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QFT in curved space

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Abstract: This is a PRELIMINARY and UNFINISHED set of notes.

Contents

1 Introduction

In QFT in curved space, one considers quantum matter fields in a classical gravity background. This is similar to studying quantum matter fields in a classical electromagnetic fields, for example. Some motivations to study QFT in curved space:

1. Although gravity is usually very weak, there are situations (black holes, early cosmology) where gravitational effects are important, and it is interesting to understand what happens to quantum fields in these backgrounds. For example, we will find particle production due to gravitational effects.

2. The study of QFT in curved space clarifies conceptual issues of QFT. Removing Poincaré invariance teaches us many things about the general structure of QFT. We will learn for example that the concept of particle is an observer-dependent concept.

3. QFT in curved space is an interesting "deformation" of conventional QFT, and sometimes is more tractable than QFT in flat space. For example, working on a compact space provides a natural IR regulator. Yang–Mills theory on flat space is plagued with IR problems, and semiclassical methods like instantons are useless. However, on a small compact space asymptotically free theories are weakly coupled and one can use semiclassical intuition.

4. QFT in curved space is mathematically interesting. New directions in mathematics have opened by considering quantum field theories on general manifolds (Donaldson theory, Chern–Simons theory).

2 Some GR ingredients

2.1 Conventions

In *Minkowski* signature, our convention is the same as in [4] and [21], i.e. the $(---)$ convention. This means that

- The signature of the metric is $(+-\cdots-)$.
- The Riemann tensor is:

$$R^{\lambda}{}_{\rho\mu\nu} = \partial_\nu \Gamma^{\lambda}_{\rho\mu} - \partial_\mu \Gamma^{\lambda}_{\rho\nu} + \Gamma^{\lambda}_{\nu\delta}\Gamma^{\delta}_{\mu\rho} - \Gamma^{\lambda}_{\mu\delta}\Gamma^{\delta}_{\nu\rho}. \tag{2.1}$$

- The Ricci tensor is defined as

$$R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu}. \tag{2.2}$$

In *Euclidean* signature, we use the following convention: the Riemann tensor is defined as above, but the Ricci tensor is

$$R_{\mu\nu} = R^{\sigma}{}_{\mu\nu\sigma}. \tag{2.3}$$

2.2 Some gravitational backgrounds

The spatial sections of our manifolds will involve flat Euclidean space $\mathbb{R}^n$, the n-sphere $\mathbb{S}^n$, or the hyperbolic space $\mathbb{H}^n$. We will sometimes denote them by $M_{k,n}$, where $k = \pm 1, 0$ for the sphere, the hyperbolic space and Euclidean space, respectively. Their metric will be denoted by

$$\mathrm{d}\Sigma_{k,n}^2 = \begin{cases} \mathrm{d}\Omega_n^2, & k = 1, \\ \sum_{i=1}^n \mathrm{d}x_i^2, & k = 0, \\ \mathrm{d}\Xi_n^2, & k = -1, \end{cases} \tag{2.4}$$

where $\mathrm{d}\Omega_n^2$ is the standard metric on the sphere $\mathbb{S}^n$ of unit radius, while $\mathrm{d}\Xi_n^2$ is the standard metric on the hyperbolic space, also with unit radius (more information on the hyperbolic space can be found in section 4.4.2). The d-dimensional manifolds

$$\mathbb{R} \times M_{k,d-1} \tag{2.5}$$

with metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2 \mathrm{d}\Sigma_{k,d-1}^2, \tag{2.6}$$

represent static backgrounds. In the case $k = +1$, this background is called the static Einstein universe. We will be particularly interested on standard Robertson–Walker spacetimes, which can be obtained from the static examples (2.6) by "inflating" the $(d-1)$-dimensional section with a time-dependent radius $a(t)$. In this way we obtain the *Robertson–Walker* (RW) metric in d dimensions:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t) \mathrm{d}\Sigma_{k,d-1}^2. \tag{2.7}$$

