\xi_v\sigma_\mu\bar{\lambda}^v - i\lambda_v\sigma_\mu\bar{\xi}^v, \\
\delta\lambda^a{}_{v\alpha} &= D^a{}_v{}^w\xi_{w\alpha} - i\xi_{v\alpha}[\phi,\phi^\dagger]^a - i\sigma^{\mu\nu}{}_\alpha{}^\beta\xi_{v\beta}F^a_{\mu\nu} + \\
&\quad + i\sqrt{2}\epsilon_{vw}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\xi}^{w\dot{\alpha}}\nabla_\mu\phi^a + 2iq^\dagger{}_{(v}T^a q_{w)}\xi^w{}_\alpha, \\
\delta D^{a\ vw} &= 2i\bar{\xi}^{(v}\bar{\sigma}^\mu\nabla_\mu\bar{\lambda}^{a\ w)} + 2i\nabla_\mu\bar{\lambda}^{a\ (v}\bar{\sigma}^\mu\xi^{w)} + 2\sqrt{2}i\xi^{(v}[\lambda^{w)},\phi^\dagger]^a \\
&\quad + 2\sqrt{2}i\bar{\xi}^{(v}[\bar{\lambda}^{w)},\phi]^a + 2q^{\dagger(v}T^a\psi\xi^{w)} + 2\sqrt{2}i\bar{\xi}^{(v}\bar{\psi}T^a q^{w)} \\
&\qquad + 2\sqrt{2}iq^{\dagger(v}T^a\bar{\chi}\bar{\xi}^{w)} - 2\sqrt{2}i\xi^{(v}\chi T^a q^{w)}, \\
\delta q_v &= \sqrt{2}\xi_v\psi + \sqrt{2}\epsilon_{vw}\bar{\xi}^w\bar{\chi}, \\
\delta\psi &= \sqrt{2}\epsilon^{vw}\xi_v F_w + i\sqrt{2}\sigma^\mu\bar{\xi}^w\nabla_\mu q_w \\
&\quad - \sqrt{2}m\epsilon^{vw}\xi_v q_w - 2i\epsilon^{vw}\xi_v\phi^\dagger q_w, \\
\delta\chi &= \sqrt{2}\xi_w F^{\dagger w} - i\sqrt{2}\epsilon_{vw}\sigma^\mu\bar{\xi}^v\nabla_\mu q^{\dagger w} \\
&\quad - \sqrt{2}mq^{\dagger v}\xi_v - 2i\xi_v q^{\dagger v}\phi^\dagger, \\
\delta F_v &= -i\sqrt{2}\nabla_\mu\psi\sigma^\mu\epsilon_{vw}\bar{\xi}^w + i\sqrt{2}\xi_v\sigma^\mu\nabla_\mu\bar{\chi} \\
&\quad + \sqrt{2}m\xi_v\psi - 2i\xi_v\phi\psi + 2i\xi_v\epsilon^{wz}\lambda_w q_z \\
&\quad + \sqrt{2}m\epsilon_{vw}\bar{\xi}^w\bar{\chi} + 2i\epsilon_{vw}\bar{\xi}^w\phi^\dagger\bar{\chi} + 2i\epsilon_{vw}\bar{\xi}^w\bar{\lambda}^z q_z.
\end{aligned}
\tag{6.20}
$$

Finally, in the massless theory $m=0$ there is a $U(1)_R$ symmetry:

$$
\begin{aligned}
&A_\mu \to A_\mu; q_R = 0, && D_{vw} \to D_{vw}; q_R = 0, \\
&\lambda_{v\alpha} \to \mathrm{e}^{i\varphi}\lambda_{v\alpha}; q_R = 1, && \phi \to \mathrm{e}^{2i\varphi}\phi; q_R = 2, \\
&\bar{\lambda}^v{}_{\dot{\alpha}} \to \mathrm{e}^{-i\varphi}\bar{\lambda}^v{}_{\dot{\alpha}}; q_R = 1, && \phi^\dagger \to \mathrm{e}^{-2i\varphi}\phi^\dagger; q_R = -2, \\
&\psi \to \mathrm{e}^{-i\varphi}\psi; q_R = -1, && q_v \to q_v; q_R = 0, \\
&\chi \to \mathrm{e}^{-i\varphi}\chi; q_R = -1,
\end{aligned}
\tag{6.21}
$$

7. Topological field theories from twisted supersymmetry

In this section, we introduce topological field theories and we give a brief general overview of their properties, focusing on the so-called theories of the Witten or cohomological type. We then explain the twisting procedure, which produces topological field theories from $\mathcal{N}=2$ theories, and put it in practice with the examples of the previous section. General introductions to topological field theories can be found in [46][18][35], among other references.

7.1. Topological field theories: basic properties

Topological field theories (TFT's) were first introduced by Witten in [47]. A quantum field theory is topological if, when put on a manifold X with a Riemannian metric $g_{\mu\nu}$, the correlation functions of some set of operators do not depend (at least formally) on the metric. We then have

$$\frac{\delta}{\delta g_{\mu\nu}}\langle \mathcal{O}_{i_1}\cdots\mathcal{O}_{i_n}\rangle = 0, \tag{7.1}$$

where $\mathcal{O}_{i_1}, \cdots, \mathcal{O}_{i_n}$ are operators in the theory. There are two different types of TFT's: in the TFT's theories of the Scwharz type, one tries to define all the ingredients in the theory (the action, the operators, and so on) without using the metric of the manifold. The most important example is Chern-Simons theory, introduced by Witten in [10]. In the TFT's of the Witten type, one has an explicit metric dependence, but the theory has an underlying scalar symmetry δ acting on the fields in such a way that the correlation functions of the theory do not depend on the background metric. More precisely, if the energy-momentum tensor of the theory $T_{\mu\nu} = (\delta/\delta g^{\mu\nu})S(\phi_i)$ can be written as

$$T_{\mu\nu} = -i\delta G_{\mu\nu} \tag{7.2}$$

where $G_{\mu\nu}$ is some tensor, then (7.1) holds for any operator $\mathcal{O}$ which is δ-invariant. This is because:

$$\begin{aligned}\frac{\delta}{\delta g^{\mu\nu}}\langle \mathcal{O}_{i_1}\mathcal{O}_{i_2}\cdots\mathcal{O}_{i_n}\rangle &= \langle \mathcal{O}_{i_1}\mathcal{O}_{i_2}\cdots\mathcal{O}_{i_n}T_{\mu\nu}\rangle \\ &= -i\langle \mathcal{O}_{i_1}\mathcal{O}_{i_2}\cdots\mathcal{O}_{i_n}\delta G_{\mu\nu}\rangle = \pm i\langle \delta(\mathcal{O}_{i_1}\mathcal{O}_{i_2}\cdots\mathcal{O}_{i_n}G_{\mu\nu})\rangle = 0.\end{aligned} \tag{7.3}$$

In this derivation we have used the fact that δ is a symmetry of the classical action $S(\phi_i)$ and of the quantum theory. Such a symmetry is called a *topological symmetry*. In some situations this symmetry is anomalous, and the theory is not strictly topological. However, in most of the interesting cases, this dependence is mild and under control. We will see a very explicit example of this in Donaldson theory on manifolds of $b_2^+ = 1$.

If the theory is topological, as we have described it, the natural operators are then those which are δ-invariant. On the other hand, operators which are δ-exact decouple from the theory, since their correlation functions vanish. The operators that are in the cohomology of δ are called *topological observables*:

$$\mathcal{O} \in \frac{\mathrm{Ker}\,\delta}{\mathrm{Im}\,\delta}. \tag{7.4}$$

The topological symmetry δ is not nilpotent: in general one has

$$\delta^2 = Z \tag{7.5}$$

where Z is a certain transformation in the theory. It can be a local transformation (a gauge transformation) or a global transformation (for example, a global $U(1)$ symmetry). The appropriate framework to analyze the observables is then *equivariant cohomology*, and for consistency one has to consider only operators that are invariant under the transformation generated by Z (for example, gauge invariant operators). Equivariant cohomology turns out to be a very natural language to describe TFT's with local and global symmetries, but we are not going to explore it in these lectures. The interested reader should look at [18][48].

The structure of topological field theories of the Witten type leads immediately to a general version of the Donaldson map [47]. Remember that, starting with the curvature of the universal bundle, this map associates cohomology classes in the instanton moduli space to homology classes in the four-manifold. One can easily see that in any theory where (7.2) is satisfied, one can define topological observables associated to homology cycles in spacetime. If (7.2) holds, then one has:

$$P_\mu = T_{0\mu} = -i\delta G_\mu, \tag{7.6}$$

where

$$G_\mu \equiv G_{0\mu}. \tag{7.7}$$

In the theories that we are goint to consider, δ is essentially given by a supersymmetric transformation, and therefore it is a Grassmannian symmetry. It follows that G_μ is an anticommuting operator. On the other hand, from the point of view of the Lorentz group, they are a scalar and a one-form, respectively. Then, topological field theories of Witten type violate the spin-statistics theorem. Consider now a δ-invariant operator $\phi^{(0)}(x)$. The *descent operators* are defined as

$$\phi^{(n)}_{\mu_1\mu_2\cdots\mu_n}(x) = G_{\mu_1}G_{\mu_2}\cdots G_{\mu_n}\phi^{(0)}(x), \quad n = 1, \cdots, d, \tag{7.8}$$

where d is the dimension of the spacetime manifold. Since the G_{μ_i} anticonmute, $\phi^{(n)}$ is antisymmetric in the indices $\mu_1, \cdots, \mu_n$, and therefore it gives an n-form:

$$\phi^{(n)} = \frac{1}{n!}\phi^{(n)}_{\mu_1\mu_2\cdots\mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}. \tag{7.9}$$

As an immediate consequence of (7.6) and the δ-invariance of $\phi^{(0)}$, one has the following *descent equations*:

$$d\phi^{(n)} = \delta\phi^{(n+1)}, \tag{7.10}$$

where d is the exterior derivative and we have taken into account that $P_\mu = -i\partial_\mu$. Now it is easy to see that the operator

$$W^{(\gamma_n)}_{\phi^{(0)}} = \int_{\gamma_n} \phi^{(n)}, \tag{7.11}$$

where $\gamma_n \in H_n(X)$, is a topological observable:

$$\delta W^{(\gamma_n)}_{\phi^{(0)}} = \int_{\gamma_n} \delta\phi^{(n)} = \int_{\gamma_n} d\phi^{(n-1)} = \int_{\partial\gamma_n} \phi^{(n-1)} = 0, \tag{7.12}$$

since $\partial\gamma_n = 0$.

Exercise. *Homology and observables*

Show that, if γ_n is trivial in homology (*i.e.*, if it is ∂-exact), then $W^{(\gamma_n)}_{\phi^{(0)}}$ is δ-exact.

Therefore, given a (scalar) topological observable, one can construct a family of topological observables

$$W^{(\gamma_{i_n})}_{\phi^{(0)}}, \quad i_n = 1, \cdots, b_n; \quad n = 1, \cdots, d, \tag{7.13}$$

in one-to-one correspondence with the homology classes of spacetime. This descent procedure is the analog of the Donaldson map in Donaldson-Witten and Seiberg-Witten theory. Notice that any family of operators $\phi^{(n)}$ that satisfies the descent equations (7.10) gives topological observables. The explicit realization (7.8) in terms of the G operator can then be regarded as a canonical solution to (7.10).

7.2. Twist of $\mathcal{N} = 2$ supersymmetry

In the early eighties, Witten noticed in two seminal papers [49][50] that supersymmetry has a deep relation to topology. The simplest example of such a relation is supersymmetric quantum mechanics, which provides a physical reformulation (and in fact a refinement) of Morse theory [50]. Another example are $\mathcal{N} = 2$ theories in two and four dimensions. In 1988 Witten discovered that, by changing the coupling to gravity of the fields in an $\mathcal{N} = 2$ theory in two or four dimensions, a theory satisfying the requirements of the previous subsection was obtained [47][51]. This redefinition of the theory is called *twisting*. We are now going to explain in some detail how this works in the four-dimensional case.

The $\mathcal{N}=2$ supersymmetry algebra (with no central charges) is:

$$
\begin{aligned}
&\{Q_{\alpha v}, \overline{Q}_{\dot\beta w}\} = 2\epsilon_{vw}\sigma^{\mu}_{\alpha\dot\beta}P_\mu && \{Q_{\alpha v}, Q_{\beta w}\} = 0 \\
&[P_\mu, Q_{\alpha v}] = 0 && [P_\mu, \overline{Q}\dot\alpha v] = 0 \\
&[M_{\mu\nu}, Q_{\alpha v}] = -(\sigma_{\mu\nu})_\alpha{}^\beta Q_{\beta v} && [M_{\mu\nu}, \overline{Q}^{\dot\alpha v}] = -(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\overline{Q}^{\dot\beta v} \\
&[Q_{\alpha v}, B^a] = -\tfrac{1}{2}(\tau^a)_v{}^w Q_{\alpha w} && [\overline{Q}_{\dot\alpha}{}^v, B^a] = \tfrac{1}{2}\overline{Q}_{\dot\alpha}{}^w(\tau^a)_w{}^v \\
&[Q_{\alpha v}, R] = Q_{\alpha v} && [\overline{Q}_{\dot\alpha v}, R] = -\overline{Q}_{\dot\alpha v}
\end{aligned}
\tag{7.14}
$$

Here, the indices $v, w \in \{1,2\}$. The twisting procedure consists of redefining the coupling to gravity of the theory, *i.e.* in redefining the spins of the fields. To do this, we couple the fields to the $SU(2)_+$ spin connection according to their isospin. This means that we *add* to the Lagrangian the term

$$J^R_\mu \omega^\mu_+, \tag{7.15}$$

where J^R_μ is the $SU(2)_R$ current of the theory, and ω^μ_+ is the $SU(2)_+$ spin connection. We then have a new rotation group $\mathcal{K}' = SU'(2)_+ \otimes SU(2)_-$, where $SU'(2)_+$ is the diagonal of $SU(2)_+ \times SU(2)_R$. In practice, the twist means essentially that the $SU(2)_R$ indices v, w become spinorial indices $\dot\alpha, \dot\beta$, and we have the change $\overline{Q}_{\dot\alpha v} \to \overline{Q}_{\dot\alpha\dot\beta}$ and $Q_{\alpha v} \to Q_{\alpha\dot\beta}$. It is easy to check that the *topological supercharge*

$$\overline{\mathcal{Q}} \equiv \epsilon^{\dot\alpha\dot\beta}\overline{Q}_{\dot\alpha\dot\beta} = \overline{Q}_{\dot1\dot2} - \overline{Q}_{\dot2\dot1}. \tag{7.16}$$

is a scalar with respect to $\mathcal{K}'$. This topological supercharge will provide the topological symmetry δ that we need for a topological theory. The $\mathcal{N}=2$ algebra also gives a natural way to construct the operator G_μ defined in (7.7). In fact, define:

$$G_\mu = \frac{i}{4}(\bar\sigma_\mu)^{\dot\alpha\gamma}Q_{\gamma\dot\alpha}. \tag{7.17}$$

Using now the $\{Q, \overline{Q}\}$ anticommutator one can show that

$$\{\overline{\mathcal{Q}}, G_\mu\} = \partial_\mu. \tag{7.18}$$

This means that the supersymmetry algebra by itself almost guarantees (7.2). In the models that we will consider, (7.2) is true (at least on-shell). Finally, notice that from the anticommutator $\{\overline{Q}, \overline{Q}\}$ in (7.14) follows that the topological supercharge is nilpotent $\overline{\mathcal{Q}}^2 = 0$ (in the absence of central charge).

Our main conclusion is that *by twisting $\mathcal{N}=2$ supersymmetry one can construct a quantum field theory that satisfies (almost) all the requirements of a topological field theory of the Witten type.* In the rest of this section we are going to analyze in detail the twist of the $\mathcal{N}=2$ theories that we described in the last section.

8. Donaldson-Witten theory

Donaldson-Witten theory (also known as topological Yang-Mills theory) is the topological field theory that results from twisting $\mathcal{N}=2$ Yang-Mills theory in four dimensions. Historically it was the first TFT of the Witten type, and as we will see it provides a physical realization of Donaldson theory.

8.1. The topological action

Remember that $\mathcal{N}=2$ super Yang-Mills theory contains a gauge field A_μ, two gluinos $\lambda_{v\alpha}$ and a complex scalar ϕ, all of them in the adjoint representation of the gauge group $\mathcal{G}$. In the off-shell formulation, we also have auxiliary fields D_{vw} in the **3** of the internal $SU(2)_R$. The total symmetry group of the theory is

$$\mathcal{H} = SU(2)_+ \times SU(2)_- \times SU(2)_R \times U(1)_R. \tag{8.1}$$

Under the twist, the fields in the $\mathcal{N}=2$ supermultiplet change their spin content as follows:

$$\begin{aligned} A_\mu\ (1/2,1/2,0)^0 &\rightarrow A_\mu\ (1/2,1/2)^0, \\ \lambda_{v\alpha}\ (1/2,0,1/2)^{-1} &\rightarrow \psi_{\dot\beta\alpha}\ (1/2,1/2)^1, \\ \bar\lambda_{v\dot\alpha}\ (0,1/2,1/2)^1 &\rightarrow \eta\ (0,0)^{-1}, \quad \chi_{\dot\alpha\dot\beta}\ (1,0)^{-1}, \\ \phi\ (0,0,0)^{-2} &\rightarrow \phi\ (0,0)^{-2}, \\ \phi^\dagger\ (0,0,0)^2 &\rightarrow \phi^\dagger\ (0,0)^2, \\ D_{vw}\ (0,0,1)^0 &\rightarrow D_{\dot\alpha\dot\beta}\ (1,0)^0, \end{aligned} \tag{8.2}$$

where we have written the quantum numbers with respect to the group H before the twist, and with respect to the group $\mathcal{H}' = SU(2)'_+ \otimes SU(2)_- \otimes U(1)_R$ after the twist. In the topological theory, the $U(1)_R$ charge is usually called *ghost number*. The η and χ fields are given by the antisymmetric and symmetric pieces of $\overline{\lambda}_{\dot\alpha\dot\beta}$, respectively. More precisely:

$$\chi_{\dot\alpha\dot\beta} = \overline{\lambda}_{(\dot\alpha\dot\beta)}, \quad \eta = \frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\overline{\lambda}_{\dot\alpha\dot\beta}. \tag{8.3}$$

From the $\mathcal{N}=2$ action it is straightforward to find:

$$\begin{aligned} S = \int d^4x\,\sqrt{g}\mathrm{Tr}\,\{ &\nabla_\mu\phi\nabla^\mu\phi^\dagger - i\psi^{\dot\beta}{}_\alpha\nabla^{\dot\alpha\alpha}\chi_{\dot\beta\dot\alpha} - i\psi_{\dot\alpha\alpha}\nabla^{\dot\alpha\alpha}\eta - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &+ \frac{1}{4}D_{\dot\alpha\dot\beta}D^{\dot\alpha\dot\beta} - \tfrac{1}{2}[\phi,\phi^\dagger]^2 - \frac{i}{\sqrt{2}}\chi^{\dot\alpha\dot\beta}[\phi,\chi_{\dot\alpha\dot\beta}] + i\sqrt{2}\eta[\phi,\eta] - \frac{i}{\sqrt{2}}\psi_{\dot\alpha\alpha}[\psi^{\dot\alpha\alpha},\phi^\dagger]\} \end{aligned} \tag{8.4}$$

where $\nabla^{\dot\alpha\alpha} = \bar\sigma^{\mu\dot\alpha\alpha}\nabla_\mu$. The $\overline{\mathcal{Q}}$-transformations are easily obtained from the $\mathcal{N}=2$ transformations:

$$\begin{aligned}
[\overline{\mathcal{Q}}, \phi] &= 0, \\
[\overline{\mathcal{Q}}, A_\mu] &= \psi_\mu, \\
\{\overline{\mathcal{Q}}, \eta\} &= [\phi, \phi^\dagger], \\
\{\overline{\mathcal{Q}}, \psi_\mu\} &= 2\sqrt{2}\nabla_\mu\phi,
\end{aligned}
\qquad
\begin{aligned}
[\overline{\mathcal{Q}}, \phi^\dagger] &= 2\sqrt{2}i\eta, \\
\{\overline{\mathcal{Q}}, \chi_{\dot\alpha\dot\beta}\} &= i(F^+_{\dot\alpha\dot\beta} - D_{\dot\alpha\dot\beta}), \\
[\overline{\mathcal{Q}}, D] &= (2\nabla\psi)_+ + 2\sqrt{2}[\phi, \chi].
\end{aligned} \tag{8.5}$$

In (8.5), $\psi_\mu = \sigma_{\mu\alpha\dot\beta}\psi^{\alpha\dot\beta}$ and $F^+_{\dot\alpha\dot\beta} = \bar\sigma^{\mu\nu}_{\dot\alpha\dot\beta}F_{\mu\nu}$ is the selfdual part of $F_{\mu\nu}$. We have also denoted It is not difficult to show that the action of Donaldson-Witten theory is $\overline{\mathcal{Q}}$-exact up to a topological term, *i.e.*

$$S = \{\overline{\mathcal{Q}}, V\} - \frac{1}{2}\int F\wedge F, \tag{8.6}$$

where

$$V = \int d^4x\, \mathrm{Tr}\,\Big\{\, \frac{i}{4}\chi_{\dot\alpha\dot\beta}(F^{\dot\alpha\dot\beta} + D^{\dot\alpha\dot\beta}) - \frac{1}{2}\eta[\phi,\eta] + \frac{1}{2\sqrt{2}}\psi_{\alpha\dot\alpha}\nabla^{\dot\alpha\alpha}\phi^\dagger\,\Big\}. \tag{8.7}$$

As we will see in a moment, this has important implications for the quantum behavior of the theory.

Exercise. *The Lagrangian of Donaldson-Witten theory*

Derive (8.4) and (8.6).

One of the most interesting aspects of the twisting procedure is the following: if we put the original $\mathcal{N}=2$ Yang-Mills theory on an arbitrary Riemannian four-manifold, using the well-known prescription of minimal coupling to gravity, we find global obstructions to have a well-defined theory. The reason is very simple: not every four-manifold is Spin, and therefore the fermionic fields $\lambda_{\alpha v}$ are not well-defined unless $w_2(X) = 0$. However, after the twisting, all fields are differential forms on X, and therefore the twisted theory makes sense globally on an arbitrary Riemannian four-manifold.

8.2. The observables

The observables of Donaldson-Witten theory can be constructed by using the topological descent equations. As we have emphasized, these equations have a canonical solution given by the operator (7.17). Using again the $\mathcal{N}=2$ supersymmetry transformations, one can work out the action of G_μ on the different fields of the theory. The result is:

$$
\begin{aligned}
[G_\mu, \phi] &= \frac{1}{2\sqrt{2}}\psi_\mu, & [G, \overline{\phi}] &= 0, \\
[G_\nu, A_\mu] &= \frac{i}{2} g_{\mu\nu}\eta - i\chi_{\mu\nu}, & [G, F^+] &= i\nabla\chi + \frac{3i}{2} * \nabla\eta, \\
[G, \eta] &= -\frac{i\sqrt{2}}{4}\nabla\overline{\phi}, & \{G, \chi\} &= -\frac{3i\sqrt{2}}{8} * \nabla\overline{\phi}, \\
\{G_\mu, \psi_\nu\} &= -(F^-_{\mu\nu} + D^+_{\mu\nu}), & [G, D] &= -\frac{3i}{4} * \nabla\eta + \frac{3i}{2}\nabla\chi,
\end{aligned}
\tag{8.8}
$$

We can now construct the topological observables of the theory by using the descent equations. The starting point must be a gauge-invariant, $\overline{\mathcal{Q}}$-closed operator which is not $\overline{\mathcal{Q}}$-trivial. Since $[\overline{\mathcal{Q}}, \phi] = 0$, the simplest candidates are the operators

$$
\mathcal{O}_n = \mathrm{Tr}(\phi^n), \quad n = 2, \cdots, N. \tag{8.9}
$$

Here we are going to restrict ourselves to $SU(2)$, therefore the starting point for the descent procedure will be the operator,

$$
\mathcal{O} = \mathrm{Tr}(\phi^2). \tag{8.10}
$$

It is easy to see that the following operators satisfy the descent equations (7.10):

$$
\begin{aligned}
\mathcal{O}^{(1)} &= \mathrm{Tr}\,(\frac{1}{\sqrt{2}}\phi\psi_\mu)dx^\mu, \\
\mathcal{O}^{(2)} &= -\frac{1}{2}\mathrm{Tr}\,(\frac{1}{\sqrt{2}}\phi F_{\mu\nu} - \frac{1}{4}\psi_\mu\psi_\nu)dx^\mu \wedge dx^\nu, \\
\mathcal{O}^{(3)} &= -\frac{1}{8}\mathrm{Tr}\,(\psi_\lambda F_{\mu\nu})dx^\lambda \wedge dx^\mu \wedge dx^\nu, \\
\mathcal{O}^{(4)} &= \frac{1}{32}\mathrm{Tr}\,(F_{\lambda\tau}F_{\mu\nu})dx^\lambda \wedge dx^\tau \wedge dx^\mu \wedge dx^\nu.
\end{aligned}
\tag{8.11}
$$

This is, however, not the canonical solution to the descent equations provided by G, which in this case is a little bit more complicated.