In this metric, the spatial sections contract or expand according to the function $a(t)$. If $a(t) =$ constant we have a static spacetime. It is useful to introduce the *conformal time*

$$\eta = \int^t \frac{\mathrm{d}t'}{a(t')}. \tag{2.8}$$

With this coordinate, we can write (2.7) as

$$\mathrm{d}s^2 = C(\eta) \left(\mathrm{d}\eta^2 - \mathrm{d}\Sigma_k^2 \right) \tag{2.9}$$

where

$$C(\eta) = a^2(\eta). \tag{2.10}$$

2.3 Hypersurfaces

An excellent reference for this material is the book by Eric Poisson [23].

Let M be a d-dimensional Lorentzian manifold. A hypersurface $\Sigma \subset M$ is defined by *parametric equations* of the form

$$x^\mu = x^\mu(y^a), \tag{2.11}$$

where x^μ are coordinates on M, and

$$y^a, \qquad a = 1, \cdots, d-1. \tag{2.12}$$

are intrinsic coordinates on Σ. We can also define it by *implicit equations*

$$\Phi(x^\mu) = 0. \tag{2.13}$$

The vector $\Phi_{,\alpha}$ is always normal to the hypersurface, as in the standard Euclidean geometry. The hypersurface Σ is a *null* hypersurface if

$$g^{\alpha\beta}\Phi_{,\alpha}\Phi_{,\beta} = 0. \tag{2.14}$$

If the hypersurface is not null, we will call it *space-like* (respectively, *time-like*) if the vectors in the tangent space at each point are space-like (respectively, time-like). When the hypersurface Σ is not null, we can introduce on it a unit normal vector, which is defined by

$$n^\mu n_\mu = \epsilon = \begin{cases} +1, & \text{if } \Sigma \text{ is space-like} \\ -1, & \text{if } \Sigma \text{ is time-like.} \end{cases} \tag{2.15}$$

When the hypersurface is defined by the implicit equations (2.13), we have that

$$n_\alpha \propto \Phi_{,\alpha}. \tag{2.16}$$

We will require that n^α points in the direction of increasing Φ, i.e.

$$n^\alpha \Phi_{,\alpha} > 0. \tag{2.17}$$

It can be easily checked that

$$n_\alpha = \frac{\epsilon\Phi_{,\alpha}}{|g^{\alpha\beta}\Phi_{,\alpha}\Phi_{,\beta}|^{1/2}} \tag{2.18}$$

has the required properties.

Define now the vectors

$$e^\mu_a = \frac{\partial x^\mu}{\partial y^a} \tag{2.19}$$

which are tangent to Σ, and in particular they satisfy

$$n_\mu e^\mu_a = 0, \qquad a = 1, \cdots, d-1. \tag{2.20}$$

The metric restricted to Σ gives

$$\mathrm{d}s^2\big|_\Sigma = g_{\mu\nu} e^\mu_a e^\nu_b \mathrm{d}y^a \mathrm{d}y^b. \tag{2.21}$$

This defines the *induced metric*, or *first fundamental form*, which is a tensor on Σ, as

$$h_{ab} = -\epsilon g_{\mu\nu} e^\mu_a e^\nu_b. \tag{2.22}$$

The sign is chosen so that the restriction to a space-like surface gives a Riemannian metric with an overall positive sign, since our conventions for the metric are $(+---)$ (and therefore different from the ones in [23]). We then write

$$\mathrm{d}s^2\big|_\Sigma = -\epsilon h_{ab} \mathrm{d}y^a \mathrm{d}y^b. \tag{2.23}$$

The *surface element* is defined, when the surface is non-null as

$$\mathrm{d}\Sigma = |h|^{1/2} \mathrm{d}^{n-1} y \tag{2.24}$$

where h is the induced metric, and

$$\mathrm{d}\Sigma_\mu = \epsilon n_\mu \mathrm{d}\Sigma. \tag{2.25}$$

One has the following result, which is a pseudo-Riemannian version of Stokes theorem:

$$\int_M A^\mu{}_{;\mu}\sqrt{-g}\mathrm{d}^d x = \int_{\partial M} A^\mu \mathrm{d}\Sigma_\mu. \tag{2.26}$$