Exercise. *Descent equations in topological Yang-Mills theory*

Show that (8.11) satisfy the descent equations. Compare with the canonical solution.

In these lectures we will restrict ourselves to the observables:

$$I_1(\delta) = \int_\delta \mathcal{O}^{(1)}, \qquad I_2(S) = \int_S \mathcal{O}^{(2)}, \tag{8.12}$$

where $\delta \in H_1(X)$, $S \in H_2(X)$. They correspond to the differential forms on the moduli space of ASD connections that were introduced in (3.54) through the use of the Donaldson map (3.53) (and this is why we have used the same notation for both). Notice that the ghost number of the operators in (8.11) is in fact their degree as differential forms in moduli space. The operators (8.11) are naturally interpreted as the decomposition of the Pontriagin class of the universal bundle (4.20) with respect to the bigrading of $\Omega^*(\mathcal{B}^* \times X)$. In fact, the Grasmannian field ψ_μ can be interpreted as a $(1,1)$ form: a one-form in spacetime and also a one-form in the space $\mathcal{A}$. The operator $\overline{\mathcal{Q}}$ is then interpreted as the equivariant differential in $\mathcal{A}$ with respect to gauge transformations. This leads to a beautiful geometric interpretation of topological Yang-Mills theory in terms of equivariant cohomology [52] and the Mathai-Quillen formalism [53], which is reviewed in detail in [18].

8.3. Evaluation of the path integral

We now consider the topological theory defined by the topological Yang-Mills action, $S_{\rm TYM} = \{\overline{\mathcal{Q}}, V\}$. The evaluation of the path integral of the theory defined by the Donaldson-Witten action can be drastically simplified by taking into account the following fact. The (unnormalized) correlation functions of the theory are defined by

$$\langle \phi_1 \cdots \phi_n \rangle = \int \mathcal{D}\phi \, \phi_1 \cdots \phi_n \, {\rm e}^{-\frac{1}{g^2} S_{\rm TYM}}, \tag{8.13}$$

where $\phi_1, \cdots, \phi_n$ are generic fields. Since $S_{\rm TYM}$ is $\overline{\mathcal{Q}}$-exact, one has:

$$\frac{\partial}{\partial g} \langle \phi_1 \cdots \phi_n \rangle = \frac{2}{g^3} \langle \phi_1 \cdots \phi_n S_{\rm TYM} \rangle = 0, \tag{8.14}$$

where we have used the fact that δ is a symmetry of the theory, and therefore the insertion of a δ-exact operator in the path integral gives zero. The above result is remarkable: it says that, in a topological field theory in which the action is $\overline{\mathcal{Q}}$-exact, the computations do not depend on the value of the coupling constant. In particular, *the semiclassical approximation is exact!* [47]. We can then evaluate the path integral in the saddle-point approximation as follows: first, we look at zero modes, *i.e.* classical configurations that minimize the action. Then we look at nonzero modes, *i.e.* we consider quantum fluctuations around these configurations. Since the saddle-point approximation is exact, it is enough to consider the quadratic fluctuations. The integral over the zero modes gives a finite integral over the space of bosonic collective coordinates, and a finite Grassmannian integral over the zero modes of the fermi fields. The integral over the quadratic fluctuations gives a bunch of determinants. Since the theory has a bose-fermi $\overline{\mathcal{Q}}$ symmetry, it is easy to see that the determinants cancel (up to a sign), as in supersymmetric theories.

Let us then analyze the bosonic and fermionic zero modes. A quick way to find the bosonic zero modes is to look for supersymmetric configurations. These are classical configurations such that $\{\overline{\mathcal{Q}}, \text{fermi}\} = 0$ for all Fermi fields in the theory, and they give minima of the Lagrangian. Indeed, it was shown in [54][55] that in topological field theories with a fermionic symmetry $\overline{\mathcal{Q}}$ one can compute by localization on the fixed points of this symmetry. In this case, by looking at $\{\overline{\mathcal{Q}}, \chi\} = 0$, one finds

$$F^+ = D^+. \tag{8.15}$$

But on-shell $D^+ = 0$, and therefore (8.15) reduces to the usual ASD equations. The zero modes of the gauge field are then instanton configurations. In addition, by looking at $\{\overline{\mathcal{Q}}, \psi\} = 0$, we find the equation of motion for the ϕ field,

$$d_A \phi = 0. \tag{8.16}$$

This equation is also familiar: as we saw in chapter 2, its nontrivial solutions correspond to reducible connections. Let us assume for simplicity that we are in a situation in which no reducible solutions occur, so that $\phi = 0$. In that case, (8.15) tells us that the integral over the collective coordinates reduces to an integral over the instanton moduli space $\mathcal{M}_{ASD}$.

Let us now look at the fermionic zero modes in the background of an instanton. The kinetic terms for the ψ, χ and η fermions fits precisely into the instanton deformation complex (3.45). Therefore, using the index theorem we can compute:

$$N_\psi - N_\chi = \dim \mathcal{M}_{\text{ASD}}, \tag{8.17}$$

where $N_{\psi,\chi}$ denotes the number of zeromodes of the corresponding fields, and we have used the fact that the connection A is irreducible, so that η (which is a scalar) has no zero modes (in other words, $d_A \eta = 0$ only has the trivial solution). Finally, if we assume that the connection is regular, then one has that $\text{Coker}\, p^+ d_A = 0$, and there are not χ zero modes. In this situation, the number of ψ zero modes is simply the dimension of the moduli space of ASD instantons. If we denote the bosonic and the fermionic zero modes by da_i, $d\psi_i$, respectively, where $i = 1, \cdots, D$ and $D = \dim\mathcal{M}_{\rm ASD}$, then the zero-mode measure becomes:

$$\prod_{i=1}^{D} da_i d\psi_i. \tag{8.18}$$

This is in fact the natural measure for integration of differential forms on $\mathcal{M}_{\rm ASD}$, and the Grassmannian variables ψ_i are then interpreted as a basis of one-forms on $\mathcal{M}_{\rm ASD}$.

We can already discuss how to compute correlation functions of the operators $\mathcal{O}$, $I_2(S)$, and $I_1(\delta)$. These operators contain the fields ψ, A_μ and ϕ. In evaluating the path integral, it is enough to replace ψ and A_μ by their zero modes, and the field ϕ (with no zero modes) by its quantum fluctuations, that we then integrate out at quadratic order. Further corrections are higher order in the coupling constant and do not contribute to the saddle-point approximation, which in this case is exact. We have then to compute the one-point function $\langle \phi^a \rangle$. The relevant terms in the action are

$$S(\phi, \phi^\dagger) = \int d^4x {\rm Tr}\{\nabla_\mu \phi \nabla^\mu \phi^\dagger - \frac{i}{\sqrt{2}} \phi^\dagger [\psi_\mu, \psi^\mu]\}, \tag{8.19}$$

since we are only considering quadratic terms. We then have to compute

$$\langle \phi^a(x) \rangle = \int \mathcal{D}\phi \mathcal{D}\phi^\dagger \phi^a(x) \exp -S(\phi, \phi^\dagger). \tag{8.20}$$

If we take into account that

$$\langle \phi^a(x) \phi^{b\dagger}(y) \rangle = -G^{ab}(x - y), \tag{8.21}$$

where $G^{ab}(x - y)$ is the Green's function of the Laplacian $\nabla_\mu \nabla^\mu$, we find:

$$\langle \phi^a(x) \rangle = -\frac{i}{\sqrt{2}} \int d^4y \sqrt{g} G^{ab}(x, y) [\psi(x)_\mu, \psi(y)^\mu]_b. \tag{8.22}$$

This expresses ϕ in terms of zero modes. It turns out that this is precisely (up to a multiplicative constant) the component along $\mathcal{B}^*$ of the curvature $K_{\mathbf{P}}$ of the universal bundle (see for example [7], p. 196). This is in perfect agreement with the correspondence between the observables (8.11) and the differential forms on moduli space (3.54) constructed in Donaldson theory.

The main conclusion of this analysis is that, up to possible normalizations,

$$\begin{aligned}&\langle \mathcal{O}^{\ell} I_2(S_{i_1}) \cdots I_2(S_{i_p}) I_1(\delta_{j_1}) \cdots \ I_1(\delta_{j_q})\rangle \\ &= \int_{\mathcal{M}_{\rm ASD}} \mathcal{O}^{\ell} \wedge I_2(S_{i_1}) \wedge \cdots \wedge I_2(S_{i_p}) \wedge I_1(\delta_{j_1}) \wedge \cdots \wedge \ I_1(\delta_{j_q}),\end{aligned} \tag{8.23}$$

i.e. the correlation function of the observables of twisted $\mathcal{N} = 2$ Yang-Mills theory is precisely the corresponding Donaldson invariant. The requirement that the differential form in the r.h.s. has top degree (otherwise the invariant is zero) corresponds, in the field theory side, to the requirement that the correlation has ghost number equal to $\dim\mathcal{M}_{\rm ASD}$, *i.e.* that the operator in the correlation function soaks up all the fermionic zero modes, which is the well-known 't Hooft rule [56]. (8.23) was one of the most important results of Witten's seminal work [47], and it opened a completely different approach to Donaldson theory by means of topological quantum field theory.

9. Twisted $\mathcal{N} = 2$ theories with hypermultiplets

9.1. The twisted free hypermultiplet

In this subsection, we are going to analyze the twist of the free $\mathcal{N} = 2$ hypermultiplet. Twisted hypermultiplets have been considered in detail in many papers, for example [57][58][59][60][61][45]. Remember that the field content of the hypermultiplet is given by a doublet of complex scalar fields, q_v, two Weyl fermions ψ y χ, and a doublet of complex auxiliaries F_v. After the twist, the fields in the theory become

$$\begin{array}{rlcl} q_v & (0,0,1/2) & \longrightarrow & M_{\dot\alpha} \ \ (1/2,0), \\ \psi_\alpha & (1/2,0,0) & \longrightarrow & \nu_\alpha \ \ (1/2,0), \\ \bar\chi_{\dot\alpha} & (0,1/2,0) & \longrightarrow & \mu_{\dot\alpha} \ \ (0,1/2), \\ F_v & (0,0,1/2) & \longrightarrow & K_{\dot\alpha} \ \ (1/2,0), \\ q^{\dagger v} & (0,0,1/2) & \longrightarrow & \overline{M}^{\dot\alpha} \ \ (1/2,0), \\ \bar\psi_{\dot\alpha} & (0,1/2,0) & \longrightarrow & \bar\mu_{\dot\alpha} \ \ (0,1/2), \\ \chi_\alpha & (1/2,0,0) & \longrightarrow & \bar\nu_\alpha \ \ (1/2,0), \\ F^{\dagger v} & (0,0,1/2) & \longrightarrow & \overline{K}^{\dot\alpha} \ \ (1/2,0), \end{array} \tag{9.1}$$

where we have written the quantum numbers with respect to $\mathcal{H}$ and $\mathcal{H}'$. The twisted free action is

$$S = \int d^4x \, (\partial_\mu \overline{M}^{\dot\alpha} \partial^\mu M_{\dot\alpha} + m\overline{M}^{\dot\alpha} K_{\dot\alpha} + m\overline{K}^{\dot\alpha} M_{\dot\alpha} + \overline{K}^{\dot\alpha} K_{\dot\alpha}$$
$$- i\bar\nu_\alpha \partial^{\dot\alpha\alpha} \mu_{\dot\alpha} - i\bar\mu^{\dot\alpha} \partial_{\alpha\dot\alpha} \nu^\alpha - m\bar\mu^{\dot\alpha}\mu_{\dot\alpha} - m\bar\nu_\alpha \nu^\alpha). \tag{9.2}$$

The topological transformations are:

$$\begin{aligned}
\delta M_{\dot\alpha} &= -\sqrt{2}\epsilon \mu_{\dot\alpha}, & \delta \overline{M}_{\dot\alpha} &= \sqrt{2}\epsilon \bar\mu_{\dot\alpha}, \\
\delta \mu_{\dot\alpha} &= -\sqrt{2}\epsilon K_{\dot\alpha}, & \delta \bar\mu_{\dot\alpha} &= -\sqrt{2}\epsilon \overline{K}_{\dot\alpha}, \\
\delta \nu_\alpha &= i\sqrt{2}\epsilon \partial_{\alpha\dot\alpha} M^{\dot\alpha}, & \delta \bar\nu_\alpha &= i\sqrt{2}\epsilon \partial_{\alpha\dot\alpha} \overline{M}^{\dot\alpha}, \\
\delta K_{\dot\alpha} &= -i\sqrt{2}\epsilon \partial_{\alpha\dot\alpha} \nu^\alpha, & \delta \overline{K}_{\dot\alpha} &= i\sqrt{2}\epsilon \partial_{\alpha\dot\alpha} \bar\nu^\alpha,
\end{aligned} \tag{9.3}$$

This off-shell formulation of the theory is not particularly suitable, since the action turns out to be not δ-exact. However, another off-shell formulation is possible which cures this problem, and requires the introduction of new auxiliary fields h y $\bar h$. The new, δ' transformations turn out to be:

$$\begin{aligned}
\delta' M_{\dot\alpha} &= -\sqrt{2}\epsilon \mu_{\dot\alpha}, & \delta' \overline{M}_{\dot\alpha} &= \sqrt{2}\epsilon \bar\mu_{\dot\alpha}, \\
\delta' \mu_{\dot\alpha} &= -\sqrt{2}\epsilon m M_{\dot\alpha}, & \delta' \bar\mu_{\dot\alpha} &= -\sqrt{2}\epsilon m \overline{M}_{\dot\alpha}, \\
\delta' \nu_\alpha &= \sqrt{2}\epsilon h_\alpha, & \delta' \bar\nu_\alpha &= \sqrt{2}\epsilon \bar h_\alpha, \\
\delta' h_\alpha &= \sqrt{2}\epsilon m \nu_\alpha, & \delta' \bar h_\alpha &= -\sqrt{2}\epsilon m \bar\nu_\alpha,
\end{aligned} \tag{9.4}$$

In this formulation, the central charge is realized off-shell as a global $U(1)$ symmetry with parameter m, and the action can be written as $\delta'\Lambda$, where

$$\Lambda = \int d^4x \, \frac{1}{\sqrt{2}} \{ -i\bar\nu_\alpha \partial^{\dot\alpha\alpha} M_{\dot\alpha} - i\overline{M}^{\dot\alpha} \partial_{\alpha\dot\alpha} \nu^{\dot\alpha} - \tfrac{1}{2} m\bar\mu^{\dot\alpha} M_{\dot\alpha} - \tfrac{1}{2}\bar\nu_\alpha h^\alpha - \tfrac{1}{2} m\overline{M}^{\dot\alpha}\mu_{\dot\alpha} - \tfrac{1}{2}\bar h_\alpha \nu^\alpha \} \tag{9.5}$$

and reads

$$S = \int d^4x \, \{ -i\bar h_\alpha \partial^{\alpha\alpha} M_{\dot\alpha} - i\overline{M}^{\dot\alpha} \partial_{\alpha\dot\alpha} h^\alpha - i\bar\nu_\alpha \partial^{\dot\alpha\alpha}\mu_{\dot\alpha} - i\bar\mu^{\dot\alpha}\partial_{\alpha\dot\alpha}\nu^\alpha$$
$$- m^2 \overline{M}^{\dot\alpha} M_{\dot\alpha} - m\bar\nu_\alpha \nu^\alpha - m\bar\mu^{\dot\alpha}\mu_{\dot\alpha} - \bar h_\alpha h^\alpha \} \tag{9.6}$$

After integrating out the new auxiliary fields, one finds the twisted on-shell action for the massive hypermultiplet (9.2).

Notice that, in contrast to $\mathcal{N}=2$ Yang-Mills, where the twisting procedure made the theory well-defined on any four-manifold, the twisted free hypermultiplet theory contains spinors. In fact, the twisting has converted the two scalar fields q_v into a new spinor field $M_{\dot\alpha}$. Strictly speaking, the twisted free hypermultiplet only makes sense globally on Spin manifolds. We will see later that, under certain circumstances, the coupling to gauge fields can solve this problem.

9.2. Twisted $\mathcal{N}=2$ gauge theory coupled to hypermultiplets

We can now put together the results for twisted $\mathcal{N}=2$ Yang-Mills theory and for the free hypermultiplet and consider the coupled theory. Again, it is convenient to redefine the auxiliary fields for the hypermultiplet in the way that we have just described. The results are easy to derive. For the action, we find:

$$
\begin{aligned}
S = &\int d^4x \,\mathrm{Tr}\,\{\nabla_\mu\phi\nabla^\mu\phi^\dagger + i\psi^{\dot\beta}{}_\alpha\nabla^{\dot\alpha\alpha}\chi_{\dot\alpha\dot\beta} + i\psi_{\alpha\dot\alpha}\nabla^{\dot\alpha\alpha}\eta + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
&- \frac{1}{4}D_{\dot\alpha\dot\beta}D^{\dot\alpha\dot\beta} + \tfrac{1}{2}[\phi,\phi^\dagger]^2 - \frac{i}{\sqrt{2}}\chi^{\dot\alpha\dot\beta}[\phi,\chi_{\dot\alpha\dot\beta}] - i\sqrt{2}\eta[\phi,\eta] - \frac{i}{\sqrt{2}}\psi_{\alpha\dot\alpha}[\psi^{\alpha\dot\alpha},\phi^\dagger]\} \\
&+ \int d^4x\{-i\bar h_\alpha\nabla^{\alpha\alpha}M_{\dot\alpha} - i\overline{M}^{\dot\alpha}\nabla_{\alpha\dot\alpha}h^\alpha - i\bar\nu_\alpha\nabla^{\dot\alpha\alpha}\mu_{\dot\alpha} - i\bar\mu^{\dot\alpha}\nabla_{\alpha\dot\alpha}\nu^\alpha \\
&- m^2\overline{M}^{\dot\alpha}M_{\dot\alpha} - m\bar\nu_\alpha\nu^\alpha - m\bar\mu^{\dot\alpha}\mu_{\dot\alpha} - \bar h_\alpha h^\alpha - \overline{M}^{\dot\alpha}F^+_{\dot\alpha\dot\beta}M^{\dot\beta}\,\} \\
&+ \int d^4x\,\{\, i\sqrt{2}(\overline{M}^{\dot\alpha}\chi_{\dot\alpha}{}^{\dot\beta}\mu_{\dot\beta} + \bar\mu^{\dot\alpha}\chi_{\dot\alpha}{}^{\dot\beta}M_{\dot\beta}) + i\sqrt{2}(\overline{M}^{\dot\alpha}\psi_{\alpha\dot\alpha}\nu^\alpha - \bar\nu_\alpha\psi^{\dot\alpha\alpha}M_{\dot\alpha}) \\
&+ \overline{M}^{\dot\alpha}\{\phi,\phi^\dagger\}M_{\dot\alpha} + i\sqrt{2}(\overline{M}^\alpha\eta\mu_\alpha - \bar\mu^\alpha\eta M_{\dot\alpha}) - i\sqrt{2}\bar\nu_\alpha\phi\nu^\alpha + i\sqrt{2}\bar\mu^{\dot\alpha}\phi^\dagger\mu_{\dot\alpha} \\
&+ (\overline{M}^{(\dot\alpha}T^aM^{\dot\beta)})(\overline{M}_{(\dot\alpha}T^aM_{\dot\beta)}) - i\sqrt{2}m\overline{M}^{\dot\alpha}\phi M_{\dot\alpha} + i\sqrt{2}m\overline{M}^{\dot\alpha}\phi^\dagger M_{\dot\alpha}\,\}
\end{aligned}
\tag{9.7}
$$

and the δ' transformations are:

$$
\begin{aligned}
&\delta'\phi = 0, \qquad \delta'\phi^\dagger = 2\sqrt{2}i\epsilon\eta, \qquad \delta'\eta = i\epsilon[\phi,\phi^\dagger], \\
&\delta' A_\mu = \epsilon\psi_\mu, \qquad \delta'\psi_\mu = 2\sqrt{2}\epsilon\nabla_\mu\phi, \\
&\delta'\chi^a{}_{\alpha\beta} = i\epsilon(F^{+a}_{\dot\alpha\dot\beta} - D^a_{\dot\alpha\dot\beta} + 2i\overline{M}_{(\dot\alpha}T^aM_{\dot\beta)}), \\
&\delta' D^a_{\dot\alpha\dot\beta} = \epsilon(d_A\psi)^a_{\dot\alpha\dot\beta} + 2\sqrt{2}i\epsilon[\chi^{\dot\alpha\dot\beta},\phi]^a + 2\sqrt{2}\epsilon(\overline{M}^{(\dot\alpha}T^a\mu^{\dot\beta)} + \bar\mu^{(\dot\alpha}T^aM^{\dot\beta)}), \\
\\
&\delta' M_{\dot\alpha} = -\sqrt{2}\epsilon\mu_{\dot\alpha}, \quad \delta'\overline{M}_{\dot\alpha} = \sqrt{2}\epsilon\bar\mu_{\dot\alpha}, \\
&\delta'\mu^j{}_{\dot\alpha} = \sqrt{2}\epsilon m M^j{}_{\dot\alpha} + 2i\epsilon\phi^{jk}M^k{}_{\dot\alpha}, \\
&\delta'\bar\mu^j{}_{\dot\alpha} = \sqrt{2}\epsilon m\overline{M}^j{}_{\dot\alpha} + 2i\epsilon\overline{M}^k{}_{\dot\alpha}\phi^{kj}, \\
&\delta'\nu_\alpha = \sqrt{2}\epsilon h_\alpha, \qquad \delta'\bar\nu_\alpha = \sqrt{2}\epsilon\bar h_\alpha, \\
&\delta' h^j{}_\alpha = -\sqrt{2}\epsilon m\nu^j{}_\alpha - 2i\epsilon\phi^{jk}\nu^k{}_\alpha, \\
&\delta'\bar h^j{}_\alpha = \sqrt{2}\epsilon m\bar\nu^j{}_\alpha + 2i\epsilon\bar\nu^k{}_\alpha\phi^{kj}.
\end{aligned}
\tag{9.8}
$$

Notice that $\delta' = \epsilon\{\overline{\mathcal{Q}}, \}$. With this topological symmetry, the action is δ'-exact,

$$
\begin{aligned}
\epsilon S = \delta' \int d^4x \, \mathrm{Tr} \{ \, & \frac{i}{4}\chi_{\dot\alpha\dot\beta}F^{\dot\alpha\dot\beta} + \frac{1}{4}\chi_{\dot\alpha\dot\beta}D^{\dot\alpha\dot\beta} - i\eta[\phi,\eta] + \frac{i}{\sqrt{2}}\psi_{\alpha\dot\alpha}\nabla^{\dot\alpha\alpha}\eta \, \} \\
& + \int d^4x \, \frac{1}{\sqrt{2}}\{ -i\bar\nu_\alpha\nabla^{\dot\alpha\alpha}M_{\dot\alpha} - i\overline{M}^{\dot\alpha}\nabla_{\alpha\dot\alpha}\nu^\alpha - \tfrac{1}{2}m\bar\mu^{\dot\alpha}M_{\dot\alpha} - \tfrac{1}{2}\bar\nu_\alpha h^\alpha - \tfrac{1}{2}m\overline{M}^\alpha\mu_{\dot\alpha} \\
& - \tfrac{1}{2}\bar h_\alpha\nu^\alpha + \frac{i}{\sqrt{2}}(\,\bar\mu^{\dot\alpha}\phi^\dagger M_{\dot\alpha} + \overline{M}^{\dot\alpha}\phi^\dagger\mu_{\dot\alpha}\,) - \frac{i}{\sqrt{2}}\overline{M}_{\dot\alpha}\chi^{\dot\alpha\dot\beta}M_{\dot\beta} \, \}
\end{aligned}
\tag{9.9}
$$

up to a topological term $\sim \int F\wedge F$.

9.3. Recovering the Seiberg-Witten equations

Let us consider the twisted $\mathcal{N}=2$ theory coupled to a hypermultiplet that we have just analyzed, but in the simple case of an abelian group $U(1)$ and with zero mass $m=0$. Since the action is $\overline{\mathcal{Q}}$-exact, the theory can be analyzed very much like Donaldson-Witten theory. In order to compute correlation functions, we can restrict ourselves to supersymmetric configurations. The auxiliary fields D and h satisfy the algebraic equations of motion $D_{\dot\alpha\dot\beta}=0$ and $h_\alpha = \nabla_{\alpha\dot\alpha}M^{\dot\alpha}$. Therefore, $\{\overline{Q},\chi\}=0$ gives:

$$
F^+_{\dot\alpha\dot\beta} + 2i\overline{M}_{(\dot\alpha}M_{\dot\beta)} = 0,
\tag{9.10}
$$

while $\{\overline{Q},\nu_\alpha\}=0$ gives

$$
\nabla_{\alpha\dot\alpha}M^{\dot\alpha} = 0.
\tag{9.11}
$$

These are precisely the Seiberg-Witten monopole equations (4.2). The only subtlety is, of course, that in the original equations (4.2) M is a section of $S^+\otimes L^{1/2}$, where L is the determinant line bundle of a Spin_c structure, while here M is a section of $S^+\otimes U$, where U is in principle a conventional $U(1)$ bundle. In fact, as we remarked before, the twisted hypermultiplet is not globally well-defined on an arbitrary four-manifold, unless $w_2(X)=0$. As we will see, in the relevant physical realization of the $U(1)$ theory, a subtle topological effect will make U the square root of the determinant line bundle L associated to a Spin_c connection, in such a way that the twisted abelian theory will be well-defined. In any case, if we assume that the $U(1)$ connection is really a Spin_c connections, we see that supersymmetric configurations are given by solutions to the SW equations, and the space of bosonic zero modes is precisely the SW moduli space. We have assumed here, as in our discussion of Donaldson-Witten theory, that there are no reducible pairs, and therefore $\phi=0$. The analysis of the fermionic coordinates is also very similar to the analysis of

Donaldson theory. The fermionic zero modes fit into the deformation complex (4.10), and in a generic situation their number equals the dimension of the SW moduli space.

Let us now analyze the observables of the abelian theory. The starting point for the descent procedure is:

$$\mathcal{O} = \phi, \tag{9.12}$$

and the descendants are simply:

$$\mathcal{O}^{(1)} = \frac{1}{2\sqrt{2}}\psi, \quad \mathcal{O}^{(2)} = -\frac{1}{\sqrt{2}}F. \tag{9.13}$$

The only non trivial observables are ϕ itself and

$$\nu_i = \int_{\delta_i} \mathcal{O}^{(1)}, \quad i = 1, \cdots, b_1. \tag{9.14}$$

The observables ϕ and ν_i, $i = 1, \cdots, b_1$ are precisely the differential forms on moduli space (4.22) and (4.23), respectively. As in Donaldson-Witten theory, it is straightforward to show that correlation functions in this topological field theory are precisely the Seiberg-Witten invariants (assuming, of course, that the $U(1)$ connection is really a Spin_c connection). More precisely, one has

$$\langle \phi^\ell \nu_{i_1} \cdots \nu_{i_r} \rangle = \int_{\mathcal{M}_{\rm SW}} \phi^\ell \wedge \nu_{i_1} \wedge \cdots \wedge \nu_{i_r}. \tag{9.15}$$

It is understood here that ϕ is obtained by integrating out, as in Donaldson-Witten theory, and is therefore expressed in terms of fermionic zero modes.

10. The Seiberg-Witten solution

The Seiberg-Witten solution [5][62] provides the exact low energy effective action (LEEA) of $\mathcal{N} = 2$ super Yang-Mills theory up to two derivatives and four fermi terms. In these lectures we can not make a detailed analysis of this remarkable achievement, and we will just provide some starting points and the final results, which will be needed in the topological applications of the remaining lectures. There are excellent reviews that the reader can consult for further information, for example [44][63][64][65]. We will start by reviewing some semiclassical aspects of the theory that motivate the ingredients for the exact solution. $\mathcal{N} = 2$ supersymmetry reduces the problem of finding the LEEA to the problem of finding a single holomorphic function, the so-called prepotential. The duality properties of the prepotential are reviewed in the second subsection. Since the SW solution is written in terms of an auxiliary elliptic curve, we will review in some detail some of the basic aspects of elliptic curves and the theory of elliptic functions. With this machinery, we will present the exact solution in subsection 4 and explores some of its properties. Finally, we will reexpress some aspects of the solution in terms of modular forms, something that will be very useful to analyze the u-plane integral.

10.1. Low-energy effective action: semiclassical aspects

1) Classical potential and moduli space of vacua

In order to analyze the $\mathcal{N}=2$, $SU(2)$ theory, the first step is to perform a classical analysis. The classical action containts the scalar potential

$$V = \frac{1}{2g^2}\mathrm{Tr}[\phi^\dagger, \phi]^2 \,. \tag{10.1}$$

The vacua are determined by $V=0$, *i.e.* $[\phi, \phi^\dagger]=0$. This means that ϕ is gauge-equivalent to an element in the Cartan subalgebra, and

$$\phi = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \tag{10.2}$$

When ϕ is of this form, the gauge symmetry is broken to $U(1)$. The vev a gives mass to the $W^\pm$ bosons in the usual way. The final outcome of this process is that, from the three $\mathcal{N}=2$ vector superfields of the $SU(2)$ theory, two of them (the $\mathcal{N}=2$ supermultiplets of the $W^\pm$ bosons) get a mass, while one of them remains massless. Classically, the only light degree of freedom is a $U(1)$, $\mathcal{N}=2$ multiplet.

Notice that different vacuum values of ϕ correspond to different physical theories, and therefore the complex number a parametrizes the space of physically inequivalent vacua, or the classical moduli space of the theory, which has complex dimension one. However, this parametrization of the moduli space is not the desired one as after using the gauge symmetry to put ϕ in this form, there is still some residual gauge invariance: a Weyl reflection will take $a \to -a$. It is more convenient to parametrize the moduli space by the vacuum expectation value of a gauge-invariant operator,

$$u = \langle \mathrm{Tr}(\phi^2) \rangle. \tag{10.3}$$

Classically, $u = 2a^2$, but as we will see this gets corrected in the full quantum theory.

2) Constraints from $\mathcal{N}=2$ supersymmetry

Our main goal is to find the effective quantum theory for the massless degree of freedom, which classically is a $U(1)$ $\mathcal{N}=2$ multiplet $\mathcal{A}$. In $\mathcal{N}=1$ language, this contains an $\mathcal{N}=1$ chiral multiplet

$$A = a - \sqrt{2}\theta\lambda_1 + \theta^2 F, \tag{10.4}$$

and an $\mathcal{N}=1$ vector multiplet

$$V = -2\theta\sigma^\mu\bar{\theta}A_\mu - 2i\bar{\theta}^2\theta\lambda_2 + 2i\theta^2\bar{\theta}\bar{\lambda}^2 + \theta^2\bar{\theta}^2 D. \tag{10.5}$$

Notice that we have denoted by a both the lowest component of the $U(1)$ chiral multiplet, in (10.4), and its vacuum expectation value in (10.2).

In principle, the low-energy effective action should be obtained by integrating out the massive degrees of freedom, *i.e.* by performing the path integral over all the modes of the massive multiplets. Since this procedure gives in general a nonlocal action, one has to expand in derivatives to obtain a local effective theory. The expansion is then in powers of p/Λ, where p is the momentum and Λ is in this case the dynamically generated scale.

Exercise. *Integrating out massive particles*

Here is a simple model for an effective action which is obtained by integrating out a massive scalar field [66]. Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}\big(\partial_\mu H \partial^\mu H - m_H^2 H^2\big) + JH \tag{10.6}$$

where H, J are scalar fields. Compute the effective Lagrangian for J by integrating out H in the path integral. This gives a nonlocal Lagrangian

$$\mathcal{L}_{\text{eff}} = -\frac{1}{2}\int d^4x d^4y J(x) G(x-y) J(y), \tag{10.7}$$

where $G(x-y)$ is the usual Green function. Expand $J(y)$ around $y=x$ to obtain

$$\mathcal{L}_{\text{eff}} = \int d^4x \frac{1}{2m_H^2} J(x)J(x) + \cdots. \tag{10.8}$$

Notice that this Lagrangian becomes singular as $m_H \to 0$.

Of course, in real life it is extremely hard to integrate out the massive degrees of freedom explicitly, and one obtains the structure of low energy effective actions by symmetry considerations (as for example in chiral perturbation theory). One of the big advantages of $\mathcal{N}=2$ supersymmetry is that the leading terms in the derivative expansion (the terms that contain at most two derivatives and four fermions) can be obtained from a single holomorphic function $\mathcal{F}(a)$ called the *prepotential*. In the $\mathcal{N}=1$ language, the low energy effective Lagrangian at this order takes the form:

$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi}\,\text{Im}\left[\int d^4\theta\,\frac{\partial\mathcal{F}(A)}{\partial A}\bar{A} + \int d^2\theta\,\frac{1}{2}\frac{\partial^2\mathcal{F}}{\partial A^2}W^\alpha W_\alpha\right]. \tag{10.9}$$

In terms of $\mathcal{F}(a)$, the Kähler potential is given by

$$K = \text{Im}\left(\bar{A}\frac{\partial\mathcal{F}(A)}{\partial A}\right) \tag{10.10}$$

. If a denotes the scalar component of the chiral superfield A, then the metric on the space of fields, and therefore, the one on the space of Higgs vacuua, is given by

$$ds^2 = g_{a\bar{a}}\,da d\bar{a} = \text{Im}\tau\,da d\bar{a}, \tag{10.11}$$

where

$$\tau(a) = \frac{\partial^2\mathcal{F}}{\partial a^2}. \tag{10.12}$$

Due to $\mathcal{N}=2$ supersymmetry, the metric is then given by the gauge coupling τ which appears in front of the $F^i_{\mu\nu}F^{\mu\nu i}$ term in the above Lagrangian. Notice that, in terms of the usual coupling constant $1/g^2$ and the theta angle, one has

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}. \tag{10.13}$$

Exercise. *The effective Lagrangian in detail*

Expand the above Lagrangian in component fields to obtain:

$$\begin{aligned} &-\frac{1}{16\pi}\text{Im}\left[\tau(F^2_{\mu\nu} - iF_{\mu\nu} * F_{\mu\nu})\right] - \frac{1}{4\pi}\left[\tau\partial_\mu a\partial^\mu\bar{a} + i\tau\lambda^v\sigma^\mu\partial_\mu\bar{\lambda}_v\right] \\ &-\frac{\sqrt{2}}{16\pi}\text{Im}\left[\frac{d\tau}{da}\lambda^v\sigma^{\mu\nu}\lambda_v F_{\mu\nu}\right] - \frac{\sqrt{2}}{16\pi}\text{Im}\left[\frac{d\tau}{da}\lambda^v\lambda^w D_{vw}\right] \\ &+\frac{1}{16\pi}\text{Im}\tau D_{vw}D^{vw} + \frac{1}{48\pi}\text{Im}\left[\frac{d^2\tau}{da^2}(\lambda^v\lambda^w)(\lambda_v\lambda_w)\right]. \end{aligned} \tag{10.14}$$

---

3) The prepotential to one-loop

The classical analysis and the constraints of $\mathcal{N}=2$ supersymmetry give then the above general picture of the LEEA. In order to make further progress, we have to investigate the quantum dynamics. The first thing we can do is to evaluate the perturbative one-loop correction to $\mathcal{F}$. The effective coupling of the theory can be computed as follows. At energies larger than a, the masses of the $W^{\pm}$ bosons are negligible, and we can use the β-function of the $SU(2)$ theory. The β-function for a $SU(N_c)$ gauge theory with N_f Weyl fermions in the representations R_f and N_s complex scalar fields in the representations R_s is given by

$$\beta(g) \equiv \mu \frac{dg}{d\mu} = -\frac{g^3}{48\pi^2}\big(11N_c - 2\sum_f C_f - \frac{1}{4}\sum_s C_s\big), \qquad (10.15)$$

where $C_{f,s}$ are the Casimirs of the representations $R_{f,s}$, normalized in such a way that the Casimir in the adjoint is given by N_c. Since $\mathcal{N}=2$ pure $SU(2)$ Yang-Mills contains one complex scalar and two Weyl fermions, both in the adjoint representation, the β-function is given by,

$$\beta(g) = -\frac{1}{4\pi^2} g^3. \qquad (10.16)$$

The $SU(2)$ theory is then asymptotically free, and generates a scale Λ. Below the scale a, the $W^{\pm}$ bosons decouple, and we are left with an effective $U(1)$ theory. The effective coupling constant is obtained by matching the running coupling constant of the $SU(2)$ theory at the scale $\mu = a$.

Fig. 1: The effective coupling constant is obtained by matching the running coupling constant of $\mathcal{N}=2$ super Yang-Mills to the scale a set by the mass of the $W^{\pm}$.

We can then use standard perturbative calculations as long as $a >> \Lambda$, to find

$$\tau(a) = \frac{2i}{\pi}\log\Big(\frac{a}{\Lambda}\Big), \tag{10.17}$$

where we used holomorphy of τ. If we now integrate (10.12) we find

$$\mathcal{F}_{1-\text{loop}}(a) = \frac{i}{2\pi}a^2\ln\frac{a^2}{\Lambda^2}\,. \tag{10.18}$$

It is known that, due to $\mathcal{N} = 2$ supersymmetry, the above one-loop expression for the prepotential does not receive higher order perturbative corrections, and (10.18) gives the full perturbative prepotential.

4) Instanton effects

As we are dealing with a non-abelian gauge theory, we should expect nonperturbative effects due to instantons. The first possible effect of instanton backgrounds is to break some of the classical symmetries, as in the $U(1)$ problem of QCD [56]. Classically, our theory has the full global $SU(2)_R\times U(1)_R$ as a symmetry group. However, in an instanton background $U(1)_R$ is broken to a discrete subgroup. This can be easily understood as follows: by the index theorem, in the presence of an $SU(N)$ instanton, there are $2N$ zero-modes for each fermion λ_α in the adjoint representation, and since there is an $SU(2)_R$ doublet of fermions, there are in total $4N$ zero modes. The measure of the path-integral

$$\mathcal{D}\bar{\lambda}_{1\dot{\alpha}}\mathcal{D}\lambda_{1\alpha}\mathcal{D}\bar{\lambda}_{2\dot{\alpha}}\mathcal{D}\lambda_{2\alpha} \tag{10.19}$$

is not invariant under the $U(1)_R$ symmetry (6.8): by the usual argument a la Fujikawa [67], it picks a factor

$$\mathrm{e}^{2i\phi\,\mathrm{index}(D)} = \mathrm{e}^{-4iN\phi}, \tag{10.20}$$

where D is the Dirac operator coupled to the adjoint bundle. Therefore, the measure is only invariant under the discrete group $\mathbf{Z}_{4N}$. The surviving discrete R-symmetry is broken spontaneously by the Higgs vacuum. The field ϕ has charge 2 under $\mathbf{Z}_{4N}$ and transforms to $e^{\pi i/N}\phi^2$. Therefore, for the $SU(2)$ theory, if the vacuum is characterized by non-zero ϕ, then $\mathbf{Z}_8$ is broken down to $\mathbf{Z}_4$. The spontaneously broken symmetries have a nontrivial action on the moduli space, and they act as a $\mathbf{Z}_2$ on the u-plane: $u \to -u$.

As it was shown in [68], instantons also give nonperturbative corrections to the prepotential. The structure of these corrections can be determined as follows: first, a correction to $\mathcal{F}$ coming from a configuration of instanton number k should be proportional to the k-instanton factor $\exp(-8\pi^2 k/g^2)$ (since the prepotential is a holomorphic function, it cannot receive corrections from anti-instanton configurations). Using the explicit expression for the β-function of the theory (10.16), the k-instanton factor can be written as

$$e^{-8\pi^2 k/g^2} = \left(\frac{\Lambda}{a}\right)^{4k}. \tag{10.21}$$

To make further progress, we notice following [69] that although the R-symmetry is broken by instanton effects, one can formally restore it by assigning charge 2 to the dynamically generated scale Λ. In this way, the instanton factor (10.21) is neutral. The prepotential has charge 4, and this finally implies that the k-instanton correction should also be proportional to a^2. Putting these together, the prepotential including generic non-perturbative corrections can be written as

$$\mathcal{F} = \frac{i}{2\pi} a^2 \ln\frac{a^2}{\Lambda^2} + \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{a}\right)^{4k} a^2 \,. \tag{10.22}$$

The coefficients $\mathcal{F}_k$ are constants (this is due to the fact that, in a supersymmetric theory, instantons contribute to the path integral only through zero-modes). It turns out that the $\mathcal{F}_k$ are all nonzero. The Seiberg-Witten exact solution will give an explicit procedure to determine all $\mathcal{F}_k$.

5) BPS states

Spontaneously broken $SU(N)$ gauge theories with scalar fields in the adjoint representations contain 't Hooft-Polyakov monopoles and dyons in their semiclassical spectrum. These are time-independent, finite energy-solutions of the classical equations of motion, and they are characterized by the value (10.2) of the Higgs field at spatial infinity. A dyon of electric charge n_e and magnetic charge n_m has a mass

$$M = \sqrt{2}|a(n_e + \tau_{\rm cl} n_m)|. \tag{10.23}$$

The spectrum of $\mathcal{N} = 2$, $SU(2)$ Yang-Mills theory has a monopole of charge $n_m = 1$ and dyons of charge $(n_e, n_m) = (n, 1)$, and quantization of these solitons leads to states which preserve half of the supersymmetries and arrange themselves into an $\mathcal{N} = 2$ massive hypermultiplet, with central charge given semiclassically by

$$Z = a(n_e + \tau_{\rm cl} n_m). \tag{10.24}$$

These states satisfy the Bogomolnyi bound $M = \sqrt{2}|Z|$ and are called BPS states. As we will see, (10.24) gets corrected in the full quantum theory.

10.2. Sl$(2, \mathbf{Z})$ *duality of the effective action*

One of the most important aspects of the LEEA described by a $\mathcal{N} = 2$ prepotential is that one can perform an Sl$(2, \mathbf{Z})$ transformation to obtain another description of the same low energy theory. These different descriptions involve different parametrizations of the quantum moduli space. Depending on the region we are looking at, some parametrizations will be more appropriate than others, and this will play a crucial role in the rest of these lectures.

The duality transformations of the $\mathcal{N} = 2$ LEEA is in fact a generalization of the usual duality of Abelian Maxwell theory. Let us then start with the Maxwell action in Minkowski space, with conventions $(F_{\mu\nu})^2 = -(\widetilde{F}_{\mu\nu})^2$ and $\widetilde{\widetilde{F}} = -F$:

$$\frac{1}{32\pi} \mathrm{Im} \int \tau(a)(F + i\widetilde{F})^2 = \frac{1}{16\pi} \mathrm{Im} \int \tau(a)(F^2 + i\widetilde{F}F) \, . \tag{10.25}$$

Usually we take the gauge connection A_μ as the basic field, with $F = dA$, and then it follows that $dF = 0$ (Bianchi identity). But we can regard F as an independent field and implement the Bianchi identity by introducing a Lagrange multiplier vector field V_D. To fix the Lagrange multiplier term, $U(1) \subset SU(2)$ is normalized such that all $SO(3)$ fields have integer charges. With this convention, a magnetic monopole satisfies $\epsilon^{0\mu\nu\rho}\partial_\mu F_{\nu\rho} = 8\pi\delta^{(3)}(x)$. The Lagrange multiplier term can now be constructed by coupling V_D to a monopole:

$$\frac{1}{8\pi} \int V_{D\mu} \epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = \frac{1}{8\pi} \int \widetilde{F}_D F = \frac{1}{16\pi} \mathrm{Re} \int (\widetilde{F}_D - iF_D)(F + i\widetilde{F}) \, , \tag{10.26}$$

with $F_{D\mu\nu} = \partial_\mu V_{D\nu} - \partial_\nu V_{D\mu}$. Adding this to the gauge field action and integrating over F, we obtain the dual theory

$$\frac{1}{32\pi} \mathrm{Im} \int \left(-\frac{1}{\tau}\right) (F_D + i\widetilde{F}_D)^2 \frac{1}{16\pi} \mathrm{Im} \int \left(-\frac{1}{\tau}\right) (F_D^2 + i\widetilde{F}_D F_D) \, . \tag{10.27}$$

We want to show that something similar happens in the effective theory described by (10.9). Let us consider first the second term in (10.9):

$$\frac{1}{8\pi} \mathrm{Im} \int d^2\theta \tau(A) W^2 \, . \tag{10.28}$$

We can in fact perform a generalization in $\mathcal{N} = 1$ superspace of the dualization procedure explained above [5]. The Bianchi identity is now replaced by $\mathrm{Im}\mathcal{D}W = 0$. This can be implemented by introducing a vector superfield V_D and the corresponding Lagrange multiplier term becomes

$$\frac{1}{4\pi}\,\mathrm{Im}\int d^4x d^4\theta\, V_D \mathcal{D}W = \frac{1}{4\pi}\,\mathrm{Re}\int d^4x d^4\theta\, i\mathcal{D}V_D W = -\frac{1}{4\pi}\,\mathrm{Im}\int d^4x d^2\theta\, W_D W\,. \tag{10.29}$$

Adding this to the action and integrating out W, we obtain the dual action

$$\frac{1}{8\pi}\,\mathrm{Im}\int d^2\theta\left(-\frac{1}{\tau(A)}W_D^2\right)\,. \tag{10.30}$$

The conclusion of this analysis is the following: the effect of the duality transformation is to replace a gauge field which couples to electric charges by a dual gauge field which couples to magnetic charges, and at the same time the gauge coupling is transformed as

$$\tau \to \tau_D = -\frac{1}{\tau}\,. \tag{10.31}$$

This is the famous electric-magnetic duality of $U(1)$ gauge theory. Notice that it transforms weak coupling in strong coupling, and viceversa. Since the action is invariant under the shift $\tau \to \tau+1$, we can consider the group $\mathrm{Sl}(2,\mathbf{Z})$ generated by these two transformations:

$$\mathrm{Sl}(2,\mathbf{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} |a,b,c,d \in \mathrm{Z},\ \ ad-bc=1 \right\}. \tag{10.32}$$

This is the full duality group of our theory, and acts on τ as

$$\tau \to \frac{a\tau+b}{c\tau+d}. \tag{10.33}$$

The transformations $\tau \to -1/\tau$ and $\tau \to \tau+1$ are implemented by the matrices

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{10.34}$$

In order to show that the effective theory (10.9) has this duality group, we have to analyze the first term in (10.9), which involves the chiral superfield A. Let us introduce $h(A) = \partial\mathcal{F}/\partial A$. In terms of this, $\tau(A) = \partial h(A)/\partial A$ and the scalar kinetic energy term becomes $\mathrm{Im}\int d^4\theta\, h(A)\bar{A}$. The dual theory corresponding to (10.31) is defined by a dual chiral field

$$A_D = h(A) = \frac{\partial\mathcal{F}}{\partial A}. \tag{10.35}$$

Under this transformation, the scalar kinetic energy term transforms to

$$\mathrm{Im} \int d^4\theta\, h(A)\bar{A} = \mathrm{Im} \int d^4\theta\, h_D(A_D)\bar{A}_D, \tag{10.36}$$

where $h_D(h(A)) = -A$, and therefore retains its form. This transformation defines in particular a *dual prepotential* $\mathcal{F}_D(A_D)$ through the relation:

$$h_D(A_D) = \frac{\partial \mathcal{F}(A_D)}{\partial A_D} = -A, \tag{10.37}$$

and therefore

$$-\frac{1}{\tau(A)} = -\frac{1}{h'(A)} = h'_D(A_D) = \tau_D(A_D), \tag{10.38}$$

and the dual coupling constant is related to the original coupling constant τ through an S-duality transformation (10.31). In conclusion, given the effective Lagrangian (10.9), which is written in an "electric frame," (*i.e.* in terms of the $U(1)$ photon of the underlying $SU(2)$ theory), one can write it through a duality transformation in a "magnetic" frame, *i.e.* in terms of a $U(1)$ gauge field that couples locally to magnetic monopoles. This transformation, induced by the matrix S, is usually called S-duality.