The ambient metric $g^{\alpha\beta}$, when restricted to Σ, can be decomposed as

$$g^{\alpha\beta} = \epsilon\left(n^\alpha n^\beta - h^{ab}e^\alpha_a e^\beta_b\right). \tag{2.27}$$

To see this, notice that it leads to the right results for the products of the d linearly independent vectors n_α, e^α_a, $a = 1, \cdots, d$:

$$\begin{aligned}
&\epsilon\left(n^\alpha n^\beta - h^{ab}e^\alpha_a e^\beta_b\right) n_\alpha n_\beta = \epsilon,\\
&\epsilon\left(n^\alpha n^\beta - h^{ab}e^\alpha_a e^\beta_b\right) n_\alpha e_{\beta a} = 0,
\end{aligned} \tag{2.28}$$

and finally

$$\begin{aligned}
\epsilon\left(n^\alpha n^\beta - h^{cd}e^\alpha_c e^\beta_d\right) e_{\alpha a}e_{\beta b} &= -\epsilon h^{cd}e^\alpha_c e^\beta_d e_{\alpha a}e_{\beta b}\\
&= -\epsilon h^{cd}\left(e^\alpha_c e_{\alpha a}\right)\left(e^\beta_d e_{\beta b}\right) = -\epsilon h^{cd}\left(-\epsilon h_{ca}\right)\left(-\epsilon h_{db}\right)\\
&= -\epsilon h_{ab}.
\end{aligned} \tag{2.29}$$

The *second fundamental form* or *extrinsic curvature* is defined as

$$K_{ab} = n_{\alpha;\beta}e^\alpha_a e^\beta_b. \tag{2.30}$$

It is easy to show that this is a symmetric tensor. We need two algebraic facts. First of all, we have

$$n_\alpha e^\alpha_b = 0 \Rightarrow n_{\alpha;\beta}e^\alpha_b + n_\alpha e^\alpha_{b;\beta} = 0. \tag{2.31}$$

Second, we have

$$\begin{aligned}
e^\alpha_{a;\beta}e^\beta_b &= \nabla_\beta\left(\frac{\partial x^\alpha}{\partial y^a}\right)\frac{\partial x^\beta}{\partial y^b} = \left(\frac{\partial}{\partial x^\beta}\left(\frac{\partial x^\alpha}{\partial y^a}\right) + \Gamma^\alpha_{\beta\gamma}\frac{\partial x^\gamma}{\partial y^a}\right)\frac{\partial x^\beta}{\partial y^b}\\
&= \frac{\partial^2 x^\alpha}{\partial y^a \partial y^b} + \Gamma^\alpha_{\beta\gamma}\frac{\partial x^\gamma}{\partial y^a}\frac{\partial x^\beta}{\partial y^b},
\end{aligned} \tag{2.32}$$

which is manifestly symmetric under exchange of a and b. Therefore,

$$K_{ab} = n_{\alpha;\beta}e^\alpha_a e^\beta_b = -n_\alpha e^\alpha_{a;\beta}e^\beta_b = -n_\alpha e^\alpha_{b;\beta}e^\beta_a = n_{\alpha;\beta}e^\alpha_b e^\beta_a = K_{ba} \tag{2.33}$$

We then have

$$K_{ab} = n_{\alpha;\beta}e^\alpha_{(a}e^\beta_{b)} = n_{(\alpha;\beta)}e^\alpha_a e^\beta_b = \frac{1}{2}\left(\mathcal{L}_n g_{\alpha\beta}\right)e^\alpha_a e^\beta_b \tag{2.34}$$

where $\mathcal{L}_n$ is the Lie derivative along the normal vector n. The trace of the extrinsic curvature is

$$K = h^{ab}K_{ab} = h^{ab}e^\alpha_a e^\beta_b n_{\alpha;\beta} = \left(n^\alpha n^\beta - \epsilon g^{\alpha\beta}\right)n_{\alpha;\beta} \tag{2.35}$$

Since

$$n^\alpha n_\alpha = \epsilon \Rightarrow n^\alpha n_{\alpha;\beta} = 0, \tag{2.36}$$

and we obtain

$$K = -\epsilon n^{\alpha}{}_{;\alpha}. \tag{2.37}$$

We can also write

$$K = h^{\alpha\beta} n_{\alpha;\beta}, \tag{2.38}$$

where

$$h^{\alpha\beta} = h^{ab} e^{\alpha}_{a} e^{\beta}_{b}. \tag{2.39}$$

Example 2.1. Let us consider $d = (n+2)$-dimensional spacetime M with a metric of the form

$$\mathrm{d}s^2 = V(r)\mathrm{d}t^2 - V(r)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega_n^2. \tag{2.40}$$

We will consider the timelike surface Σ defined by $r =$ constant, so that

$$y^a = (t, x^m), \quad m = 1, \cdots, n, \tag{2.41}$$

where x^m are coordinates of $\mathbb{S}^n$. The tangent vectors are simply

$$e^r_a = 0, \quad e^{\alpha}_a = \delta^{\alpha}_a, \qquad a = 0, \cdots, n. \tag{2.42}$$

The unit normal vector is then

$$n^{\mu} = V^{1/2}(r)\delta^{\mu}_r, \qquad n_r = -V^{-1/2}(r), \tag{2.43}$$

and it has norm $\epsilon = -1$. It corresponds to the vector field

$$n = V^{1/2}(r)\frac{\partial}{\partial r}. \tag{2.44}$$

The induced metric is

$$h_{ab}\mathrm{d}y^a\mathrm{d}y^b = V(r)\mathrm{d}t^2 - r^2\mathrm{d}\Omega_n^2. \tag{2.45}$$

Notice that $h_{rr} = 0$. The components of the extrinsic curvature can be computed in a straightforward way. We have, for the nonzero components restricted to Σ,

$$\begin{aligned} K_{tt} &= -\Gamma^r_{tt} n_r = \frac{1}{2} V^{1/2}(r)V'(r), \\ K_{pq} &= -\Gamma^r_{pq} n_r = -rV^{1/2}(r) g^{\mathbb{S}^n}_{pq}. \end{aligned} \tag{2.46}$$

where we have written

$$\mathrm{d}\Sigma^2_{k,n} = g^{\mathbb{S}^n}_{pq}\mathrm{d}x^p\mathrm{d}x^q \tag{2.47}$$

and we used that

$$\Gamma^r_{tt} = -\frac{1}{2}g^{rr}\partial_r g_{tt} = \frac{1}{2}V(r)V'(r), \qquad \Gamma^r_{pq} = -\frac{1}{2}g^{rr}\partial_r g_{pq} = -rV(r)g^{\mathbb{S}^n}_{pq}. \tag{2.48}$$

The trace is then

$$K = h^{tt}K_{tt} + h^{pq}K_{pq} = \frac{1}{2}V^{-1/2}(r)V'(r) + \frac{n}{r}V^{1/2}(r). \tag{2.49}$$

□

2.4 Variational principle

The Einstein–Hilbert action for the gravitational field is

$$S_{\rm bulk} = \frac{1}{16\pi G_N}\int \mathrm{d}^d x \sqrt{-g} R \tag{2.50}$$

and its variation leads to Einstein's equations, after including the matter fields. However, the standard derivation misses a subtle contribution due to the boundary terms. This contributions is not important if our main interest is the study of the solutions to Einstein's equations in classical GR. However, in the quantum theory, the action becomes a crucial object since it gives the weight of a field configuration to the path integral. It is then important to revisit the variational principle having in mind the applications to quantum black holes.