We can in fact consider the effect of the full $\mathrm{Sl}(2,\mathbf{Z})$ group on A and $\mathcal{F}$, or equivalently on A and $A_D = \partial\mathcal{F}/\partial A$. The transformation (10.33) implies that

$$\begin{pmatrix} A_D \\ A \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A_D \\ A \end{pmatrix} . \tag{10.39}$$

The transformation of $\mathcal{F}$ can be easily obtained from (10.39), or equivalently from

$$\begin{aligned} A'_D =& aA_D + bA\,, \\ A' =& cA_D + dA\,. \end{aligned} \tag{10.40}$$

The first equation can be integrated with respect to A' by using the second equation and the result is

$$\mathcal{F}' = \frac{1}{2}\beta\delta A^2 + \frac{1}{2}\alpha\gamma A_D^2 + \beta\gamma A A_D + \mathcal{F}\,. \tag{10.41}$$

Finally, notice that the Kähler metric on moduli space,

$$ds^2 = \mathrm{Im}\, \frac{da_D}{du}\frac{d\bar{a}}{d\bar{u}}\, du d\bar{u} = -\frac{i}{2}\left(\frac{da_D}{du}\frac{d\bar{a}}{d\bar{u}} - \frac{d\bar{a}_D}{d\bar{u}}\frac{da}{du}\right) du d\bar{u} \tag{10.42}$$

is manifestly $\mathrm{Sl}(2, \mathbf{Z})$ invariant. Duality of the effective action implies that the correct expression for the central charge for an (n_e, n_m) dyon must be

$$Z = a n_e + a_D n_m. \tag{10.43}$$

This can be easily seen as follows: by analyzing the coupling of hypermultiplets of electric charge n_e to the chiral field A in the effective action

$$\sqrt{2} n_e \widetilde{M} A M, \tag{10.44}$$

one easily finds the expected mass $M = \sqrt{2}|a n_e|$. However, by $\mathrm{Sl}(2, \mathbf{Z})$ duality, the mass of a monopole of charge n_m must be given by $M = \sqrt{2}|a_D n_m|$. This leads to (10.43) for an arbitrary dyon. Notice that in the semiclassical limit $a_D \sim \tau_{\mathrm{cl}} a^2$ one recovers (10.24).

The $\mathrm{Sl}(2, \mathbf{Z})$ covariance of the effective theory will be crucial for the structure of the Seiberg-Witten solution, and many of the quantities involved will behave as modular forms. Recall that F is a modular form of weight (n, m) if under the transformation (10.33) it behaves as

$$F \to (c\tau + d)^n (c\bar{\tau} + d)^m F. \tag{10.45}$$

For example, u is a modular form of weight $(0, 0)$, since it is the vev of a gauge-invariant operator (and as it will become clear later, it is a good global coordinate on moduli space). Therefore, it shouldn't depend on the description we use.

Exercise. *Some modular forms*

Derive (10.41), and use it to show that

$$\mathcal{F} - \frac{1}{2} a a_D \tag{10.46}$$

is a modular form of weight $(0, 0)$. Also prove that du/da is a modular form of weight $(-1, 0)$.

10.3. Elliptic curves

The exact solution of Seiberg-Witten is given in terms of quantities associated to an elliptic curve. This curve, also called *Seiberg-Witten curve*, plays a crucial role in the story, and therefore it is convenient to stop at this point and give a brief summary of the properties of elliptic curves that are needed. We will mainly follow the presentation given in [70][71] (see also [72]).

As it is well known, every algebraic curve of genus one can be written in the Weierstrass form [73],

$$Y^2 = 4X^3 - g_2 X - g_3 = 4(X - e_1)(X - e_2)(X - e_3), \tag{10.47}$$

where the coefficients g_2, g_3 are related to the roots e_i, $i = 1, 2, 3$ by the equations

$$g_2 = -4(e_2 e_3 + e_3 e_1 + e_1 e_2), \quad g_3 = 4 e_1 e_2 e_3. \tag{10.48}$$

The discriminant of the curve is

$$\Delta = 16 \prod_{i<j} (e_i - e_j)^2 = g_2^3 - 27 g_3^2. \tag{10.49}$$

To uniformize the curve, we use Abel's theorem, which states that an algebraic curve of genus one like (10.47) is of the form $\mathbf{C}/\Lambda$ for some lattice $\Lambda \subset \mathbf{C}$ with half-periods ω_1, ω_3, such that $\mathrm{Im}(\omega_3/\omega_1) > 0$. The map from $\mathbf{C}/\Lambda$ to the curve (10.47) is given by

$$\psi(z) = (\wp(z), \wp'(z)) = (X, Y), \tag{10.50}$$

where we consider (X, Y) as inhomogeneous coordinates in $\mathbf{CP}^2$, and the Weierstrass function $\wp(z)$ verifies the differential equation

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2 \wp(z) - g_3. \tag{10.51}$$

Under this correspondence, the half periods of the lattice Λ, ω_i, $i = 1, 2, 3$, $\omega_2 = \omega_1 + \omega_3$, are mapped to the roots $e_i = \wp(\omega_i)$ of the cubic equation in (10.47), and the differential dz on $\mathbf{C}/\Lambda$ is mapped to the abelian differential of the first kind dX/Y. The map (10.50) has an inverse given by

$$z = \psi^{-1}(p) = \int_\infty^p \frac{dX}{Y}, \tag{10.52}$$

which is defined modulo Λ. We can obtain an explicit expression for the inverse map (10.52) by doing the change of variable $t^2 = (e_1 - e_3)/(X - e_3)$, to obtain

$$z = -\frac{1}{\sqrt{e_1 - e_3}} F(\phi, k), \tag{10.53}$$

where $F(\phi, k)$ is the incomplete elliptic integral of the first kind, with modulus $k^2 = (e_2 - e_3)/(e_1 - e_3)$, and $\sin^2\phi = (e_1 - e_3)/(\wp(z) - e_3)$.

In fact, all these functions can be computed in terms of the roots e_i and elliptic functions. First of all we have the periods of the abelian differential dX/Y. We take the branch cut on the X-plane from e_3 to e_2, and from e_1 to infinity, so that the α_1 and α_2 cycles of the torus circle around $e_1 - e_2$ and aorund $e_3 - e_2$, respectively. Therefore, the periods of the abelian differential dX/Y are given by

$$
\begin{aligned}
2\omega_3 &= \oint_{\alpha_1} \frac{dX}{Y} = \int_{e_1}^{e_2} \frac{dX}{\sqrt{(X-e_1)(X-e_2)(X-e_3)}}, \\
2\omega_1 &= \oint_{\alpha_2} \frac{dX}{Y} = \int_{e_3}^{e_2} \frac{dX}{\sqrt{(X-e_1)(X-e_2)(X-e_3)}}.
\end{aligned}
\tag{10.54}
$$

Introducing now the complementary modulus $k'^2 = 1 - k^2$, we obtain a representation of the periods in terms of the complete elliptic integral of the first kind,

$$
\begin{aligned}
\omega_3 &= \frac{i}{\sqrt{e_1 - e_3}} K'(k), \\
\omega_1 &= \frac{1}{\sqrt{e_1 - e_3}} K(k),
\end{aligned}
\tag{10.55}
$$

where $K'(k) \equiv K(k')$. We will also need the Weierstrass ζ-function, which is defined by the equation $\zeta'(z) = -\wp(z)$. Because of this property, we have that

$$
\eta_i \equiv \zeta(\omega_i) = -\int_{\omega_j}^{\omega_2} dz\, \wp(z), \qquad i,j = 1,3,\ j \neq i,
\tag{10.56}
$$

hence their values at the half-periods can be computed in terms of complete elliptic integrals,

$$
\begin{aligned}
\eta_3 &= -\, i\frac{e_3}{\sqrt{e_1 - e_3}} K'(k) - i\sqrt{e_1 - e_3} E'(k), \\
\eta_1 &= -\, \frac{e_1}{\sqrt{e_1 - e_3}} K(k) + \sqrt{e_1 - e_3} E(k),
\end{aligned}
\tag{10.57}
$$

where $E'(k) \equiv E(k')$. These periods satisfy the so-called Legendre's relation,

$$
\eta_1 \omega_3 - \eta_3 \omega_1 = \frac{\pi i}{2}.
\tag{10.58}
$$

When one of the periods of the lattice goes to infinity, say ω_1, elliptic functions degenerate to trigonometric functions, and one can derive expressions for the roots e_i and $\zeta(\omega_1)$ in terms of ω_1:

$$
e_1 = e_2 = -\frac{e_3}{2} = -\frac{\pi^2}{12\omega_3^2}, \qquad \zeta(\frac{\omega_3}{2}) = \frac{\pi^2}{12\omega_3}.
\tag{10.59}
$$

After this lightning review of elliptic curves, we can already consider the Seiberg-Witten curve in some detail.

10.4. The exact solution of Seiberg and Witten

The exact answer for the low-energy effective action of pure $\mathcal{N}=2$ Yang-Mills theory was given by Seiberg and Witten in [5] in terms of an auxiliary elliptic curve which will play a very important role in what follows. First of all, we will use the normalization of [62]. To do that, we will rescale

$$a \to a/2, \quad \tau \to 2\tau \tag{10.60}$$

so that $\tau = \theta/\pi + (8\pi i/g^2)$. With these conventions, the Seiberg-Witten curve is

$$y^2 = x^3 - ux^2 + \frac{1}{4}x, \tag{10.61}$$

in units where $\Lambda = 1$. This elliptic curve describes topologically a torus. To further specify the solution, we need another ingredient, which is an abelian differential λ_{SW} on the elliptic curve, with the property that

$$\frac{d\lambda}{du} = \frac{\sqrt{2}}{8\pi}\frac{dx}{y}. \tag{10.62}$$

The exact expression for the LEEA is completely determined by the expressions:

$$a_D = \oint_{\alpha_1} \lambda, \quad a = \oint_{\alpha_2} \lambda. \tag{10.63}$$

This determines a, a_D in terms of u, and therefore determines implicitly the prepotential $\mathcal{F}(a)$ through $a_D = \partial\mathcal{F}/\partial a$.

Let us give a more explicit expression for these integrals. The first thing to do is to write the Seiberg-Witten curve in the Weierstrass form. This curve (as well as the generalizations to theories with matter presented in [62]) has the form:

$$y^2 = x^3 + Bx^2 + Cx + D, \tag{10.64}$$

where the coefficients B, C and D depend on the gauge-invariant parameter u and the dynamical scale of the theory Λ which we have set equal to one. To put this curve in the Weierstrass form, it suffices to redefine the variables as

$$y = 4Y, \qquad x = 4X - \frac{1}{3}B, \tag{10.65}$$

and the curve (10.64) has now the form given in (10.47) with

$$g_2 = -\frac{1}{4}\big(C - \frac{1}{3}B^2\big), \qquad g_3 = -\frac{1}{16}\big(D + \frac{2B^3}{27} - \frac{CB}{3}\big). \tag{10.66}$$

Notice that with the redefinition given in (10.65), the abelian differential of the first kind is $dX/Y = dx/y$.

Exercise. *The Seiberg-Witten curve in Weierstrass form*

Show that, for the SW curve (10.61), one has:

$$g_2 = \frac{1}{4}\left(\frac{u^2}{3} - \frac{1}{4}\right), \quad g_3 = \frac{1}{48}\left(\frac{2u^3}{9} - \frac{u}{4}\right), \tag{10.67}$$

and the determinant is:

$$\Delta = \frac{1}{4096}(u^2 - 1). \tag{10.68}$$

Also show that the roots e_i have the explicit expression:

$$e_1 = \frac{u}{24} + \frac{\sqrt{u^2-1}}{8}, \;\; e_2 = \frac{u}{24} - \frac{\sqrt{u^2-1}}{8}, \;\; e_3 = -\frac{u}{12}. \tag{10.69}$$

Of course, at this point this only holds up to permutation, but we will show in a moment that with the above conventions this is the right choice.

The next step is to obtain an explicit expression for the Seiberg-Witten abelian differential.

Exercise. *The Seiberg-Witten differential*

Show that

$$\lambda_{SW} = \frac{\sqrt{2}}{8\pi}\frac{dx}{y}(2u - 4x) \tag{10.70}$$

satisfies (10.62).

Notice that

$$\frac{da_D}{du} = \frac{\sqrt{2}}{4\pi}\omega_3, \quad \frac{da}{du} = \frac{\sqrt{2}}{4\pi}\omega_1, \tag{10.71}$$

and therefore

$$\tau = \frac{da_D}{da} = \frac{\omega_3}{\omega_1}, \tag{10.72}$$

i.e. the effective gauge coupling constant is the τ parameter of the elliptic curve. In particular, $\operatorname{Im}\tau > 0$, as required for positivity of the Kähler metric (10.11) on moduli space.

In order to compute a and a_D, we have to compute the integrals:

$$\begin{aligned} a_D =& \frac{\sqrt{2}}{\pi}\int_{\omega_1}^{\omega_2} dz\left(\frac{u}{6} - 4\wp(z)\right), \\ a =& \frac{\sqrt{2}}{\pi}\int_{\omega_3}^{\omega_2} dz\left(\frac{u}{6} - 4\wp(z)\right). \end{aligned} \tag{10.73}$$

Using (10.56) and $\zeta(\omega_2) = \zeta(\omega_1) + \zeta(\omega_3)$, we obtain

$$\begin{aligned} a_D =& \frac{\sqrt{2}}{\pi}\Big(4\zeta(\omega_3) + \frac{u}{6}\,\omega_3\Big), \\ a =& \frac{\sqrt{2}}{\pi}\Big(4\zeta(\omega_1) + \frac{u}{6}\,\omega_1\Big), \end{aligned} \tag{10.74}$$

Exercise. *Matone's relation*

Using Legendre's relation (10.58), show that [74]:

$$a\Big(\frac{da_D}{du}\Big) - a_D\Big(\frac{da}{du}\Big) = \frac{i}{\pi} \tag{10.75}$$

and integrate it to prove that

$$\mathcal{F} - \frac{1}{2} a a_D = -\frac{i}{2\pi} u. \tag{10.76}$$

This also confirms, in view of the exercise in subsection 9.2, that u is a modular form of weight $(0,0)$.

Now we will analyze in more detail the exact solution of Seiberg and Witten, by first checking that it reproduces the known weak-coupling behaviour, and then by looking at the strong coupling regime.

We first look at the weak coupling regime, *i.e.* the regime where $u \to \infty$. In this case, one finds:

$$e_1 = \frac{u}{6} - \frac{1}{16u} + \cdots, \quad e_2 = -\frac{u}{12} + \frac{1}{16u} + \cdots \tag{10.77}$$

Using now the expansions of the elliptic integral of the first kind for small k^2,

$$\begin{aligned} K(k) =& \frac{\pi}{2}\Big\{1 + \frac{1}{4}k^2 + \cdots\Big\}, \\ K'(k) =& \log\frac{4}{k} + \frac{1}{4}\Big(\log\frac{4}{k} - 1\Big)k^2 + \cdots, \end{aligned} \tag{10.78}$$

we find

$$\begin{aligned} \omega_3 =& \frac{2i}{\sqrt{u}}\Big(\log u + 3\log 2 + \mathcal{O}\Big(\frac{1}{u}\Big)\Big), \\ \omega_1 =& \frac{\pi}{\sqrt{u}}\Big(1 + \mathcal{O}\Big(\frac{1}{u}\Big)\Big), \end{aligned} \tag{10.79}$$

and

$$\tau(u) = \frac{2i}{\pi}\big(\log u + 3\log 2\big) + \mathcal{O}\Big(\frac{1}{u}\Big) \tag{10.80}$$

in agreement with the one-loop result.

Exercise. *a, a_D at weak coupling*

By using the expansion of the complete elliptic integral of the second kind,

$$\begin{aligned} E(k) =& \frac{\pi}{2}\Big\{1 - \frac{1}{4}k^2 + \cdots\Big\}, \\ E'(k) =& 1 + \frac{1}{2}\Big(\log\frac{4}{k} - \frac{1}{2}\Big)k^2 + \cdots, \end{aligned} \tag{10.81}$$

derive the leading terms of the expansions of $\zeta(\omega_{3,1})$:

$$\begin{aligned} \zeta(\omega_3) =& \frac{i}{6}\sqrt{u}\log u - \frac{i}{6}\sqrt{u} + \cdots, \\ \zeta(\omega_1) =& \frac{\pi}{12}\sqrt{u} + \cdots, \end{aligned} \tag{10.82}$$

and use this to check the weak coupling expressions

$$\begin{aligned} a =& \frac{\sqrt{2u}}{2}\Big(1 + \mathcal{O}\Big(\frac{1}{u}\Big)\Big), \\ a_D =& \frac{i}{\pi}\sqrt{2u}\Big(\log u + \mathcal{O}\Big(\frac{1}{u}\Big)\Big). \end{aligned} \tag{10.83}$$

Let us now analyze in more detail what happens at other points in the u-plane. There are clearly some special points in moduli space at which the curve (10.61) degenerates. This happens when two roots coincide, and geometrically they indicate that one of the cycles of the torus has "pinched" and the curve has a node. In the case of the SW curve, since the discriminant is (10.68), this happens at $u = \pm 1$, where $e_1 = e_2$. Let us first consider the point $u = 1$. Since

$$k^2 = \frac{u - \sqrt{u^2 - 1}}{u + \sqrt{u^2 - 1}} \tag{10.84}$$

is one at $u = 1$, the period ω_1 diverges. Using (10.59), it is easy to see that at this point

$$a_D = 0. \tag{10.85}$$

Exercise. *Strong coupling expansion*

Derive the expansion of a_D, a around the monopole point:

$$\begin{aligned} a_D =& \frac{i}{2}(u-1) + \mathcal{O}(u-1), \\ a =& -\frac{1}{4\pi}(u-1)\log(u-1) + \mathcal{O}(u-1). \end{aligned} \tag{10.86}$$

Show then that the dual coupling around $u = 1$ behaves as

$$\tau_D = -\frac{i}{2\pi}\log a_D + \frac{2i}{\pi}\log 2 - \frac{1}{4} + \mathcal{O}(a_D). \tag{10.87}$$

Using the general expression for the β-function of a $U(1)$ theory coupled to Weyl fermions of charges Q_f and scalars with charge Q_s

$$\beta(g) = \frac{g^3}{16\pi^2}\Big(\sum_f \frac{2}{3}Q_f^2 + \sum_s \frac{1}{6}Q_s^2\Big), \tag{10.88}$$

we see that the effective coupling (10.87) behaves as the coupling of $\mathcal{N}=2$ QED coupled to a massive $\mathcal{N}=2$ hypermultiplet of charge 1 and mass $\sim a_D$. Therefore, what is going on is the following: the $\mathcal{N}=2$ super Yang-Mills theory has in its spectrum a magnetic monopole with charge $(n_m, n_e) = (1,0)$ and mass $M = \sqrt{2}|a_D|$. In the semiclassical regime, when $u, a >> 1$, the monopole is rather massive, and its mass goes like $\sim |u|^{1/2} \log|u|$. However, as we go to strong coupling the monopole becomes lighter and precisely at $u=1$ it becomes exactly *massless*. The fact that the the SW solution has an apparent singularity at $u=1$ has now a simple explanation: to obtain the SW solution, one has to integrate out massive degrees of freedom. In particular, at generic points in the u-plane we have to integrate out as well the massive monopoles in the spectrum. However, integrating out a massless particle leads to singularities in the effective action (see the first exercise of this section), and at $u=1$ what we are seeing is precisely the singularity due to integrating out the massless monopole. This is shown very explicitly in (10.87), where we see a logarithmic singularity in the one-loop effective coupling as the mass of the monopole running around the loop goes to zero.

A similar story takes place at $u=-1$: there, what becomes massless is the dyon with quantum numbers $(n_e, n_m) = (1,1)$. Although this is commonly referred to as a dyon (and we will do the same), it is in fact a monopole [75]: due to Witten's effect [76], the effective electric charge is not n_e, but $q_e = n_e + \mathrm{Re}\,\tau n_m$. Using the SW exact solution, one can check that, at $u=-1$, $q_e = 0$, so the particle becoming massless at $u=-1$ is physically a monopole, as it should be in view of the $u \to -u$ symmetry.

The picture that emerges from the SW solution is then the following: the quantum moduli space of the $\mathcal{N}=2$ theory is still the u-plane, but with a corrected Kähler metric given by a nontrivial prepotential, and with a singular behavior not only at infinity, but also at $u=\pm 1$. On this u-plane there is a (flat) $\mathrm{Sl}(2,\mathbf{Z})$ vector bundle, and (a_D, a) can be regarded as a section of this bundle. Different choices of the section are related by modular transformations. There is a subgroup of $\mathrm{Sl}(2,\mathbf{Z})$ that has a special significance, and it is called the *monodromy group*. This arises as follows: the u-plane has three punctures, at $u_* = \infty, 1, -1$, which correspond to the three physical singularities (weak coupling, massless monopole and massless dyon). Since we have a flat bundle, the homotpy group of the u-plane (which is generated by the nontrivial one-cycles around the punctures) gives a subgroup of the structure group of the bundle $\mathrm{Sl}(2,\mathbf{Z})$. This is the monodromy group, and it is also called the congruence subgroup of $\mathrm{Sl}(2,\mathbf{Z})$ associated to the SW elliptic curve.

Exercise. *Monodromies*

By studying the effect on a_D, a of $u \to \mathrm{e}^{2\pi i}u$, $u-1 \to \mathrm{e}^{2\pi i}(u-1)$, show that the monodromies at $u=\infty$ and $u=1$ are given by

$$M_\infty = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}. \tag{10.89}$$

The monodromy M_{-1} at $u=-1$ can be obtained from the requirement that $M_1 M_{-1} = M_\infty$.

The matrices (10.89) generate the congruence subgroup $\Gamma^0(4)$ of $\mathrm{Sl}(2,\mathbf{Z})$, which is defined by

$$\Gamma^0(4) = \Big\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2,\mathbf{Z}) : \, b \equiv 0 \bmod 4 \Big\}. \tag{10.90}$$

This is the congruence subgroup of the Seiberg-Witten curve (10.61).

Since the SW solution tells us the exact prepotential, we can write the LEEA at any point in moduli space. At a generic point of the u-plane, the low energy degrees of freedom are the fields of an $\mathcal{N}=2$ vector multiplet, and the LEEA is given by (10.9). However, near $u=1,-1$ this action becomes singular because we are integrating out a multiplet which is becoming massless. To cure the singularity, we have to add this multiplet to the effective action. Suppose for example that we are near $u=1$. The monopole hypermultiplet couples locally to the dual $\mathcal{N}=2$ vector multiplet $\mathcal{A}_D$. Therefore, the LEEA will be the action of a $U(1)$ $\mathcal{N}=2$ theory coupled to an $\mathcal{N}=2$ hypermultiplet with charge 1. The Lagrangian is then (10.9) written in terms of the $\mathcal{A}_D$, together with

$$\int d^2\theta d^2\bar{\theta}\{M^\dagger e^{2V_D} M + \widetilde{M}^\dagger e^{-2V_D}\widetilde{M}\} + \sqrt{2}\Big(\int d^2\theta \widetilde{M} A_D M + \text{h.c.}\Big). \tag{10.91}$$

There is however an important subtlety: the LEEA in magnetic variables is determined by the dual prepotential $\mathcal{F}_D(a_D)$. This prepotential is obtained after integrating out the monopole, and this is what leads to a logarithmic divergence in the τ_D coupling near $u=1$. If we do *not* integrate out the monopole, but we include it in the action, the prepotential should be modified (and in particular the singularity of the modified prepotential should be smoothed out). The corrected prepotential $\widetilde{\mathcal{F}}_D(a_D)$ turns out to be determined by [3]

$$\frac{d^2}{da_D^2}\widetilde{\mathcal{F}}_D(a_D) = \tau_D - \frac{1}{2\pi i}\log\, a_D, \tag{10.92}$$

and it is smooth. We will justify this in section 12.