Let us then consider the gravitational action in a region M of space-time, with boundary ∂M. We want to analyze the variation of the action as we vary the metric, with the condition that the metric variation vanishes at the boundary:

$$\delta g_{\alpha\beta}|_{\partial M} = 0. \tag{2.51}$$

To study the variation of the Einstein–Hilbert action, one uses that

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\alpha\beta}\delta g^{\alpha\beta}, \tag{2.52}$$

therefore

$$\begin{aligned}\delta\left(\sqrt{-g}R\right) &= \delta\left(\sqrt{-g}g^{\alpha\beta}R_{\alpha\beta}\right) = R\delta\sqrt{-g} + \sqrt{-g}\left(\delta g^{\alpha\beta}R_{\alpha\beta} + g^{\alpha\beta}\delta R_{\alpha\beta}\right)\\ &= G_{\alpha\beta}\delta g^{\alpha\beta}\sqrt{-g} + g^{\alpha\beta}\delta R_{\alpha\beta}\sqrt{-g},\end{aligned} \tag{2.53}$$

where

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R \tag{2.54}$$

is Einstein's tensor. The vanishing of the first term leads to Einstein's equations in the vacuum,

$$G_{\alpha\beta} = 0, \tag{2.55}$$

while the last one is usually neglected by arguing that it vanishes on a boundary at infinity. However, let us look at it more carefully. By using a local frame, where the Christoffel symbols vanish, we have that

$$\delta R_{\alpha\beta} = \delta\left(R^{\mu}{}_{\alpha\mu\beta}\right) = \delta\left(\Gamma^{\mu}_{\alpha\mu,\beta} - \Gamma^{\mu}_{\alpha\beta,\mu}\right) = \left(\delta\Gamma^{\mu}_{\alpha\mu}\right)_{;\beta} - \left(\delta\Gamma^{\mu}_{\alpha\beta}\right)_{;\mu}. \tag{2.56}$$

Since the last expression is covariant, it can be used in any coordinate system. We now write,

$$g^{\alpha\beta}\delta R_{\alpha\beta} = \delta v^{\mu}_{;\mu}, \tag{2.57}$$

where

$$\delta v^{\mu} = g^{\alpha\mu}\delta\Gamma^{\beta}_{\alpha\beta} - g^{\alpha\beta}\delta\Gamma^{\mu}_{\alpha\beta}. \tag{2.58}$$

We can now use Stokes' theorem to write

$$\int_M \mathrm{d}^d x\, \sqrt{-g}g^{\alpha\beta}\delta R_{\alpha\beta} = \int_M \mathrm{d}^d x\, \sqrt{-g}\delta v^{\mu}{}_{;\mu} = \int_{\partial M}\delta v^{\mu}\mathrm{d}\Sigma_{\mu}. \tag{2.59}$$

We now compute δv^μ. Since the variation of the metric vanishes at the boundary (but *not* its derivatives) we find

$$\delta\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}\left(\delta g_{\nu\alpha,\beta} + \delta g_{\nu\beta,\alpha} - \delta g_{\alpha\beta,\nu}\right). \tag{2.60}$$

When we plug this into (2.58), we find

$$\delta v_\mu = g^{\alpha\beta}\left(\delta g_{\alpha\beta,\mu} - \delta g_{\alpha\mu,\beta}\right). \tag{2.61}$$

We should now compute $n^\mu \delta v_\mu$, where n^μ is an orthonormal vector to ∂M. We find

$$n^\mu \delta v_\mu = -n^\mu \epsilon \left(n^\alpha n^\beta - h^{ab} e^\alpha_a e^\beta_b\right)\left(\delta g_{\mu\beta,\alpha} - \delta g_{\alpha\beta,\mu}\right). \tag{2.62}$$

Now, we notice that the last bracket is antisymmetric in μ, α, while $n^\mu n^\alpha$ is symmetric. Therefore, the part of the metric which involves the normal vectors drops out. Next, we note that

$$\delta g_{\mu\beta,\alpha} e^\alpha_a = 0, \tag{2.63}$$

since $\delta g_{\alpha\beta}$ vanishes everywhere on ∂M, therefore their tangential derivatives vanish as well. We finally find,

$$n^\mu \delta v_\mu = -\epsilon h^{\alpha\beta} \delta g_{\alpha\beta,\mu} n^\mu. \tag{2.64}$$

This involves the derivative of the metric variation along the *normal* direction to ∂M, so it is in general non-vanishing. We then find

$$16\pi G_N \delta S_{\text{bulk}} = \int_M \mathrm{d}^d x \sqrt{-g}\, G_{\alpha\beta} \delta g^{\alpha\beta} - \int_{\partial M} h^{\alpha\beta} \delta g_{\alpha\beta,\mu} n^\mu |h|^{1/2} \mathrm{d}^{d-1} y. \tag{2.65}$$

Therefore, in the presence of a boundary, Einstein's equations are not sufficient to guarantee the vanishing of the variation of the action, due to the second term in (2.65). To fix this, we must add an explicit boundary term to the action, which turns out to be

$$S_{\text{bdry}} = \frac{1}{8\pi G_N} \oint_{\partial M} K |h|^{1/2} \mathrm{d}^{d-1} y. \tag{2.66}$$

To see that this is has the right properties, let us study its variation. Since the metric is fixed at the boundary, $h_{\alpha\beta}$ is fixed and the only variation comes from the extrinsic curvature K. We have,

$$\begin{aligned} \delta K &= h^{\alpha\beta} \delta\left(n_{\alpha;\beta}\right) = h^{\alpha\beta} \delta\left(-\Gamma^\gamma_{\alpha\beta} n_\gamma\right) = -\frac{1}{2} h^{\alpha\beta} n^\mu \left(\delta g_{\mu\alpha,\beta} + \delta g_{\mu\beta,\alpha} - \delta g_{\alpha\beta,\mu}\right) \\ &= \frac{1}{2} h^{\alpha\beta} \delta g_{\alpha\beta,\mu} n^\mu, \end{aligned} \tag{2.67}$$

where we used (2.63). It follows that

$$16\pi G_N \delta S_{\text{bdry}} = \oint_{\partial M} h^{\alpha\beta} \delta g_{\alpha\beta,\mu} n^\mu |h|^{1/2} \mathrm{d}^{d-1} y, \tag{2.68}$$

which cancels exactly the second term in (2.65), as we wanted.

One problem of the boundary term (2.66) is that, in general, it leads to divergences, even in flat space. In order to obtain finite values for the total action, it is necessary to add a counterterm which removes the divergences in (2.66). In [11] it was proposed to use

$$S_{\text{ct}} = -\frac{1}{8\pi G_N} \int_{\partial M} K^{(0)} |h|^{1/2} \mathrm{d}^{d-1} y \tag{2.69}$$

where $K^{(0)}$ is the extrinsic curvature of the boundary ∂M when it is embedded in *flat* space. We will see what is its precise rôle in the context of black hole physics.

3 Quantum scalar fields in curved space: the canonical approach

3.1 Review of the Minkowski case

The free Klein–Gordon field in d dimensions is governed by the action

$$S = \frac{1}{2}\int \mathrm{d}^d x \left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) \tag{3.1}$$

which leads to the EOM

$$(\Box + m^2)\phi = 0, \qquad \Box = \partial^\mu\partial_\mu. \tag{3.2}$$

An appropriately normalized solution of this equation is the plane wave

$$u_{\mathbf{k}}(t,\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^{d-1}2\omega}}\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{3.3}$$

where

$$\omega = \sqrt{\mathbf{k}^2 + m^2}. \tag{3.4}$$

Any solution to the KG equation can be Fourier expanded as

$$\phi(x) = \int \mathrm{d}^{d-1}\mathbf{k}\left[a_{\mathbf{k}}u_{\mathbf{k}}(x) + a^\dagger_{\mathbf{k}}u^*_{\mathbf{k}}(x)\right] \tag{3.5}$$

Let us define the inner product

$$\langle\phi_1,\phi_2\rangle = \mathrm{i}\int \mathrm{d}^{d-1}\mathbf{x}\Big(\phi_1^*(x)\partial_t\phi_2(x) - \partial_t\phi_1^*(x)\phi_2(x)\Big). \tag{3.6}$$