Exercise. *Smoothing out singularities in softly broken $\mathcal{N}=2$ super Yang-Mills*

The smoothing out of the singularities once the monopole field shows up very clearly in some models with soft supersymmetry breaking. In these models a scalar potential on the u-plane is generated, which has the form [77]:

$$V_u = \left(\frac{b_{01}^2}{b_{11}} - b_{01}\right) f^2, \tag{10.93}$$

where

$$b_{ij} = \frac{1}{4\pi}\mathrm{Im}\tau_{ij}, \tag{10.94}$$

and the generalized couplings are defined as follows:

$$\tau_{11} = \tau, \quad \tau_{01} = \tau_{10} = \frac{1}{\pi}\frac{du}{da}, \qquad \tau_{00} = \frac{i}{\pi}\big(2u - a\frac{du}{da}\big). \tag{10.95}$$

a) Show that V_u is invariant under $\mathrm{Sl}(2,\mathbf{Z})$ transformations (*i.e.* it is a modular form of weight $(0,0)$). After writing it in magnetic variables, analyze the singularity at $u=1$. Notice that, even though V_u is finite at $u=1$, its first derivatives diverge.

b) If one adds the monopole fields M, $\widetilde{M}$ near $u=1$ (without modifying the prepotential), one gets some extra terms

$$\begin{aligned} V_M =& 2|a_D|^2(|m|^2+|\tilde{m}|^2) + \frac{1}{2b_{11}}(|m|^2+|\tilde{m}|^2)^2 \\ &+ \frac{\sqrt{2}fb_{01}}{b_{11}}(m\tilde{m} + \overline{m}\overline{\tilde{m}}), \end{aligned} \tag{10.96}$$

where the b_{ij} are computed here in the magnetic frame. Minimize the potential with respect to the monopole fields. Show that $|m| = |\tilde{m}|$, and that $|m| \neq 0$ at $u=1$ (*i.e.* the monopole field condenses at $u=1$). Plugging the value of the condensate in V_M, show that the total potential $V_u + V_M$ is smooht at $u=1$. Finally, compute the total potential with the modified prepotential (10.92) and check that you get the same answer. Therefore, at least in this situation, incorporating the monopole hypermultiplet into the action produces the corrected prepotential automatically.

10.5. The Seiberg-Witten solution in terms of modular forms

In this subsection, we give some extra details on the Seiberg-Witten curve which will be important in the analysis of the u-plane integral.

So far we have parametrized the quantum moduli space by the u-variable. Since this is moduli space describes a family of elliptic curves, we can parametrize it in terms of the τ modulus of the tori. Of course, there is an infinite family of frames that we can choose.

In this subsection τ will denote the electric frame, which is the most convenient one for the semiclassical region. Since $\mathrm{Im}\,\tau > 0$, τ lives in the upper half-plane of the complex plane H. However, two values of τ that are related by an element of the monodromy group of the curve should be regarded as equivalent. In order to find the moduli space of the elliptic curve, we then have to quotient H by the monodromy group. If the monodromy group was $\mathrm{Sl}(2, \mathbf{Z})$, we could take the standard fundamental domain

$$\mathcal{F} = \{\tau \in H : -\frac{1}{2} \leq \mathrm{Re}\tau \leq \frac{1}{2}, \, |z| \geq 1\} \tag{10.97}$$

as our moduli space. However, the monodromy group of an elliptic curve is usually a subgroup of $\mathrm{Sl}(2, \mathbf{Z})$. In the case of the Seiberg-Witten curve, we have seen that the monodromy group is $\Gamma^0(4)$. This is a congruence subgroup of $\mathrm{Sl}(2, \mathbf{Z})$ of index 6, and $\mathrm{Sl}(2, \mathbf{Z})$ can be written as a union of cosets:

$$\mathrm{Sl}(2, \mathbf{Z}) = \cup_{i=1}^{6} \alpha_i^{-1} \Gamma^0(4), \tag{10.98}$$

where

$$\begin{aligned} &\alpha_1 = \mathbf{1}, \;\; \alpha_2 = T, \;\; \alpha_3 = T^2, \\ &\alpha_4 = T^3, \;\; \alpha_5 = S, \;\; \alpha_6 = T^2 S. \end{aligned} \tag{10.99}$$

It is not difficult to prove ([78], p. 105) that $\cup_{i=1}^{6} \alpha_i \mathcal{F}$ is a fundamental domain for $\Gamma^0(4)$. We then find that the moduli space of the Seiberg-Witten curve (when parametrized by τ) is given by the fundamental domain of $\Gamma^0(4)$:

$$\mathcal{F} \cup T \cdot \mathcal{F} \cup T^2 \cdot \mathcal{F} \cup T^3 \cdot \mathcal{F} \cup S \cdot \mathcal{F} \cup T^2 S \cdot \mathcal{F}. \tag{10.100}$$

The first four domains give the region of the cusp at $\tau \to i\infty$ and correspond to the semiclassical region. The region $S \cdot \mathcal{F}$ surrounds the cusp near $\tau = 0$ and will be referred to as the monopole cusp (notice that $\tau = 0$ is the value of the electric coupling constant at $u = 1$). The region $T^2 S \cdot \mathcal{F}$ surrounds the cusp near $\tau = 2$ and corresponds to the massless dyon at $u = -1$.

One can in fact write all the quantities involved in the SW solution in terms of (generalized) modular forms with respect to the congruence subgroup $\Gamma^0(4)$. Here, the action of the modular group is conceptually different from the action that we have discussed so far. The $\mathrm{Sl}(2, \mathbf{Z})$ action discussed in subsection 10.2 is a change of frame for each point in the moduli space, while the $\mathrm{Sl}(2, \mathbf{Z})$ transformations in (10.100) relate different regions of the moduli space.

In order to write the relevant quantities explicitly, the starting point is the relation between the roots of the Weierstrass cubic and the theta functions (see, for example, [79]):

$$
\begin{aligned}
e_1 &= \left(\frac{\pi}{2\omega_1}\right)^2 \frac{1}{3}(\vartheta_3^4(\tau) + \vartheta_4^4(\tau)), \\
e_2 &= \left(\frac{\pi}{2\omega_1}\right)^2 \frac{1}{3}(\vartheta_2^4(\tau) - \vartheta_4^4(\tau)), \\
e_3 &= -\left(\frac{\pi}{2\omega_1}\right)^2 \frac{1}{3}(\vartheta_2^4(\tau) + \vartheta_3^4(\tau)),
\end{aligned}
\tag{10.101}
$$

where the Jacobi theta functions are given in the appendix.

Exercise. *Roots and theta functions*

Use the explicit expressions for the periods and τ in the electric frame, at large u, to check that the identification between the roots and the theta functions given in (10.101) is the correct one

---

The next step is to find an expression for the period ω_1 and for u in terms of modular forms. This is easily done by using that $e_3 = -u/12$, and that

$$
e_2 - e_1 = \frac{1}{4}\sqrt{u^2 - 1} = \left(\frac{\pi}{2\omega_1}\right)^2 \vartheta_4^4(\tau). \tag{10.102}
$$

Exercise. *u and da/du in terms of modular forms*

Use the above equations for the roots to obtain:

$$
u = \frac{1}{2}\frac{\vartheta_2^4(\tau) + \vartheta_3^4(\tau)}{(\vartheta_2\vartheta_3)^2}, \quad h(\tau) = \frac{da}{du} = \frac{1}{2}\vartheta_2\vartheta_3. \tag{10.103}
$$

Notice that the sign ambiguities can be always resolved by looking at the large u-behavior.

From these explicit expressions, one can derive explicit expressions for all the quantities involved in the SW solution in terms of modular forms. The only extra ingredient is the relation between the value of the ζ-function at the half-period and the Eisenstein series,

$$
\zeta(\omega_1) = \frac{\pi^2}{12\omega_1} E_2(\tau), \tag{10.104}
$$

which gives

$$
a = \frac{1}{6}\left(\frac{2E_2 + \vartheta_2^4 + \vartheta_3^4}{\vartheta_2\vartheta_3}\right). \tag{10.105}
$$

Exercise. *Practice with modular forms*

Derive the above expression for a, as well as expressions for $du/d\tau$ and $u^2 - 1$ in terms of modular forms. By performing an S-duality transformation, derive expressions for the dual quantities listed in Appendix B (except for a_D, τ_D and q_D, these quantities are denoted by the subindex M, which refers to "monopole.")

11. The u-plane integral

In this section, we put together all the previous ingredients to compute the Donaldson-Witten generating function by using the effective theory. This computation involves in general an integration over the u-plane, which was first explained in detail by Moore and Witten in [3]. Although this integral is different from zero only when $b_2^+ = 1$, it turns out that it gives a very effective method to derive the expression for the Donaldson invariants in the general case. A short summary of the u-plane integral can be found in [80] and in [81]. The paper [82] addresses the same problems with a different point of view. The extension to the nonsimply connected case is also considered in [3][82], and it is worked out in detail in [25][83].

11.1. The basic principle (or, "Coulomb+Higgs=Donaldson")

Now that we have a low-energy description of $\mathcal{N} = 2$ Yang-Mills theory, we can use it to compute the path-integral of Donaldson-Witten theory, and therefore to compute the Donaldson invariants. In the physical theory, using the effective theory in the two-derivative approximation only gives approximate results, which are valid for energies much lower than the dynamical scale Λ. For example, the two-derivative approximation to the pion Lagrangian of QCD gives only the bare bones of the pion-pion scattering amplitude. The main difference of having a topological field theory is that the theory does not depend on the scale, and therefore the low energy approximation should be exact, without any further correction. We can then hope that the twisted two-derivative LEEA is all we need in order to compute the Donaldson invariants, and that we can compute Z_{DW} in terms of the effective theory.

What is then the general structure of the computations in the low-energy theory? In a physical theory in Minkowski space, one does not integrate over zero modes which are not normalizable. These modes are rather parameters that specify our theory. For example, the u-plane description that we have presented in the previous section gives in fact a family of theories parametrized by u, since in a noncompact space the zero mode of the scalar field u is not normalizable. However, on a compact manifold, a scalar zero mode *is* normalizable and we have to integrate over it. This means that the evaluation of Z_{DW} in the effective theory involves an integration over the u-plane, *i.e.* over the Coulomb branch of $\mathcal{N}=2$ super Yang-Mills theory.

However, there are two points in the u-plane where something special happens: these are the points $u=\pm 1$, where there are extra degrees of freedom becoming massless. At these points the contribution from the effective theory has to be different, since for example we will have to perform a path integral over these extra degrees of freedom. We then expect that Z_{DW} looks like

$$Z_{DW} = Z_u + Z_{u=1} + Z_{u=-1}, \tag{11.1}$$

where the first summand is the integral over the u-plane, and the second and third terms come from the localized contributions of the monopole and the dyon singularities. This basic structure can be summarized by the principle "Donaldson=Coulomb + Higgs." [84]. "Higgs" refers at the scalars in the monopole and dyon multiplet, of course. The rest of these lectures are devoted to spell (11.1) in detail.

11.2. Effective topological field theory on the u-plane

It turns out that it is more convenient to start with Z_u. The reason is that (modulo some subtleties that we will explain in a moment) the computation of Z_u involves just the twisted version of the effective Lagrangian (10.9). This is done as in Donaldson-Witten theory, with the simplification that the theory described by (10.9) is abelian, but with the complication that it has an arbitrary prepotential (the abelian version of Donaldson-Witten theory corresponds to the prepotential $\mathcal{F}=a^2/2$). In any case, the $\overline{\mathcal{Q}}$ symmetry is precisely the abelian version of (8.5)(after the rescaling (10.60)). We can easily obtain from (10.14) the Lagrangian density:

$$\begin{aligned}\mathcal{L} = {} & \frac{i}{16\pi}\left(\bar\tau F_+\wedge F_+ + \tau F_-\wedge F_-\right) + \frac{1}{8\pi}\mathrm{Im}\tau da\wedge * d\bar a - \frac{1}{8\pi}(\mathrm{Im}\tau) D\wedge * D \\ & -\frac{1}{16\pi}\tau\psi\wedge * d\eta + \frac{1}{16\pi}\bar\tau\eta\wedge d*\psi + \frac{1}{8\pi}\tau\psi\wedge d\chi - \frac{1}{8\pi}\bar\tau\chi\wedge d\psi + \\ & + \frac{i\sqrt{2}}{16\pi}\frac{d\bar\tau}{d\bar a}\eta\chi\wedge(D_+ + F_+) - \frac{i\sqrt{2}}{2^7\pi}\frac{d\tau}{da}(\psi\wedge\psi)\wedge(F_- + D_+) \\ & + \frac{i}{3\cdot 2^{11}\pi}\frac{d^2\tau}{da^2}\psi\wedge\psi\wedge\psi\wedge\psi - \frac{\sqrt{2}i}{3\cdot 2^5\pi}\{\bar{\mathcal{Q}}, \frac{d\bar\tau}{d\bar a}\chi_{\mu\nu}\chi^{\nu\lambda}\chi_\lambda{}^\mu\}\sqrt{g}d^4x\end{aligned} \tag{11.2}$$

This can be also written in terms of the fourth descendant of the prepotential:

$$\begin{aligned}\mathcal{L} =& \frac{i}{6\pi} G^4 \mathcal{F}(a) + \frac{1}{16\pi}\{\overline{\mathcal{Q}}, \overline{\mathcal{F}}'' \chi(D+F_+)\} - \frac{i\sqrt{2}}{32\pi}\{\overline{\mathcal{Q}}, \overline{\mathcal{F}}' d * \psi\} \\ &- \frac{\sqrt{2}i}{3\cdot 2^5 \pi}\{\overline{\mathcal{Q}}, \overline{\mathcal{F}}''' \chi_{\mu\nu}\chi^{\nu\lambda}\chi_{\lambda}{}^{\mu}\}\sqrt{g} d^4 x\end{aligned} \tag{11.3}$$

Exercise. *The effective twisted theory*

Derive (11.2) by twisting (10.14), and show that it can be written as (11.3).

However, the effective topological theory has some extra terms that can not be derived from the physical theory (10.14) in Minkowski space. There are two terms of this kind: terms due to a coupling to the gravitational background, and contact terms of the two-observables. Let us discuss first the terms due to gravitational effects.

It is well known that effective field theories in curved space contain extra couplings to the background metric, involving the curvature tensor. In the case that the theory is topological, the extension of the theory to curved space is restricted by $\overline{\mathcal{Q}}$-invariance. We can have extra terms in the effective action of the type,

$$\int d^4 x \sqrt{g} T(u) F(g_{\mu\nu}), \tag{11.4}$$

where $F(g_{\mu\nu})$ is some functional of the metric. $\overline{\mathcal{Q}}$-invariance implies that $T(u)$ should be a holomorphic function of u (since $[\overline{\mathcal{Q}}, \overline{a}] \neq 0$). Also, the only topological invariants that can be written as local functionals of the metric are

$$\int_X \mathrm{Tr}(R \wedge *R), \quad \int_X \mathrm{Tr}(R \wedge R), \tag{11.5}$$

which are proportional to the Euler characteristic χ and the signature σ, respectively. Therefore, the effective action $\exp(-S_{\mathrm{eff}})$ will contain a factor

$$\mathrm{e}^{b(u)\chi + c(u)\sigma} \tag{11.6}$$

which can be regarded as a contribution to the effective measure of the low-energy theory. Fortunately, the couplings in (11.6) can be determined by holomorphy and anomaly considerations. The basic idea is the following: the microscopic theory has an anomalous $U(1)_R$ symmetry. This means that the quantum measure has a certain weight under a $U(1)_R$ transformations, which on a general four-manifold will depend on χ and σ. On the other hand, the effective measure of the low-energy theory is its canonical measure times the factor (11.6). Therefore, requiring that the effective measure reproduces the anomaly of the microscopical measure may fix the value of (11.6).

Let us see how this works in some detail. Consider the theory at large u. The effective theory, as we well know, is a $U(1)$ gauge theory. On the other hand, the underlying microscopic theory has an anomalous $U(1)_R$ symmetry. The weight of the measure under this symmetry is given by the index of the twisted Dirac operator associated to the complex (3.45):

$$-8k + \frac{3}{2}(\chi + \sigma). \tag{11.7}$$

In flat space, the anomaly is just $8k$, and this is what explained the spontaneous breaking of the $U(1)_R$ symmetry. The gravitational part of this anomaly is $\frac{3}{2}(\chi+\sigma)$: the $U(1)$ photon multiplet and the $W^{\pm}$ contribute $\frac{1}{2}(\chi + \sigma)$ each. However, the canonical measure of the effective theory only gives $\frac{1}{2}(\chi + \sigma)$, and the remaining anomaly should be reproduced by the effective interaction (11.6) (since it comes from integrating out the massive $W^{\pm}$ multiplets which carry the remaining anomaly). Since u has R-charge 4, we must have (for $u \to \infty$):

$$e^{b\chi + c\sigma} \sim u^{(\chi+\sigma)/4}. \tag{11.8}$$

Let us now consider the effective theory near $u = 1, -1$. Near $u = 1$, there is an extra light degree of freedom, the monopole. The measure of the twisted, effective theory including the monopole has the R-charge

$$\frac{1}{2}(\chi + \sigma) - \frac{c_1(L)^2}{4} + \frac{\sigma}{4}. \tag{11.9}$$

Again, the effective theory on the u-plane does not include the monopole, and therefore it misses the $\sigma/4$. Since the u-plane theory near $u = 1$ is obtained by integrating out the monopole, the effective measure (11.6) has to reproduce this anomaly. Remember that in the monopole theory a_D has R-charge 2, and since $a_D \sim u - 1$, we have, for $u \to 1$,

$$e^{b\chi + c\sigma} \sim (u - 1)^{\sigma/8}. \tag{11.10}$$

In the same way, we have that

$$e^{b\chi + c\sigma} \sim (u + 1)^{\sigma/8}, \tag{11.11}$$

for $u \to -1$. Looking at (11.8), (11.11) and (11.11), we see that the function

$$e^{c\sigma} = \beta^{\sigma}(u^2 - 1)^{\sigma/8}, \tag{11.12}$$

where β is a constant, satisfies our requirements. Notice that this holomorphic function is invariant under duality transformations (since u has modular weight $(0,0)$), and it involves in a natural way the discriminant of the SW curve.

The function $b(u)$ is trickier. It shouldn't have zeros or poles at $u = \pm 1$, since the anomaly at those points does not have any χ-dependence. It turns out that

$$\left(\frac{du}{da}\right)^{\chi/2} \tag{11.13}$$

satisfies all the requirements. Notice that this is not invariant under duality transformations: it transforms as a modular form of weight $-\chi/2$. When we choose the appropriate local coordinate, it is clear that du/da has not zeroes or poles at $u = \pm 1$ (for example, at the monopole point $du/da_D = -2i$). Since $u \sim a^2$ as $u \to \infty$, it is clear that (11.13) behaves as $u^{\chi/4}$ in the semiclassical region. We then have

$$\mathrm{e}^{b\chi} = \alpha^\chi \left(\frac{du}{da}\right)^{\chi/2}. \tag{11.14}$$

We will see later that in fact the modular weight of $\mathrm{e}^{b\chi}$ is the required one to have a consistent u-plane integral.

There is another topological term that we can add to the effective action. This is the coupling

$$\frac{i}{4}\int_X F \wedge w_2(X), \tag{11.15}$$

where $w_2(X)$ is interpreted as a gravitational background. We will see that in fact such a term is required for consistency of the u-plane integral (and this fixes the coefficient in (11.15)), and is the responsible for the transmutation of the line bundle of the magnetic theory into a Spin_c structure. It can be also derived directly, by integrating out the massive fermions [85][82].

The main result of this discussion is that the effective measure of the low energy field theory is given by the canonical measure, times

$$A^\chi B^\sigma = \alpha^\chi \beta^\sigma (u^2-1)^{\sigma/8} \left(\frac{du}{da}\right)^{\chi/2}. \tag{11.16}$$

The values of α, β can be obtained by comparing to mathematical results. It turns out that

$$\alpha^4 = -\frac{2}{\pi}, \quad \beta^4 = -\frac{16}{\pi}. \tag{11.17}$$

In addition to the measure and the action itself, we have to include the observables of the theory. The microscopical observables are obtained by topological descent on the operator $\mathcal{O} = \mathrm{Tr}(\phi^2)$. The corresponding observables in the effective theory can be obtained by using the canonical topological descent in the abelian theory, starting from the operator $2u$ (the 2 is chosen for the normalization to agree with the usual normalization in Donaldson theory). We then have

$$\begin{aligned}
&\mathcal{O} \to 2u \\
&I(\gamma) \to \widetilde{I}(\gamma) = \frac{a_1}{4\sqrt{2}} \frac{du}{da} \int_\gamma \psi, \\
&I(S) \to \widetilde{I}(S) = \frac{i}{\pi\sqrt{2}} \int_S G^2 u = \frac{i}{\pi\sqrt{2}} \int_S \left\{ \frac{1}{32} \frac{d^2u}{da^2} \psi \wedge \psi - \frac{\sqrt{2}}{4} \frac{du}{da} (F_- + D_+). \right\}
\end{aligned} \tag{11.18}$$

In this equation, we have rescaled the action of G in the way prescribed by (10.60). The overall normalizations of the observables are chosen again in order to agree with the normalizations in Donaldson theory, and

$$a_1 = \pi^{-1/2} 2^{3/4} \mathrm{e}^{-\frac{i\pi}{4}}. \tag{11.19}$$

When we go to low energies, the product of two microscopic operators $I(S_1)I(S_2)$ does not go necessarily to the product of the macroscopic operators, since the intersection points give singularities in the propagators that can induce extra contributions. Therefore, as it was anticipated in [86], in the low-energy theory we expect contact terms located at the intersection locus of the two-dimensional homology classes. We then have:

$$I(S_1)I(S_2) \to \widetilde{I}(S_1)\widetilde{I}(S_2) + \sum_{P \in S_1 \cap S_2} \epsilon_P T(P). \tag{11.20}$$

Here T is some operator that one should determine based on a series of requirements. For example, (11.20) must be $\overline{\mathcal{Q}}$-invariant, and since $\tilde{I}(S_1)\tilde{I}(S_2)$ is already $\overline{\mathcal{Q}}$-invariant, the operator T must be separately $\overline{\mathcal{Q}}$-invariant. This means that T is a holomorphic function of u, as the contributions to the effective measure that we determined above. We will see later that T must have a very precise behavior under $\mathrm{Sl}(2, \mathbf{Z})$ transformations, and using this we will be able to find it explicitly. The main conclusion of this short discussion is then that there is potentially in the effective theory a contact term with the form

$$\exp\bigl(T(u)S^2\bigr), \tag{11.21}$$

where we have written the geometric intersection in (11.20) as $(S, S) = S^2$.

Right now we have determined almost all the ingredients of the u-plane integral: the twisted action, the observables (together with their contact terms) and the effective measure. The net result is then:

$$
\begin{aligned}
Z_u =& 2\int [da\, d\bar{a}\, d\eta\, d\chi\, d\psi\, dD] A^{\chi} B^{\sigma} y^{-1/2} \\
&\times \exp\left[\frac{1}{8\pi}\int (\mathrm{Im}\tau) D\wedge * D\right] \exp\left[-\frac{i\pi}{16\pi}\bar{\tau}F_+^2 - \frac{i\pi}{16\pi}\tau F_-^2 + \frac{\pi i}{4}(F, w_2(X))\right] \\
&\times \exp\left[-\frac{i\sqrt{2}}{16\pi}\int \frac{d\bar{\tau}}{d\bar{a}}\eta\chi\wedge(D_+ + F_+) + \frac{i\sqrt{2}}{2^7\pi}\int \frac{d\tau}{da}(\psi\wedge\psi)\wedge(F_- + D_+)\right. \\
&\left. + \frac{1}{3\cdot 2^{11}\pi i}\int \frac{d^2\tau}{da^2}\psi\wedge\psi\wedge\psi\wedge\psi + 2pu + S^2 T(u) + \frac{i}{\sqrt{2}\pi}\int_S G^2 u + a_1\int_{\gamma} Gu + \cdots\right].
\end{aligned}
\tag{11.22}
$$

In the above expression we have skipped some terms (like the kinetic terms for the dynamic fields). The global factor of 2 is to correct a standard discrepancy between physical and mathematical computations of the invariants, since physicist divide by the order of the center of the gauge group in the Fadeev-Popov gauge fixing, while mathematicians don't.

In order to perform the above path-integral, we have to divide the fields into zero modes and quantum fluctuations:

$$
\begin{aligned}
&a = a_0 + a', \quad \psi = \psi_0 + \psi', \\
&\eta = \eta_0 + \eta', \quad \chi = \chi_0 + \chi' \\
&A = A_0 + A',
\end{aligned}
\tag{11.23}
$$

and then we have to determine the nature and the measure of the zero modes. The geometric content of the zero-modes is easy to obtain by looking at the kinetic terms of the effective theory: a_0 is just a constant, A_0 is a $U(1)$ connection with harmonic curvature, η_0 is a constant Grassmann variable, ψ_0 is given by a Grassmann variable times a harmonic one-form, and χ_0 is a Grassmann variables times a harmonic, self-dual two-form.

One would be tempted to follow the analysis we performed in Donaldson-Witten theory and localize the path integral on supersymmetric configurations. However, in this case this is not a good idea: the above theory is an effective theory with a $\mathrm{Sl}(2,\mathbf{Z})$ covariance, and by localizing on classical solutions we break this covariance. It turns out that life is almost as simple as in the Donaldson-Witten case: a careful analysis of the path-integral shows that, for $b_2^+ = 1$, the only contributions come from the zero modes [3]. This analysis is based on the topological invariance of the theory under global rescalings of the metric, $g \to t^2 g$. This analysis also shows that for $b_2^+ > 1$, there are too many zero-modes of the χ field to be soaked up, and the path-integral vanishes. Therefore,

$$Z_u = 0 \quad \text{for } b_2^+ > 1. \tag{11.24}$$

This does not mean that Donaldson-Witten theory is trivial for $b_2^+ > 1$: remember that Z_u is just the "Coulomb" contribution to the final answer. This vanishing result means that, for manifolds of $b_2^+ > 1$, the Donaldson invariants are given entirely in terms of the "Higgs" contribution $Z_{u=\pm 1}$. Another outcome of the analysis in [3] is that for $b_2^+ = 0$ the path integral has also one-loop contributions. The analysis of the u-plane integral for $b_2^+ = 0$ remains an interesting open problem.

Since the path-integral can be computed by zero-modes, we are dealing effectively with a finite integral. We just have to find the appropriate measure for the fields and perform the integral. Due to (11.24), we can restrict ourselves to manifolds with $b_2^+ = 1$. On these manifolds there is only one harmonic self-dual two-form ω (whose normalization is fixed by $\omega^2 = 1$), and we will write:

$$\chi = \chi_0 \omega + \chi', \quad \psi = \sum_{i=1}^{b_1} c_i \beta_i + \psi', \tag{11.25}$$

where the $\{\beta_i\}_{i=1,\cdots,b_1}$ is a basis for the harmonic one-forms. Notice that the kinetic terms for the fields in (11.2) are not canonical, since they involve a factor $\operatorname{Im}\tau$. Therefore, the measure includes a factor $(\operatorname{Im}\tau)^{1/2}$ for the commuting fields and $(\operatorname{Im}\tau)^{-1/2}$ for the anticommuting fields (this is just the Jacobian for the change of variables that takes us to canonical kinetic terms). After combining all the factors, we find that the measure for the zero-modes of the fields (except for the gauge field) is given by:

$$(\operatorname{Im}\tau)^{-b_1/2} da\, d\bar{a}\, d\eta_0\, d\chi_0 \prod_{i=1}^{b_1} dc_i. \tag{11.26}$$

11.4. Zero modes and topological sectors for the gauge field

It remains to discuss the zero modes for the gauge field. This is a little bit trickier. The solutions to the equations of motion, $dF = d * F = 0$, are harmonic two-forms. The space of collective coordinates for the zero modes will split into different topological sectors, corresponding to different line bundles T over the four manifold X, with $c_1(T) = F/2\pi \in H^2(X, \mathbf{Z})$. Now we have to remember that the gauge bundle we are considering is an $SO(3)$ bundle V with a nontrivial Stiefel-Whitney class $w_2(V)$. On the u-plane the nonabelian gauge symmetry is broken down to $U(1)$, and this means that the $SO(3)$ bundle V has the structure (3.33). Since by (3.36) $c_1(T)$ is congruent to $w_2(V)$ mod 2, we will write

$$F = 4\pi\lambda, \tag{11.27}$$

where $\lambda \in H^2(X, \mathbf{Z}) + \frac{1}{2}w_2(V)$.

For each topological sector specified by λ, the zero modes of the gauge field are the collective coordinates for the space of flat connections $F = 0$. This space is the Jacobian

$$\mathbf{T}^{b_1} = \frac{H^1(X, \mathbf{R})}{H^1(X, \mathbf{Z})}, \tag{11.28}$$

of dimension b_1, which can be also regarded as the moduli space of Wilson lines of the flat gauge connections. The measure is in this case much more subtle [85]. The Jacobi torus (11.28) has a canonical measure which is independent of τ. However, when one considers the nonzero modes of the gauge field, one has to introduce a factor $(\mathrm{Im}\,\tau)^{-1/2}$ for each non zero mode. Let us denote by B_1 and B_0 the number of one-forms and zero-forms on X. These numbers are in principle infinite, but we could make them finite through a lattice regularization. The number of nonzero modes would then be:

$$B_1 - B_0 + 1 - b_1. \tag{11.29}$$

Here we have subtracted from B_1 the number of nonzero modes which are pure gauge. There are $B_0 - 1$ of these, since the constant mode acts trivially on the gauge connection. We have also subtracted the number of zero modes, which is b_1. This gives in principle a factor:

$$(\mathrm{Im}\,\tau)^{\frac{1}{2}(b_1-1)}(\mathrm{Im}\,\tau)^{\frac{1}{2}(B_0-B_1)}. \tag{11.30}$$

In a local regularization of the theory one should eliminate the second factor. The final result for the measure is then [85]

$$(\mathrm{Im}\,\tau)^{\frac{1}{2}(b_1-1)} \prod_{i=1}^{b_1} dA_i, \tag{11.31}$$

where the A_i are the (commuting) modes of the gauge connection, and are therefore coordinates on the Jacobian (11.28). We can now write the total measure by putting together (11.26) and (11.31), to get:

$$\frac{1}{(\mathrm{Im}\,\tau)^{1/2}} da\, d\bar{a}\, d\eta_0\, d\chi_0 \prod_{i=1}^{b_1} dA_i dc_i. \tag{11.32}$$

The measure for the commuting and anticommuting collective coordinates for the Jacobi torus simply gives the usual measure for integrating differential forms on $\mathbf{T}^{b_1}$ (this is again a manifestation of the "basic tautology" [18] which says that functions on superspace = differential forms). We can then consider the c_i as a basis of one-forms $\beta_i^\sharp \in H^1(\mathbf{T}^{b_1}, \mathbf{Z})$, dual to $\beta_i \in H^1(X, \mathbf{Z})$. In this way we can identify

$$\psi = \sum_{i=1}^{b_1} \beta_i \otimes \beta_i^\sharp. \tag{11.33}$$

Finally, notice that the above identification leaves room for a factor C^{b_1}, that comes from the different normalization of the measures for the b_1 zero modes between the mathematical definition of Donaldson invariants and the one coming out from physics. It turns out that [25]:

$$C = 2^{9/4} \mathrm{e}^{\pi i/2}. \tag{11.34}$$

11.5. Evaluating the integral

We are now ready to perform the integral over the zero modes and the auxiliary fields. First of all, notice that the kinetic terms for the gauge field give the term

$$\exp\left(-i\pi\bar{\tau}(\lambda_+)^2 - i\pi\tau(\lambda_-)^2\right), \tag{11.35}$$

which, after summing over topological sectors, gives a generalization of a theta function sometimes called Siegel-Narain theta functions. These are generalizations of the usual theta functions and they include, among other things, a dependence on $\bar{\tau}$. They are very much like the generalized theta functions that one finds in Narain compactifications of heterotic string theory.

Exercise. *Lattice sums as modular forms*

Show that the theta function obtained by summing (11.35) over the lattice $H^2(X,\mathbf{Z})$ (we put $w_2(V)=0$ for simplicity) is a modular form of weight (b_2^-,b_2^+). The key ingredient in the proof is the Poisson resummation formula, which says that

$$\sum_{\vec{m}\in\mathbf{Z}^n} g(m) = \sum_{\vec{m}\in\mathbf{Z}^n} \hat{g}(m), \tag{11.36}$$

where

$$\hat{g}(y) = \int d^n x \mathrm{e}^{-2\pi i x\cdot y} g(x) \tag{11.37}$$

is the Fourier transform of g. In order to work out the details, it is useful to decompose the lattice vectors as

$$\lambda = \sum_{i=1}^{b_2} m_i e_i, \tag{11.38}$$

where m_i are integers and $\{e_i\}_{i=1,\cdots,b_2}$ is a basis of $H^2(X,\mathbf{Z})$. The function g turns out to be a Gaussian whose quadratic forms are proportional to the matrices

$$P_{ij}^{\pm} = (e_i,e_j)_{\pm} \tag{11.39}$$

which are projectors onto the $H^{2,\pm}(X,\mathbf{R})$.

In order to soak up the zero modes of χ and η, we have to bring down from the action the vertex

$$-\frac{i\sqrt{2}}{16\pi}\int_X \frac{d\bar{\tau}}{d\bar{a}}\eta\wedge\chi\wedge(F_+ + D_+). \tag{11.40}$$

Now we integrate out the auxiliary field D, which has a Gaussian action with linear term

$$-\frac{i}{4\pi}\frac{du}{da}\Big[(4\pi\lambda_- + D_+)\wedge\tilde{S}\Big] \tag{11.41}$$

where

$$\tilde{S} = S - \frac{\sqrt{2}}{32}\frac{d\tau}{du}\psi\wedge\psi. \tag{11.42}$$

Therefore, after integrating out D, the vertex becomes[2]

$$-\frac{\sqrt{2}}{16\pi}\int_X \frac{d\bar{\tau}}{d\bar{a}}\eta\wedge\chi\wedge(F_+ + i\frac{(du/da)}{\mathrm{Im}\tau}\tilde{S}_+), \tag{11.43}$$

[2] The D field is a Gaussian field with the wrong sign, and this introduces an extra factor of $-i$.

and we also get

$$\exp\left(\tilde{S}_+^2\left(\frac{(du/da)^2}{8\pi \mathrm{Im}\,\tau}\right)\right). \tag{11.44}$$

Now we can integrate the zero modes in the vertex insertion. Since F_+ coincides with $4\pi\lambda_+$, and $\chi = \chi_0\omega$, we find an overall factor

$$-\frac{\sqrt{2}}{4}\frac{d\bar{\tau}}{d\bar{a}}\cdot\exp\left(\tilde{S}_+^2\left(\frac{(du/da)^2}{8\pi y}\right)\right)\cdot\left((\lambda,\omega)+\frac{i}{4\pi y}\frac{du}{da}(\omega,S)\right), \tag{11.45}$$

where $\tau = x + iy$.

Before writing the final answer, we have to remember that the effective action includes an extra coupling to the "gravitational" background given by the second Stiefel-Whitney class of the four-manifold (11.15). In addition, there is an overall phase which encodes the dependence of the Donaldson invariants on a choice of orientation of the moduli space. This choice is specified by a choice of a lifting of $w_2(V)$ to $H^2(X,\mathbf{Z})$ [7]. We will specify such a lifting by an arbitrary and fixed $\lambda_0 \in \frac{1}{2}w_2(V)+\Gamma$. This overall sign, together with (11.15), give the phase

$$(-1)^{(\lambda-\lambda_0)\cdot w_2(X)}e^{2\pi i\lambda_0^2}. \tag{11.46}$$

There is no canonical choice of λ_0 (unless $w_2(V)=0$, in which case one takes $\lambda_0=0$). If λ_0 is replaced by $\tilde{\lambda}_0$, then (11.46) is multiplied by

$$(-1)^{\beta\cdot w_2(X)} \tag{11.47}$$

where β is the integral class $\beta = \lambda_0 - \tilde{\lambda}_0$. Thus, with the factor (11.46) included, the overall sign of the Donaldson invariants depends on a choice of λ_0, corresponding to a choice of orientation.

If we now define the following lattice sum:

$$\begin{aligned}\Psi = \exp\Big[-\frac{1}{8\pi y}\Big(\frac{du}{da}\Big)^2\tilde{S}_-^2\Big]e^{2\pi i\lambda_0^2}&\sum_{\lambda\in H^2+\frac{1}{2}w_2(V)}(-1)^{(\lambda-\lambda_0)\cdot w_2(X)}\\ \left[(\lambda,\omega)+\frac{i}{4\pi y}\frac{du}{da}(\tilde{S},\omega)\right]\cdot\times\exp&\left[-i\pi\bar{\tau}(\lambda_+)^2-i\pi\tau(\lambda_-)^2-i\frac{du}{da}(\tilde{S},\lambda_-)\right],\end{aligned} \tag{11.48}$$

where the sum over λ is the sum over topological sectors, we see that the final result of the integration is

$$Z = -\frac{\sqrt{2}}{4}\frac{d\bar{\tau}}{d\bar{a}}y^{-1/2}\exp\left(\tilde{S}^2\frac{(du/da)^2}{8\pi y}\right)\cdot\Psi. \tag{11.49}$$

Combining all the ingredients, we obtain the following expression for the u-plane integral as follows:

$$\begin{aligned} Z_u = \int_{\Gamma^0(4)\backslash H} \frac{dxdy}{y^{1/2}}\mu(\tau) \int_{\mathbf{T}^{b_1}} \exp\Big[& 2pu + \tilde{S}^2\hat{T}(u) + H(u)(\tilde{S},\psi^2) \\ & + \frac{a_1}{4\sqrt{2}}\frac{du}{da}\int_\gamma \psi + K(u)\psi^4\Big]\Psi(\tilde{S}), \end{aligned} \tag{11.50}$$

where

$$\begin{aligned} \mu(\tau) &= -\frac{\sqrt{2}}{2}\frac{da}{d\tau}A^\chi B^\sigma C^{b_1}, \\ \Psi(\tilde{S}) &= \exp(2i\pi\lambda_0^2)\exp\big[-\frac{1}{8\pi y}(\frac{du}{da})^2\tilde{S}_-^2\big] \\ &\quad\cdot \sum_{\lambda\in H^2+\frac{1}{2}w_2(V)} \exp\Big[-i\pi\bar{\tau}(\lambda_+)^2 - i\pi\tau(\lambda_-)^2 + \pi i(\lambda-\lambda_0, w_2(X))\Big] \\ &\quad\cdot \exp\big[-i\frac{du}{da}(\tilde{S}_-,\lambda_-)\big]\Big[(\lambda_+,\omega) + \frac{i}{4\pi y}\frac{du}{da}(\tilde{S}_+,\omega)\Big], \\ \hat{T}(u) &= T(u) + \frac{1}{8\pi \mathrm{Im}\tau}\left(\frac{du}{da}\right)^2, \\ H(u) &= \frac{i\sqrt{2}}{64\pi}\left(\frac{d^2u}{da^2} - 4\pi i\frac{d\tau}{du}T(u)\right), \\ K(u) &= -\frac{i}{3\cdot 2^{11}\pi}\left(\frac{d^2\tau}{da^2} - 6\frac{d\tau}{du}\frac{d^2u}{da^2} + 12\pi i(\frac{d\tau}{du})^2T(u)\right). \end{aligned} \tag{11.51}$$

Although the full modular properties of this integral will be explained in the next section, we can already use them to determine the contact term T, and therefore the unknown functions $H(u)$, $K(u)$. If we consider ψ as a modular form of weight $(1,0)$, it follows from (11.42) that $\widetilde{S}$ is a modular form of weight $(0,0)$. On the other hand, in order to have a well-defined behavior under $\mathrm{Sl}(2,\mathbf{Z})$, all the summands in the exponent of (11.50) should have the same modular weight, which we can read from the first term $2pu$ and is therefore $(0,0)$. It follows that $\hat{T}(u)$ must be a modular form of weight $(0,0)$ as well. Let us explore the consequences of this fact. Under an S transformation, the term $G(u) = (du/da)^2/8\pi\mathrm{Im}\tau$ in $\hat{T}(u)$ transforms inhomogeneously:

$$G(u) \to G(u) - \frac{i}{4\pi\tau}\left(\frac{du}{da}\right)^2. \tag{11.52}$$

It follows that T must also transform inhomogeneously, but with the opposite sign, in such a way that $\hat{T}$ is truly invariant. Under $\tau \to -1/\tau$ one must then have

$$T \to T + \frac{i}{4\pi\tau}\left(\frac{du}{da}\right)^2. \tag{11.53}$$

A comparison to the standard transformation law for the Eisenstein series $E_2(\tau)$ shows that these conditions are equivalent to the statement that

$$T = -\frac{1}{24} E_2(\tau) \left(\frac{du}{da} \right)^2 + H(u) \tag{11.54}$$

where H is modular invariant and so is an ordinary holomorphic function of u. However, there is an extra constraint that $T(u)$ must satisfy: since it vanishes at tree level, it must vanish in the semiclassical region $u \to \infty$

Exercise. *Contact term*

Show that (11.54) satisfies (11.53). By requiring that $T(u)$ vanishes semiclassically, prove that $H(u) = u/3$.

We then obtain the final expression for the contact term:

$$T = -\frac{1}{24} \left(E_2(\tau) \left(\frac{du}{da} \right)^2 - 8u \right). \tag{11.55}$$

A very elegant method to determine the structure of this and other contact terms has been proposed in [82] and applied in [25], but here we are not going to explore this issue. Using now the explicit expression for $T(u)$, one can easily show that $H(u)$ and $K(u)$ are modular forms of weight $(-2,0)$ and $(-4,0)$. Since ψ is a modular form of weight $(1,0)$, we see that all the terms in the exponent of (11.50) are in fact modular forms of weight $(0,0)$, as required.

Exercise. *Computing $H(u)$ and $K(u)$*

Define the functions $f_{1,2}$ by the following equations:

$$\frac{d^2u}{da^2} = 4 f_1(q), \qquad \frac{d\tau}{da} = \frac{16i}{\pi} f_2(q). \tag{11.56}$$

Show that they have the explicit expression

$$\begin{aligned} f_1(q) &= \frac{2E_2 + \vartheta_2^4 + \vartheta_3^4}{3\vartheta_4^8} = 1 + 24 q^{1/2} + \cdots, \\ f_2(q) &= \frac{\vartheta_2 \vartheta_3}{2\vartheta_4^8} = q^{1/8} + 18 q^{5/8} + \cdots. \end{aligned} \tag{11.57}$$

Use the above result to deduce that

$$\begin{aligned} H(u) &= \frac{\sqrt{2}}{32} u \frac{d\tau}{du} = \frac{i\sqrt{2}}{16\pi} \frac{\vartheta_2^4 + \vartheta_3^4}{\vartheta_4^8}, \\ K(u) &= \frac{7}{3 \cdot 2^{10}} u \left(\frac{d\tau}{du} \right)^2 = -\frac{7}{3 \cdot 2^7 \pi^2} \frac{(\vartheta_2 \vartheta_3)^2 (\vartheta_2^4 + \vartheta_3^4)}{\vartheta_4^{16}}. \end{aligned} \tag{11.58}$$

As a last step, we will write (11.50) in a more compact form. We explained before that this integral is both an integral over the fundamental domain of $\Gamma^0(4)$ and an integral of a differential form over the Jacobi torus $\mathbf{T}^{b_1}$ (remember that ψ is a one-form on this torus). This last aspect can be incorporated in a much more convenient way by using the following simple mathematical facts: first, on a manifold of $b_2^+ = 1$, for any β_1, β_2, β_3 and β_4 in $H^1(X, \mathbf{Z})$, one has $\beta_1 \wedge \beta_2 \wedge \beta_3 \wedge \beta_4 = 0$ (this means, in particular, that the last term in the exponent of (11.50) vanishes). Second, the image of the map

$$\wedge : H^1(X, \mathbf{Z}) \otimes H^1(X, \mathbf{Z}) \longrightarrow H^2(X, \mathbf{Z}) \tag{11.59}$$

is generated by a single rational cohomology class Λ. We introduce now the antisymmetric matrix a_{ij} associated to the basis β_i of $H^1(X, \mathbf{Z})$, $i = 1, \ldots, b_1$, as $\beta_i \wedge \beta_j = a_{ij}\Lambda$. Finally, we introduce the two-form on $\mathbf{T}^{b_1}$ as

$$\Omega = \sum_{i<j} a_{ij} \beta_i^\sharp \wedge \beta_j^\sharp, \tag{11.60}$$

which does not depend on the choice of basis. This is a volume element for the torus, hence

$$\mathrm{vol}(\mathbf{T}^{b_1}) = \int_{\mathbf{T}^{b_1}} \frac{\Omega^{b_1/2}}{(b_1/2)!}. \tag{11.61}$$

Notice that, for $SU(2)$ and $b_2^+ = 1$, b_1 has to be even for the Donaldson invariants to be different from zero.

Exercise.

Prove the above statements by looking at the map (11.59). Find Λ in the case $X = \Sigma_g \times \mathbf{P}^1$. *Hint*: Notice that, if $\beta_1 \wedge \beta_2 \wedge \beta_3 \wedge \beta_4 \neq 0$, then $H^2(X)$ contains the subspace generated by $\beta_i \wedge \beta_j$. The proof is in [32].

We can now write the u-plane integral in a more convenient way. If we define $\delta^\sharp = \sum_{i=1}^{b_1} \zeta_i \beta_i^\sharp$ as the image of δ in (3.58) under the isomorphism $H_1(X, \mathbf{Z}) \simeq H^1(\mathbf{T}^{b_1}, \mathbf{Z})$, we find the final expression:

$$Z_u = -4\pi i \int_{\Gamma^0(4)\backslash H} \frac{dxdy}{y^{1/2}} \int_{\mathbf{T}^{b_1}} h\widehat{f}(p, \delta, S, \tau, y)\Psi(\tilde{S}), \tag{11.62}$$

with

$$\widehat{f}(p, \delta, S, \tau, y) = \frac{\sqrt{2}}{64\pi} h^{b_1-3} \vartheta_4^\sigma f_2^{-1} \mathrm{e}^{2pu+S^2\hat{T}} \exp\Big[2f_1(S, \Lambda)\Omega + ih^{-1}\delta^\sharp\Big], \tag{11.63}$$

where $f_{1,2}$ were defined in (11.56). One can also write $\tilde{S}$ in a more compact way:

$$\tilde{S} = S - 16 f_2 h(\Lambda \otimes \Omega). \tag{11.64}$$

This gives the final expression of the u-plane integral. We now proceed to study its properties and what are its most immediate mathematical applications.

12. Properties and applications of the u-plane integral

12.1. Behavior under monodromy and duality

In order to understand better the modular properties of the u-plane integral and to write it in a more convenient way, it is useful to introduce the Siegel-Narain theta function of [3]. The formal definition is as follows. Let Λ be a lattice of signature (b_+, b_-). Let P be a decomposition of $\Lambda \otimes \mathbf{R}$ as a sum of orthogonal subspaces of definite signature:

$$P : \Lambda \otimes \mathbf{R} \cong \mathbf{R}^{b_+,0} \perp \mathbf{R}^{0,b_-} \tag{12.1}$$

Let $P_\pm(\lambda) = \lambda_\pm$ denote the projections onto the two factors. We also write $\lambda = \lambda_+ + \lambda_-$. With our conventions $P_-(\lambda)^2 \leq 0$.

Let $\Lambda+\gamma$ denote a translate of the lattice Λ. We then define the following Siegel-Narain theta function:

$$\begin{aligned}\Theta_{\Lambda+\gamma}(\tau,\alpha,\beta;P,\xi) \equiv& \exp[\frac{\pi}{2y}(\xi_+^2 - \xi_-^2)] \\ \sum_{\lambda\in\Lambda+\gamma} \exp\Big\{ i\pi\tau(\lambda+\beta)_+^2 &+ i\pi\bar\tau(\lambda+\beta)_-^2 + 2\pi i(\lambda+\beta,\xi) - 2\pi i(\lambda+\tfrac{1}{2}\beta,\alpha)\Big\} \\ =& e^{i\pi(\beta,\alpha)}\exp[\frac{\pi}{2y}(\xi_+^2 - \xi_-^2)] \\ \sum_{\lambda\in\Lambda+\gamma} \exp\Big\{ i\pi\tau(\lambda+\beta)_+^2 &+ i\pi\bar\tau(\lambda+\beta)_-^2 + 2\pi i(\lambda+\beta,\xi) - 2\pi i(\lambda+\beta,\alpha)\Big\}\end{aligned} \tag{12.2}$$

The main transformation law of this theta function is:

$$\Theta_\Lambda(-1/\tau,\alpha,\beta;P,\frac{\xi_+}{\tau}+\frac{\xi_-}{\bar\tau}) = \sqrt{\frac{|\Lambda|}{|\Lambda'|}}(-i\tau)^{b_+/2}(i\bar\tau)^{b_-/2}\Theta_{\Lambda'}(\tau,\beta,-\alpha;P,\xi) \tag{12.3}$$

where Λ' is the dual lattice. If there is a characteristic vector, call it w_2, such that

$$(\lambda,\lambda) = (\lambda,w_2) \bmod 2 \tag{12.4}$$

for all λ, then we have in addition:

$$\Theta_\Lambda(\tau+1,\alpha,\beta;P,\xi) = e^{-i\pi(\beta,w_2)/2}\Theta_\Lambda(\tau,\alpha-\beta-\tfrac{1}{2}w_2,\beta;P,\xi) \tag{12.5}$$

Let us write the u-plane integral in terms of this theta function. Here we follow closely [3]. We denote

$$\Theta = \kappa^{-(w_2(X),w_2(V))}\Theta_{H^2}(\tau,\tfrac{1}{2}w_2(X),\tfrac{1}{2}w_2(E);P_\omega,\xi) \tag{12.6}$$

with $\kappa = e^{2\pi i/8}$ and

$$\xi = \rho y \frac{d\bar{a}}{d\bar{u}} \omega + \frac{1}{2\pi} \frac{d\bar{u}}{d\bar{a}} \widetilde{S}_- . \tag{12.7}$$

It is easy to see that ξ has the behaviour under $\mathrm{Sl}(2, \mathbf{Z})$ prescribed by (12.3). We now introduce the auxiliary integral $\mathcal{G}(\rho)$

$$\mathcal{G}(\rho) \equiv \int_{\Gamma^0(4)\backslash H} \frac{dxdy}{y^{3/2}} \widehat{f}(p, S, \tau, y) \bar{\Theta}, \tag{12.8}$$

where $\widehat{f}(p, S, \tau, y)$ is the almost holomorphic modular form introduced in (11.63). The integral (12.8) is related to the u-plane integral as follows:

$$Z_u = (\widetilde{S}, \omega) \mathcal{G}(\rho) \Big|_{\rho=0} + 2 \frac{d\mathcal{G}}{d\rho} \Big|_{\rho=0} . \tag{12.9}$$

Denote the integrand of (12.8) by $\frac{dxdy}{y^2} \mathcal{J}$, where $\mathcal{J} = \hat{f} \cdot y^{1/2} \bar{\Theta}$. Remember now that the fundamental domain of $\Gamma^0(4)$ contains six copies of the fundamental domain $\mathcal{F}$. In order to write the u-plane integral explicitly, we have to map the integrand in these 6 regions to the domain $\mathcal{F}$, and therefore we get six functions:

$$\begin{aligned} \mathcal{J}_{(\infty,0)}(\tau) &\equiv \mathcal{J}(\tau) \\ \mathcal{J}_{(\infty,1)}(\tau) &\equiv \mathcal{J}(\tau+1) \\ \mathcal{J}_{(\infty,2)}(\tau) &\equiv \mathcal{J}(\tau+2) \\ \mathcal{J}_{(\infty,3)}(\tau) &\equiv \mathcal{J}(\tau+3) \\ \mathcal{J}_M(\tau) &\equiv \mathcal{J}(-1/\tau) \\ \mathcal{J}_D(\tau) &\equiv \mathcal{J}(2-1/\tau) \end{aligned} \tag{12.10}$$

(the subscript D refers to "dyon," and should not be confused with the subscript for dual quantities.) In general we will denote $\Gamma^0(4)$-modular forms F transformed as in (12.10) by F_I where

$$I = (\infty, 0), (\infty, 1), (\infty, 2), (\infty, 3), M, D \tag{12.11}$$

It is now straightforward to bring the integral to the form

$$\mathcal{G}(\rho) = \int_{\mathcal{F}} \frac{dxdy}{y^{3/2}} \sum_I \hat{f}_I(p, S, \tau) \bar{\Theta}_I \tag{12.12}$$

where

$$\Theta_I = \mathrm{e}^{i\phi_I} \Theta_{H^2}(\tau, \alpha_I, \beta_I; \xi_I) \tag{12.13}$$

are the transforms of the Siegel-Narain theta function implied by (12.10), and the ϕ_I are the appropriate phases. One can check that:

$$
\begin{aligned}
\mathrm{e}^{i\phi_{(\infty,n)}} =& \kappa^{-(n+1)(w_2(X),w_2(V))}, \quad n=0,1,2,3, \\
\mathrm{e}^{i\phi_M} =& \mathrm{e}^{i\phi_D} = \kappa^{-(w_2(X),w_2(V))},
\end{aligned}
\tag{12.14}
$$

and that

$$
\begin{aligned}
\alpha_{(\infty,0)} =& \frac{1}{2}w_2(X), & \beta_{(\infty,0)} =& \frac{1}{2}w_2(V), \\
\alpha_{(\infty,1)} =& -\frac{1}{2}w_2(V), & \beta_{(\infty,1)} =& \frac{1}{2}w_2(V), \\
\alpha_{(\infty,2)} =& -w_2(V) - \frac{1}{2}w_2(X), & \beta_{(\infty,2)} =& \frac{1}{2}w_2(V), \\
\alpha_{(\infty,3)} =& -\frac{3}{2}w_2(V) - \frac{1}{2}w_2(X), & \beta_{(\infty,3)} =& \frac{1}{2}w_2(V), \\
\alpha_M =& \frac{1}{2}w_2(V), & \beta_M =& -\frac{1}{2}w_2(X), \\
\alpha_D =& \frac{1}{2}w_2(V), & \beta_D =& w_2(V) + \frac{1}{2}w_2(X).
\end{aligned}
\tag{12.15}
$$

In order to show that the integrand of (12.12) is a modular form of weight $(0,0)$, one has to check that $\widehat{f}_I$ and Θ_I are in the same unitary representation of $\mathrm{Sl}(2,\mathbf{Z})$. The behavior of the $\widehat{f}_I$ under the generators T and S is as follows:

$$
\begin{aligned}
\widehat{f}_{(\infty,0)}(\tau+1) =& \widehat{f}_{(\infty,1)}(\tau), & \widehat{f}_{(\infty,0)}(-1/\tau) =& (-i\tau)^{\sigma/2}\widehat{f}_M(\tau), \\
\widehat{f}_{(\infty,1)}(\tau+1) =& \widehat{f}_{(\infty,2)}(\tau), & \widehat{f}_{(\infty,1)}(-1/\tau) =& (-i\tau)^{\sigma/2}\widehat{f}_{(\infty,3)}(\tau), \\
\widehat{f}_{(\infty,2)}(\tau+1) =& \widehat{f}_{(\infty,3)}(\tau), & \widehat{f}_{(\infty,2)}(-1/\tau) =& (-i\tau)^{\sigma/2}\widehat{f}_D(\tau), \\
\widehat{f}_{(\infty,3)}(\tau+1) =& \widehat{f}_{(\infty,0)}(\tau), & \widehat{f}_{(\infty,3)}(-1/\tau) =& (-i\tau)^{\sigma/2}\widehat{f}_{(\infty,1)}(\tau), \\
\widehat{f}_M(\tau+1) =& \kappa^\sigma \widehat{f}_M(\tau), & \widehat{f}_M(-1/\tau) =& (-i\tau)^{\sigma/2}\widehat{f}_{(\infty,0)}(\tau), \\
\widehat{f}_D(\tau+1) =& \kappa^\sigma \widehat{f}_D(\tau), & \widehat{f}_D(-1/\tau) =& (-i\tau)^{\sigma/2}\widehat{f}_M(\infty,2)(\tau).
\end{aligned}
\tag{12.16}
$$

Using (12.14) and (12.15), one can show that the Θ_I are in the same unitary representation than $\widehat{f}_I$, therefore the integrand is a modular invariant. To prove these properties, it is crucial to take into account Wu formula (2.21) and also that $w_2(X)^2 \equiv \sigma \bmod 8$, a consequence of (2.25).

Exercise. *Modular invariance of the u-plane integral*

Prove that the $\widehat{f}_I$ transform as stated in (12.16). Show that the different Θ_I are specified by (12.14) and (12.15), and then prove that Θ_I and $\widehat{f}_I$ transform in the same representation of $\mathrm{Sl}(2,\mathbf{Z})$.

As a corollary of the above analysis, one can show that for every I, $\widehat{f}_I \bar{\Theta}_I$ is invariant under the monodromy associated to the corresponding cusp, as expected. Let us consider for example the cusp $(\infty, 0)$. The monodromy is given by T^4, which maps $\widehat{f}_{(\infty,0)} \to \widehat{f}_{(\infty,0)}$ and $\Theta_{(\infty,0)} \to \Theta_{(\infty,0)}$, therefore the product is clearly invariant. If we now look at the monopole cusp, the monodromy in magnetic variables is given by T, and we see that $\widehat{f}_M$ and Θ transform with the same phase, so that $\widehat{f}_M \bar{\Theta}_M$ is invariant. In this analysis (as well as in the analysis of modular invariance), the presence of the coupling (11.15) is crucial for consistency. Just by looking at the monodromy at infinity one can show that this coupling must be there with the right coefficient, and this provides in fact a very easy way to obtain it that can be also generalized to more complicated situations (see for example [87].)

12.2. Wall crossing

We have seen that the u-plane integral has good properties in what respects duality transformations and monodromy invariance. However, this does not mean that it is well-defined: the integration region is noncompact, and the integrand is ill-behaved in general as $\tau \to \infty$. We then have to define the integral carefully, in such a way that it makes sense and that (hopefully) we recover the results of Donaldson theory.

The precise recipe was worked out in detail by Moore and Witten [3]: first, to obtain a Donaldson invariant of some given order, we expand Z_u to the required order in p and S. This gives an integral that computes a particular Donaldson invariant. To define that particular integral, we write $\tau = x + iy$, we perform the integral for $y < y_0$, for some cutoff y_0, and then take the limit as $y_0 \to \infty$ only at the end. A similar procedure is followed near the cusps at $u = \pm 1$, introducing the dual τ-parameters and integrating first over $\mathrm{Im}\, \tau_D < y_0$, before taking the limit as $y_0 \to \infty$. This procedure eliminates the infinities. Let us see this in some detail. Set $q = \exp(2\pi i \tau)$. Then the integral to a given order in p and S is a sum of terms, each of which is a power of y times a sum of the form

$$\sum_{\nu,\mu} q^\nu \bar{q}^\mu. \tag{12.17}$$

Although ν has no lower bound, μ is bounded below by zero. This is because negative exponents in (12.17) come only from factors in the integrand, such as u and $(d\tau/du)^{-\sigma/4}$, which are singular at the cusps. But these factors are holomorphic and so contribute to ν but not μ. Consider now an integral of the following form:

$$\lim_{y_0 \to \infty} \int_{y_1}^{y_0} \frac{dy}{y^c} \int_0^k dx \sum_{\nu,\mu} q^\nu \bar{q}^\mu, \tag{12.18}$$

where y_1 is an arbitrary lower cutoff. We want to find a prescription to compute this integral in such a way that it converges when $y_0 \to \infty$. The x integral runs from 0 to k where (for $\Gamma^0(4)$) $k = 4$ for the cusp at infinity, and $k = 1$ for the other cusps. If we now look at the integrand and to the definition of the function Ψ, we see that in all cases $c > 1$ or there are, for a *generic metric* on X, no terms with $\nu = \mu = 0$. If we now integrate first over x, we project the sum in (12.18) onto terms with $\nu = \mu$, and hence (as μ is non-negative) onto terms that vanish exponentially or, if $\nu = \mu = 0$, are constant at infinity. For a generic metric on X, the y integral converges as $y_0 \to \infty$, since all terms that have survived the x integral have $c > 1$ or $\nu, \mu > 0$. With this prescription, the integral becomes for a generic metric a well-defined formal power series in p, S.

However, for special metrics there are terms with $c = 1/2$ and $\nu = \mu = 0$. These terms are of the form

$$\mathcal{I}(\omega) \equiv \int_{\mathcal{F}} \frac{dxdy}{y^{1/2}} c(d) e^{2\pi i x d - 2\pi y d} e^{-i\pi x(\lambda_+^2 + \lambda_-^2)} e^{-\pi y(\lambda_+^2 - \lambda_-^2)} (\omega, \lambda) \tag{12.19}$$

for some integer d and some λ. In (12.19), $c(d)$ is the coefficient of some modular object. We want to study the integral in (12.19) for fixed λ as the decomposition $\lambda = \lambda_+ + \lambda_-$ varies. Since $\lambda_+ = \omega(\omega, \lambda)$, this decomposition is determined by the period point ω. According to the regularization procedure that we have just proposed, after doing the x integral one projects onto d such that $2d = \lambda^2$. For this value of d, the y integral looks like

$$\int_{y_1}^{\infty} \frac{dy}{y^{1/2}} c(\lambda^2/2) e^{-2\pi y \lambda_+^2} \lambda_+. \tag{12.20}$$

Notice first that if $\lambda^2 = \lambda_+^2 + \lambda_-^2$ is strictly positive, then $\lambda_+^2 > 0$, since $\lambda_-^2 \leq 0$. Therefore, if $\lambda^2 > 0$ the above integral is always convergent. Let us then analyze what happens when $\lambda^2 < 0$. In this case, there can be a discontinuous behavior at $\lambda_+ = 0$. The discontinuity as ω crosses from $(\omega, \lambda) = 0^-$ to $(\omega, \lambda) = 0^+$ comes from the large y part of the integral and is easily computed to be (for a single copy of the $SL(2, \mathbf{Z})$ fundamental domain)

$$\mathcal{I}(\omega_+) - \mathcal{I}(\omega_-) = \sqrt{2} c(d) = \sqrt{2} \big[q^{-\lambda^2/2} c(q) \big]_{q^0}. \tag{12.21}$$

The notation $[\cdot]_{q^0}$ indicates the constant term in a Laurent expansion in powers of q. Since $\lambda_+ = 0$ we may put $(S_+, \lambda_+) = 0$ and $(S_-, \lambda_-) = (S, \lambda)$ in the function $c(q)$. It is interesting to notice that, in order to compute the wall-crossing behaviour of the u-plane integral, we don't have to compute the integral itself, but we only need information about the integrand.

Let us look at the four cusps at infinity. In all cases one has $2\lambda \equiv w_2(V)$ mod 2, and we see that the conditions $\lambda^2 < 0$, $\lambda_+ = 0$ for a discontinuity in the integral are precisely the conditions for wall-crossing of the Donaldson invariants that we explained in (3.71) (with $\zeta = 2\lambda$). The general formula (12.21) gives

$$\begin{aligned} WC(\lambda) = & -\frac{i}{2}(-1)^{(\lambda-\lambda_0, w_2(X))} e^{2\pi i \lambda_0^2} \\ & \cdot \Bigg[q^{-\lambda^2/2} h_\infty(\tau)^{b_1-2} \vartheta_4^\sigma f_{2\infty}^{-1} \exp\Big\{ 2pu_\infty + S^2 T_\infty - i(\lambda, S)/h_\infty \Big\} \\ & \cdot \int_{\mathbf{T}^{b_1}} \exp\left(2f_{1\infty}(q)(S,\Lambda)\Omega + 16 i f_{2\infty}(q)(\lambda,\Lambda)\Omega + i\frac{\delta^\sharp}{h_\infty} \right) \Bigg]_{q^0}, \end{aligned} \tag{12.22}$$

where we have included a factor of 4 to take into account the contributions from the four cusps. This expression gives in fact the wall-crossing behaviour for the Donaldson invariants of manifolds with $b_1 \geq 0$. An obvious consequence of this expression is that the wall-crossing formula only depends on the cohomology ring of the four-manifold X. For simply-connected manifolds, the wall-crossing formula (12.22) reproduces the result of Göttsche in [31].

Exercise. *Exploring the wall-crossing formula*

a) Check that the wall-crossing term is different from zero only if $0 > \lambda^2 \geq p_1/4$, where p_1 is the Pontriagin number of the gauge bundle (and $p_1 \equiv w_2(E)^2$ mod 4).

b) Show that the wall-crossing term for the Donaldson invariant corresponding to $p^r S^{d-2r}$, where d is half the dimension of moduli space, and for walls with $\lambda^2 = p_1/4$, is given by:

$$\begin{aligned} WC_\zeta(p^r S^{d-2r}) = & \frac{1}{2}(-1)^{(\lambda-\lambda_0, w_2(X))} 2^{3b_1/2-b-d}(-1)^{r+d} p^r \mathrm{vol}(\mathbf{T}^{b_1}) \\ & \sum_{b=0}^{b_1/2} \frac{(b_1/2)!}{(b_1/2-b)!} \binom{d-2r}{b} (S,\zeta)^{d-2r-b}(S,\zeta)^b(\zeta,\Lambda)^{b_1/2-b}, \end{aligned} \tag{12.23}$$

where $\zeta = 2\lambda$. This expression was fist obtained by V. Muñoz in [32] (Theorem 13 of that paper).

c) Compute the wall-crossing for $\lambda^2 = p_1/4 + 1$. Compare with Theorem 15 of [32].

In the previous section, we have computed the wall-crossing formula only for the cusps at infinity. However, there is also wall-crossing at the monopole and the dyon cusps. Let us consider the monopole cusp in some detail (the results for the dyon cusp can be obtained in a very similar way). The first thing to notice is that, after performing the S-duality transformation, the lattice points λ live in $H^2(X, \mathbf{Z}) + w_2(X)/2$. This means that 2λ corresponds to a Spin_c-structure. The walls are still defined by

$$\lambda^2 < 0, \quad (\lambda, \omega) = 0, \tag{12.24}$$

and the wall-crossing term is easily computed as:

$$\begin{aligned} &\frac{i}{8} e^{2i\pi(\lambda_0\cdot\lambda+\lambda_0^2)} \Bigg[q_D^{-\lambda^2/2} h_M^{b_1-3} \vartheta_2^{8+\sigma} \exp\Big\{ 2pu_M - i(S,\lambda)/h_M + S^2 T_M(u) \Big\} \\ &\qquad \cdot \int_{\mathbf{T}^{b_1}} \exp\Big(2f_{1M}(S,\Lambda)\Omega + 16 i f_{2M}(\lambda,\Lambda)\Omega + \frac{i}{h_M}\delta^\sharp \Big) \Bigg]_{q_D^0}, \end{aligned} \tag{12.25}$$

where the "dual" modular forms are explicitly given by

$$\begin{aligned} u_M(q_D) &= \frac{\vartheta_3^4 + \vartheta_4^4}{2(\vartheta_3\vartheta_4)^2} = 1 + 32 q_D + 256 q_D + \cdots, \\ h_M(q_D) &= \frac{1}{2i}\vartheta_3\vartheta_4 = \frac{1}{2i}(1 - 4q_D + 4q_D^2 + \cdots), \\ f_{1M}(q_D) &= \frac{2E_2 - \vartheta_3^4 - \vartheta_4^4}{3\vartheta_2^8} = -\frac{1}{8}(1 - 6q_D + 24 q_D^2 + \cdots), \\ f_{2M}(q_D) &= \frac{\vartheta_3\vartheta_4}{2i\vartheta_2^8} = \frac{1}{2^9 i}(\frac{1}{q_D} - 12 + 72 q_D + \cdots), \\ T_M(q_D) &= -\frac{1}{24}\left(\frac{E_2}{h_M^2} - 8u_M\right) = \frac{1}{2} + 8q_D + 48 q_D^2 + \cdots, \end{aligned} \tag{12.26}$$

and $q_D = \exp(2\pi i \tau_D)$.

Exercise. *SW wall-crossing and dimension of the moduli space*

Compute the leading power of q_D in (12.25) for a wall given by λ. Deduce that monopole wall-crossing only takes place if

$$d_\lambda \geq 0, \tag{12.27}$$

where d_λ is the dimension of SW moduli space for the Spin_c structure given by 2λ.

The conditions for a discontinuity of the u-plane integral at the monopole cusp (12.24) and (12.27) are precisely the conditions for wall-crossing in the SW invariants (12.25). The picture that emerges from this analysis is the following: according to the basic principle "Donaldson= Coulomb+Higgs," the Donaldson-Witten generating functions is a sum of three terms:

$$Z_{DW} = Z_u + Z_{u=1} + Z_{u=-1}. \tag{12.28}$$

When there is a Donaldson wall, the u-plane integral jumps due to the behavior at the cusps at infinity. Therefore, this jump gives the Donaldson wall-crossing, as we saw in the previous subsection. When there is a SW wall, Z_u jumps at the monopole and dyon cusps, but since Z_{DW} has no wall-crossing at these walls (generically), the jump should be compensated by an identical jump but with opposite sign in $Z_{u=\pm 1}$. Since, as we will see in a moment, $Z_{u=\pm 1}$ involves the SW invariants, in order to have a cancellation of wall-crossings we better have the same conditions for the walls. Happily, and as we have just seen, this turns out to be the case.

As Moore and Witten realized, cancellation of wall-crossings can be used to determine the detailed form of $Z_{u=\pm 1}$. We saw in section 10 that the effective Lagrangian near $u=1$ involves both the $\mathcal{N}=2$ vector multiplet (in the dual frame) and the monopole hypermultiplet. This Lagrangian comes from a prepotential $\widetilde{\mathcal{F}}_D(a_D)$ and can be written (after twisting) as:

$$\begin{aligned} &\{\overline{\mathcal{Q}}, W\} + \frac{i}{16\pi}\tilde{\tau}_D F\wedge F + p(u)\mathrm{Tr} R\wedge R + \ell(u)\mathrm{Tr} R\wedge \tilde{R} \\ &- \frac{i\sqrt{2}}{32\cdot\pi}\frac{d\tilde{\tau}_D}{da_D}(\psi\wedge\psi)\wedge F + \frac{i}{3\cdot 2^7\pi}\frac{d^2\tilde{\tau}_D}{da_D^2}\psi\wedge\psi\wedge\psi\wedge\psi. \end{aligned} \tag{12.29}$$

The part of the Lagrangian involving the monopole hypermultiplet can also be written as a $\overline{\mathcal{Q}}$-exact term after twisting (this is essentially the matter piece in (8.9)), and has been included in W. Our normalizations are such that M, the monopole field, is a section of $L^{1/2}$ (the dual line bundle), and then F is the curvature of L (this can be checked by taking into account the rescaling (10.60), which also explains why some of the factors differ from (11.2)). If $\lambda = c_1(L)$, we then have $F = 4\pi\lambda$. The exponentiation of the terms involving the densities $\mathrm{Tr}R\wedge R$, $\mathrm{Tr}R\wedge\tilde{R}$ and $F\wedge F$ gives, after integration on X, the factors

$$P(u)^{\sigma/8} L(u)^{\chi/4} C(u)^{\lambda^2/2}, \tag{12.30}$$

where $C(u) = \mathrm{e}^{-2\pi i \tilde{\tau}_D}$. In the monopole theory, the ψ field is a one-form on the SW moduli space, and a one-form on the four-manifold X. We can then write it as

$$\psi = \frac{c}{2} \sum_{i=1}^{b_1} \nu_i \beta_i, \tag{12.31}$$

where $\beta_i \in H^1(X, \mathbf{Z})$, $i = 1, \ldots, b_1$ is the basis of one-forms considered before, the ν_i are the one-forms on moduli space defined in (4.23), and c is a normalization constant (in order to agree with mathematical normalizations, one has to put $c = 2^{-9/4}\pi^{-1/2} i$ [25]). Finally, we have to consider how to insert observables in this theory. This is just as in the u-plane theory, by using the descent procedure (again, since a_D is not rescaled, we hae to use the original G action (8.8)). Therefore, $p\mathcal{O} \to 2pu_M$, and

$$\exp(I(S)) \to \exp\left(-\frac{i}{4\pi} \int_S \frac{du}{da_D} F + S^2 T_M(u) + \frac{i}{8\sqrt{2}\pi} \frac{d^2u}{da_D^2} \int_S \psi \wedge \psi \right). \tag{12.32}$$

Notice that, since a_D is invariant under the rescaling (10.60), we have to use the original G action in (8.8) (in contrast to (11.18)). Finally, we also have:

$$I_1(\delta) \to -\frac{\sqrt{2} a_1}{4} \frac{1}{h_M} \int_\delta \psi. \tag{12.33}$$

Let us now evaluate the contribution from the monopole theory. As we discussed in 9.3, the path integral reduces to an integral over the space of collective coordinates for the classical configurations, which is nothing but the SW moduli space. The observables, together with the measure factors and the terms in the Lagrangian that are no $\overline{\mathcal{Q}}$-exact, give a function of a_D and the ν_i. a_D is nothing but the operator ϕ in (4.22), and ψ gives insertions of ν_i operators. The contribution to $Z_{u=1,\lambda}$ from the Spin$_c$-structure associated to λ is given by:

$$\begin{aligned} Z_{u=1,\lambda} = \int_{\mathcal{M}_\lambda} & 2e^{2i\pi(\lambda_0^2 - \lambda_0 \cdot \lambda)} C(u)^{\lambda^2/2} P(u)^{\sigma/8} L(u)^{\chi/4} \\ & \cdot \exp\big(2pu_M + i(S,\lambda)/h_M + S^2 T_M(u)\big) \exp\Big[c^2 (P_M(u), \Lambda) \sum_{i,j=1}^{b_1} a_{ij} \nu_i \nu_j \Big], \end{aligned} \tag{12.34}$$

where

$$P_M(u) = \frac{i\sqrt{2}}{16\pi} f_{1M} S + \frac{i\sqrt{2}}{32} \frac{d\widetilde{\tau}_D}{da_D} \lambda, \tag{12.35}$$

and we have taken into account that $da_D/du = -h_M$. The factor of 2 comes from the same normalization issue that we discussed in (11.22). The overall phase depending on λ_0 can be obtained by performing carefully the duality transformation of the effective Lagrangian in the presence of a nontrivial Stiefel-Whitney class [85]. As we will see in a moment, it is clearly required to match the wall-crossings. In the above expression we have included, for simplicity, only the four and two-observables. The integral (12.34) can be evaluated in terms of SW invariants by expanding the exponential, and extracting the different powers of a_D and the ν_i. By doing this we find:

$$
\begin{aligned}
Z_{u=1,\lambda} = \sum_{b=0}^{b_1/2} \frac{1}{b!} \mathrm{Res}_{a_D=0} \Big[& 2e^{2i\pi(\lambda_0^2-\lambda_0\cdot\lambda)} C(u)^{\lambda^2/2} P(u)^{\sigma/8} L(u)^{\chi/4} \\
& \cdot \exp\left(2pu_M + i(S,\lambda)/h_M + S^2 T_M(u)\right) a_D^{-d_\lambda/2+b-1} (P_M(u),\Lambda)^b \Big] \qquad (12.36) \\
& c^{2b} \cdot \sum_{i_p,j_p=1}^{b_1} a_{i_1 j_1} \cdots a_{i_b j_b} SW(\lambda, \beta_{i_1} \wedge \beta_{j_1} \wedge \cdots \wedge \beta_{i_b} \wedge \beta_{j_b}).
\end{aligned}
$$

Let us now compare to (12.25) (and use the fact that (12.25) is odd under the change $\lambda \to -\lambda$). We see that, in order to match (12.25) and (12.36), consistency requires that λ lives in the lattice $H^2(X, \mathbf{Z}) + w_2(X)/2$, so the dual line bundle $L^{1/2}$ is in fact a Spin_c structure. The reason behind this is again the crucial coupling (11.15), since after the S-duality transformation it generates the appropriate shift in the dual λ. The effect of (11.15) in the dual theory can be analyzed directly, as in [85].

We can now determine the unknown factors in (12.36) by requiring that the wall-crossing of the above expression matches the wall-crossing at the monopole cusp. This is easily done by considering first the simply-connected case, since in that case the SW invariant jumps by ± 1 in crossing a wall. We can then write $\chi = 4 - \sigma$ for manifolds of $b_2^+ = 1$, and by comparing the terms involving σ and λ^2, one finds:

$$
\begin{aligned}
C &= \frac{a_D}{q_D}, \\
P &= -\frac{4\vartheta_2(\tau_D)^8}{h_M^4} a_D^{-1}, \qquad (12.37) \\
L &= -\frac{2i}{h_M^2}.
\end{aligned}
$$

The first relation tells us that the gauge coupling $\tilde{\tau}_D$ appearing in (12.29) is given by

$$
\tilde{\tau}_D = \tau_D - \frac{1}{2\pi i} \log\, a_D, \qquad (12.38)
$$

and therefore it is smooth at the monopole cusp. This defines the prepotential $\widetilde{\mathcal{F}}_D(a_D)$ through the equation $\widetilde{\mathcal{F}}''_D(a_D) = \tilde{\tau}_D$, and finally proves our claim in (10.92). With this information we can already compute the remaining couplings in (12.29). Since

$$\frac{d\tau_D}{da_D} = -f_{2M}(q_D), \quad a_D = -\frac{i}{6}\frac{2E_2(\tau_D) - \vartheta_3^4 - \vartheta_4^4}{\vartheta_3\vartheta_4} = -\frac{1}{4}\frac{f_{1M}}{f_{2M}}, \tag{12.39}$$

one can derive:

$$P_M(u) = \frac{i\sqrt{2}}{32\pi}\left[2f_{1M}S - 16i\left(\frac{1+8f_{1M}}{8f_{1M}}\right)f_{2M}\lambda\right]. \tag{12.40}$$

Exercise. *Matching the wall-crossing at the monopole cusp*

a) Prove (12.37).

b) Derive the wall-crossing formula for the SW invariant when no insertions of the ν_i are made (which does not involve the normalization constant c introduced in (12.31)).

$$WC_\lambda = (-1)^{b_1/2}(\lambda, \Lambda)^{b_1/2}\mathrm{vol}(\mathbf{T}^{b_1}). \tag{12.41}$$

This formula was first obtained in [42][40].

c) Find the SW walls for a product ruled surface $\Sigma_g \times \mathbf{P}^1$.

This gives in principle all the information that is needed in order to write the SW contribution at $u = 1$ to the Donaldson invariants, for any four-manifold with $b_2^+ \geq 1$. For example, for a manifold with $b_2^+ = 1$, and after including the one-observables, one obtains:

$$\begin{aligned} Z_{u=1,\lambda} =& \frac{i^{b_1+1}}{8}\sum_{b\geq 0}\sum_{n=0}^{b}\frac{(-1)^n}{n!(b-n)!}2^{-6n-5b+b_1/2}e^{2i\pi(\lambda_0^2-\lambda_0\cdot\lambda)} \\ &\cdot\Bigg[q_D^{-\lambda^2/2}h_M^{b_1-3}\vartheta_2^{8+\sigma}\Big(\frac{f_{1M}}{if_{2M}}\Big)^{(b+n-b_1)/2}\exp\Big\{2pu_M + i(S,\lambda)/h_M + S^2T_M(u)\Big\} \\ &\left(2f_{1M}(S,\Lambda) - 16i\left(\frac{1+8f_{1M}}{8f_{1M}}\right)f_{2M}(\lambda,\Lambda)\right)^n\Bigg]_{q_D^0} \\ &\cdot\sum_{i_p,j_p=1}^{b_1}a_{i_1j_1}\cdots a_{i_nj_n}SW(\lambda, \beta_{i_1}\wedge\beta_{j_1}\wedge\cdots\wedge\beta_{i_n}\wedge\beta_{j_n}\wedge\delta_*^{b-n}), \end{aligned} \tag{12.42}$$

where δ_* denotes the image of δ under the map $\delta_i \to \beta_i$.

The contribution $Z_{u=-1}$ at $u = -1$ is identical to the contribution at $u = 1$, with the only difference that one has to use the modular forms

$$u_D = -u_M, \qquad h_D = ih_M, \qquad f_{1D} = f_{1M}, \qquad f_{2D} = if_{2M}, \qquad T_D = -T_M \tag{12.43}$$

and include an extra factor $\exp(-2\pi i\lambda_0^2)$. It is easy to check that

$$Z_{u=-1}(p,\zeta_i,v_i) = \mathrm{e}^{-2\pi i\lambda_0^2} i^{(\chi+\sigma)/4} Z_{u=1}(-p,i\zeta_i,-iv_i). \tag{12.44}$$

Exercise. *R-symmetry and the dyon contribution*

Prove (12.44) by using the unbroken R-symmetry. Notice that, although this is a symmetry on $\mathbf{R}^4$, it has a gravitational anomaly on an arbitrary four-manifold, and this explains the phase in (12.44) (see also [86]).

In the above analysis of the SW contribution there is a small subtlety that we would like to comment. When we twist the effective theory with a prepotential $\widetilde{\mathcal{F}}(a_D)$, the equations for supersymmetric configurations are really:

$$F^+_{\dot\alpha\beta} + \frac{8\pi i}{\mathrm{Im}\,\tilde\tau_D}\overline{M}_{(\dot\alpha}M_{\dot\beta)} = 0, \tag{12.45}$$

where $\tilde\tau_D$ is evaluated at $a_D = 0$. Of course, in order to recover the SW monopole equations, the precise value of $\mathrm{Im}\,\tilde\tau_D$ is irrelevant as long as it is finite and different from zero (since we can always reabsorb it in the monopole field). The SW solution shows indeed that $\mathrm{Im}\,\tilde\tau_D = \log 16/2\pi$ (see for example (10.87)), and everything is consistent.

Notice that, for a general four-manifold, the SW contributions at $u = \pm 1$ are rather complicated. For example, in a manifold with $b_2^+ > 1$, $b_1 > 0$ and which is not of simple type, the term ψ^4 in (12.29) gives a contribution. For a simply connected manifold (but not necessarily of SW simple type), the above expressions give [3]:

$$\begin{aligned} Z_{u=1,\lambda} = {} & \frac{SW(\lambda)}{16}\cdot e^{2i\pi(\lambda_0^2-\lambda_0\cdot\lambda)} \\ & \cdot\left[q_D^{-\lambda^2/2}\frac{\vartheta_2^{8+\sigma}}{a_D h_M}\left(-2i\frac{a_D}{h_M^2}\right)^{(\chi+\sigma)/4} \exp\bigl(2pu_M + i(\lambda,S)/h_M + S^2T_M\bigr)\right]_{q^0}. \end{aligned} \tag{12.46}$$

Let us now consider the case of manifolds of $b_2^+ > 1$ and of SW simple type. Since $Z_u = 0$, the Donaldson-Witten generating function is given by $Z_{DW} = Z_{u=1} + Z_{u=-1}$. The SW simple type condition means that the only contribution comes from basic classes λ with $d_\lambda = 0$, therefore we only have to take the leading terms in the q-expansions in (12.46). Using $u_M = 1+\cdots$, $T_M = 1/2+\cdots$, $h_M = 1/(2i)+\cdots$ and $a_D = 16iq_D+\cdots$ we find that (12.46) reduces to

$$(-1)^\Delta 2^{1+\frac{7\chi}{4}+\frac{11\sigma}{4}}\mathrm{e}^{2p+S^2/2}\mathrm{e}^{-2(S,\lambda)}\mathrm{e}^{2i\pi(\lambda_0^2-\lambda_0\cdot\lambda)}SW(\lambda). \tag{12.47}$$

We then obtain, after summing over all λ and taking into account (4.25) and (12.44):

$$Z_{DW} = 2^{1+\frac{7\chi}{4}+\frac{11\sigma}{4}} \sum_{\lambda} \mathrm{e}^{2i\pi(\lambda_0\cdot\lambda+\lambda_0^2)} \Big[\mathrm{e}^{2p+S^2/2} \mathrm{e}^{2(S,\lambda)} + i^{(\chi+\sigma)/4-w_2(V)^2} \mathrm{e}^{-2p-S^2/2} \mathrm{e}^{-2i(S,\lambda)} \Big] SW(\lambda). \tag{12.48}$$

This is the famous Witten's magic formula [4] which expresses the Donaldson invariants in terms of SW invariants [2]. The Donaldson series reads then,

$$\mathbf{D}^{w_2(V)}(S) = 2^{1+\frac{7\chi}{4}+\frac{11\sigma}{4}} \sum_{\lambda} \mathrm{e}^{2i\pi(\lambda_0\cdot\lambda+\lambda_0^2)} \mathrm{e}^{S^2/2} e^{2(S,\lambda)} SW(\lambda). \tag{12.49}$$

Comparing to the structure theorem of Kronheimer and Mrowka (3.63), we find that the characteristic elements κ_λ are precisely the SW basic classes 2λ, and the corresponding rational numbers a_λ are given by:

$$a_\lambda = 2^{1+\frac{7\chi}{4}+\frac{11\sigma}{4}} \mathrm{e}^{2i\pi(\lambda_0\cdot\lambda+\lambda_0^2)} SW(\lambda). \tag{12.50}$$

It also follows that the SW simple type condition implies the Donaldson simple type condition. We should mention that the relation (12.48) has been essentially proven mathematically in a series of papers by Feehan and Lenness [88]. The basic idea of the proof is to use the moduli space of $SO(3)$ non-abelian monopoles. This is the moduli space associated to twisted $\mathcal{N}=2$ supersymmetric $S0(3)$ Yang-Mills theory with a hypermultiplet in the fundamental representation, and it contains both Donaldson and SW moduli spaces.

Exercise. *Generalized simple type condition*

Suppose that X (with $b_2^+ > 1$) is not of SW simple type. Using (12.46), show that

$$\left(\frac{\partial^2}{\partial p^2} - 4 \right)^n Z_{DW} = 0, \tag{12.51}$$

for some n. A manifold that satisfies this condition is called of *generalized simple type*. What is the value of n? Does (12.51) still hold if $b_2^+ = 1$?

In conclusion, the u-plane integral provides a physical way of computing the Donaldson invariants of four-manifolds of $b_2^+ > 0$, not necessarily of SW simple type. In general it is the sum of two terms, the u-plane integral Z_u given for example in (11.62), and the SW contributions at $u = \pm 1$ (which is essentially given by the expression (12.34) together with its transform (12.44)). We should finally mention that there are important aspects of the u-plane integral that we have not discussed in these lectures, like the behavior under blowup, the vanishing properties and the explicit evaluations in certain chambers. These aspects are discussed in detail in [3]. The nonsimply connected case is further discussed in [25][83][89], and a short summary of the mathematical results that have been obtained in the context of the u-plane integral can be found in [90].

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Appendix A. Conventions for spinors

In this appendix we collect our conventions for spinors (both in Minkowski and Euclidean space). We follow almost strictly [44].

The Minkowski flat metric is $\eta_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$. We raise and lower spinor indices with the antisymmetric tensor $\epsilon_{\alpha\beta}$, $\epsilon_{\dot\alpha\dot\beta}$:

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta, \quad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta, \tag{A.1}$$

where the ϵ tensor is chosen as follows:

$$\epsilon_{21} = \epsilon^{12} = -\epsilon_{12} = -\epsilon^{21} = 1, \tag{A.2}$$

Contractions satisfy the perverse rule:

$$\psi^\alpha \phi_\alpha = -\psi_\alpha \phi^\alpha. \tag{A.3}$$

We define the matrices:

$$(\sigma^\mu)_{\alpha\dot\alpha} \equiv (1, \vec\sigma), \tag{A.4}$$

where $\vec\sigma$ are the Pauli matrices, and after raising indices we find

$$(\bar\sigma^\mu)^{\dot\alpha\alpha} = \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot\beta} = (1, -\vec\sigma). \tag{A.5}$$

The continuation from Minkowski space is made via $x^0 = -ix^4$, $p_0 = ip_4$. The conventions for Euclidean spinors are as follows:

$$\sigma^\mu_{\ \alpha\dot\beta} = (i, \vec\sigma), \quad \bar\sigma^{\mu\dot\alpha\beta} = (i, -\vec\sigma). \tag{A.6}$$

The following identities are useful:

$$\begin{aligned} \bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu =& - 2g^{\mu\nu}\delta^{\dot\alpha}_{\ \dot\beta}, \\ (\bar\sigma^\mu)^{\dot\alpha\beta}(\sigma^\mu)_{\gamma\dot\rho} =& - 2\delta^\beta_{\ \gamma}\delta^{\dot\alpha}_{\ \dot\rho}. \end{aligned} \tag{A.7}$$

The (A)SD projectors are

$$\begin{aligned} \sigma^{\mu\nu} =& \frac{1}{4}\left(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu\right), \\ \overline{\sigma}^{\mu\nu} =& \frac{1}{4}\left(\overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu\right), \end{aligned} \tag{A.8}$$

where $\sigma^{\mu\nu}_{\alpha\beta}$ is ASD, while $\overline{\sigma}^{\mu\nu}_{\dot\alpha\dot\beta}$ is SD. We have, explicitly:

$$\begin{aligned} \sigma^{\mu\nu} =& \begin{pmatrix} 0 & -\frac{i}{2}\sigma^3 & \frac{i}{2}\sigma^2 & \frac{i}{2}\sigma^1 \\ \frac{i}{2}\sigma^3 & 0 & -\frac{i}{2}\sigma^1 & \frac{i}{2}\sigma^2 \\ -\frac{i}{2}\sigma^2 & \frac{i}{2}\sigma^1 & 0 & \frac{i}{2}\sigma^3 \\ -\frac{i}{2}\sigma^1 & -\frac{i}{2}\sigma^2 & -\frac{i}{2}\sigma^3 & 0 \end{pmatrix}, \\ \overline{\sigma}^{\mu\nu} =& \begin{pmatrix} 0 & -\frac{i}{2}\sigma^3 & \frac{i}{2}\sigma^2 & -\frac{i}{2}\sigma^1 \\ \frac{i}{2}\sigma^3 & 0 & -\frac{i}{2}\sigma^1 & -\frac{i}{2}\sigma^2 \\ -\frac{i}{2}\sigma^2 & \frac{i}{2}\sigma^1 & 0 & -\frac{i}{2}\sigma^3 \\ \frac{i}{2}\sigma^1 & \frac{i}{2}\sigma^2 & \frac{i}{2}\sigma^3 & 0 \end{pmatrix}. \end{aligned} \tag{A.9}$$

We finally define:

$$v_{\dot\alpha\dot\beta} = \overline{\sigma}^{\mu\nu}_{\dot\alpha\dot\beta} v^+_{\mu\nu}. \tag{A.10}$$

This implies that, if we consider a self-dual tensor

$$F^+_{\mu\nu} = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & c & -b \\ -b & -c & 0 & a \\ -c & b & -a & 0 \end{pmatrix}, \tag{A.11}$$

one has

$$F_{\dot\alpha\dot\beta} = 2i \begin{pmatrix} c - ib & -a \\ -a & -c - ib \end{pmatrix}. \tag{A.12}$$

Appendix B. Elliptic functions and modular forms

In this appendix, we recall the definitions and properties of some of the elliptic functions and modular forms used in the lectures, and we list some useful formulae. More information can be found in [78][91] and specially in [79].

Given a lattice with half-periods ω, ω', one can define the following elliptic functions:

$$\begin{aligned} \sigma(z) =& z \prod{}' \left(1 - \frac{z}{s}\right) \mathrm{e}^{\frac{z}{s} + \frac{z^2}{2s^2}}, \\ \zeta(z) =& \frac{1}{z} + \sum{}' \left(\frac{1}{z-s} + \frac{1}{s} + \frac{z}{s^2}\right) = \frac{d}{dz} \log \sigma(z), \\ \wp(z) =& \frac{1}{z^2} + \sum{}' \left(\frac{1}{(z-s)^2} - \frac{1}{s^2}\right) = -\zeta'(z), \end{aligned} \tag{B.1}$$

where $s = 2m\omega + 2m'\omega'$, and m, m' are integers, and $'$ indicates that the sum is over all pairs of integers (m, m') except the $(0, 0)$. The above functions are called, respectively, sigma, zeta, and Weierstrass functions. In the limit in which one of the periods goes to infinity, say ω', they become elementary functions:

$$\begin{aligned} \sigma(z) =& \frac{2\omega}{\pi} \mathrm{e}^{\frac{1}{6}\left(\frac{\pi z}{2\omega}\right)^2} \sin \frac{\pi z}{2\omega}, \\ \zeta(z) =& \frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2 z + \frac{\pi}{2\omega} \cot \frac{\pi z}{2\omega}, \\ \wp(z) =& -\frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2 + \frac{\pi}{2\omega} \frac{1}{\sin^2 \frac{\pi z}{2\omega}}. \end{aligned} \tag{B.2}$$

The incomplete elliptic integrals of the first and the second kind are, respectively,

$$
\begin{aligned}
F(\phi,k) &= \int_0^{\sin\phi} \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}},\\
E(\phi,k) &= \int_0^{\sin\phi} \sqrt{\frac{1-k^2t^2}{1-t^2}}.
\end{aligned}
\tag{B.3}
$$

The complete elliptic integrals are defined by:

$$
K(k) = F(\frac{\pi}{2},k), \qquad E(k) = E(\frac{\pi}{2},k). \tag{B.4}
$$

The Eisenstein function of weight two E_2 is defined by

$$
E_2 = 1 - 24\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} = 1 - 24q + \cdots \tag{B.5}
$$

and transforms under $\mathrm{Sl}(2,\mathbf{Z})$ as follows:

$$
E_2(\tau) \to (c\tau+d)^2\left(E_2(\tau) + \frac{12c}{2\pi i(c\tau+d)}\right). \tag{B.6}
$$

Our conventions for the Jacobi theta functions are:

$$
\begin{aligned}
\vartheta_2 &= 2q^{1/8}\prod(1-q^n)(1+q^n)^2\\
&= \sum_{n\in\mathbf{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2} = 2q^{1/8} + \cdots,\\
\vartheta_3 &= \prod(1-q^n)(1+q^{n-\frac{1}{2}})^2\\
&= \sum_{n\in\mathbf{Z}} q^{\frac{1}{2}n^2} = 1 + 2q^{\frac{1}{2}} + \cdots,\\
\vartheta_4 &= \prod(1-q^n)(1-q^{n-\frac{1}{2}})^2\\
&= \sum_{n\in\mathbf{Z}} q^{\frac{1}{2}n^2}(-1)^n = 1 - 2q^{\frac{1}{2}} + \cdots,
\end{aligned}
\tag{B.7}
$$

and they have the following properties under modular transformations:

$$
\begin{aligned}
\vartheta_2(-1/\tau) =& \sqrt{\frac{\tau}{i}}\theta_4(\tau), \qquad & \vartheta_2(\tau+1) =& e^{i\pi/4}\theta_2(\tau),\\
\vartheta_3(-1/\tau) =& \sqrt{\frac{\tau}{i}}\theta_3(\tau), \qquad & \vartheta_3(\tau+1) =& \theta_4(\tau),\\
\vartheta_4(-1/\tau) =& \sqrt{\frac{\tau}{i}}\theta_2(\tau), \qquad & \vartheta_4(\tau+1) =& \theta_3(\tau).
\end{aligned}
\tag{B.8}
$$

The expressions of the quantities involved in the SW solution are, in terms of modular forms:

$$
\begin{aligned}
u &= \tfrac{1}{2}\frac{\vartheta_2^4+\vartheta_3^4}{(\vartheta_2\vartheta_3)^2} \\
u^2-1 &= \frac{1}{4}\frac{\vartheta_4^8}{(\vartheta_2\vartheta_3)^4} = \frac{\vartheta_4^8}{64h^4(\tau)} \\
\frac{du}{d\tau} &= \frac{\pi}{4i}\frac{\vartheta_4^8}{(\vartheta_2\vartheta_3)^2} \\
h(\tau) &= \frac{da}{du} = \tfrac{1}{2}\vartheta_2\vartheta_3
\end{aligned}
\tag{B.9}
$$

The first few terms in the q-expansions are:

$$
\begin{aligned}
u &= \frac{1}{8q^{1/4}}\left(1+20q^{1/2}-62q+216q^{3/2}+\cdots\right) \\
&= \frac{1}{8\,q^{\frac{1}{4}}}+\frac{5\,q^{\frac{1}{4}}}{2}-\frac{31\,q^{\frac{3}{4}}}{4}+27\,q^{\frac{5}{4}}-\frac{641\,q^{\frac{7}{4}}}{8}+\frac{409\,q^{\frac{9}{4}}}{2}+\cdots
\end{aligned}
\tag{B.10}
$$

$$
u_M(q_D) = 1+32\,q_D+256\,q_D^2+1408\,q_D^3+6144\,q_D^4+22976\,q_D^5+76800\,q_D^6+\cdots \tag{B.11}
$$

$$
\begin{aligned}
T(u) &= -\frac{1}{24}\left[\frac{E_2}{h(\tau)^2}-8u\right] \\
&= q^{1/4}-2\;q^{3/4}+6\;q^{5/4}-16\;q^{7/4}+37\;q^{9/4}-78\;q^{11/4} \\
&+158\;q^{13/4}-312\;q^{15/4}+594\;q^{17/4}+\cdots \\
T_M(q_D) &= \tfrac{1}{2}+8\;q_D+48\;q_D^2+224\;q_D^3+864\;q_D^4+2928\;q_D^5+9024\;q_D^6+\cdots
\end{aligned}
\tag{B.12}
$$

$$
\begin{aligned}
h &= \tfrac{1}{2}\vartheta_2\vartheta_3 = \frac{1}{4}\vartheta_2^2(\tau/2) \\
&= q^{1/8}+2\;q^{5/8}+q^{9/8}+2\;q^{13/8}+2\;q^{17/8}+3\;q^{25/8}+2\;q^{29/8}+\cdots \\
h_M &= \frac{1}{2i}\vartheta_3\vartheta_4 = \frac{1}{2i}\vartheta_4^2(2\tau_D) \\
&= \frac{1}{2i}(1-4q_D+4q_D^2+4q_D^4-8q_D^5+\cdots)
\end{aligned}
\tag{B.13}
$$

For a_D we also have the following expansion:

$$
a_D(q_D) = 16iq_D(1+6q_D+24q_D^2+76q_D^3+\cdots) \tag{B.14}
$$

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