The main property of this product is that, if $\phi_{1,2}$ solve the KG equation, the product is time-independent, i.e. conserved. This product is anti-linear in the first variable and linear in the second variable, and it satisfies

$$\langle\phi_1,\phi_2\rangle^* = -\langle\phi_1^*,\phi_2^*\rangle = \langle\phi_2,\phi_1\rangle. \tag{3.7}$$

The plane waves are orthonormal w.r.t. this product,

$$\langle u_{\mathbf{k}},u_{\mathbf{k}'}\rangle = \delta(\mathbf{k}-\mathbf{k}'), \qquad \langle u_{\mathbf{k}},u^*_{\mathbf{k}'}\rangle = 0, \tag{3.8}$$

which is easy to check by using the integral

$$\int\frac{\mathrm{d}^{d-1}\mathbf{x}}{(2\pi)^{d-1}}\mathrm{e}^{\mathrm{i}(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} = \delta(\mathbf{k}-\mathbf{k}'). \tag{3.9}$$

Then, we have the useful property that

$$a_{\mathbf{k}} = \langle u_{\mathbf{k}},\phi\rangle. \tag{3.10}$$

The plane wave $u_{\mathbf{k}}$ is called a *positive frequency* solution with respect to t, and it is an eigenfunction of the time translation operator

$$\frac{\partial}{\partial t}u_{\mathbf{k}} = -\mathrm{i}\omega u_{\mathbf{k}}. \tag{3.11}$$

Remark 3.1. Sometimes it is useful, in order to avoid volume divergences, to consider periodic boundary conditions on a torus $\mathbb{T}^{d-1}$ of volume L^{d-1}. In this case, the wavevectors are quantized

$$\mathbf{k} = \frac{2\pi}{L}\mathbf{n}, \tag{3.12}$$

where $\mathbf{n}$ is a vector of integer entries, and the appropriately normalized basis of solutions is

$$u_{\mathbf{k}}(t, \mathbf{x}) = \frac{1}{\sqrt{L^{d-1}2\omega}} \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}. \tag{3.13}$$

The Fourier decomposition is in terms of discrete modes,

$$\phi(t, \mathbf{x}) = \sum_{\mathbf{k}} \left(a_{\mathbf{k}} u_{\mathbf{k}} + a^{\dagger}_{\mathbf{k}} u^{*}_{\mathbf{k}} \right). \tag{3.14}$$

The standard dictionary between discrete and continuous theories is

$$\int \mathrm{d}^{d-1}\mathbf{k} \to \left(\frac{2\pi}{L}\right)^{d-1} \sum_{\mathbf{k}} \tag{3.15}$$

and the orthonormality property of the modes is given by

$$\langle u_{\mathbf{k}}, u_{\mathbf{k}'} \rangle = \delta_{\mathbf{k}\mathbf{k}'}, \tag{3.16}$$

where one has to use the relationship

$$\int \mathrm{d}^{d-1}\mathbf{x}\, \mathrm{e}^{\mathrm{i}(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} = L^{d-1}\delta_{\mathbf{k}\mathbf{k}'}. \tag{3.17}$$

□

In order to quantize this theory, we introduce a Lagrangian leading to the above EOM,

$$\mathcal{L} = \frac{1}{2}\left(\eta^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi - m^2\phi^2\right). \tag{3.18}$$

The canonical momentum is defined as

$$\pi = \frac{\delta\mathcal{L}}{\delta\partial_0\phi} = \partial_t\phi. \tag{3.19}$$

and it has the expansion

$$\pi(t, \mathbf{x}) = -\mathrm{i}\int \mathrm{d}^{d-1}\mathbf{k}\, \omega_k \left(a_{\mathbf{k}} u_{\mathbf{k}} - a^{\dagger}_{\mathbf{k}} u^{*}_{\mathbf{k}} \right). \tag{3.20}$$

The equal-time canonical commutation relations (we set $\hbar = 1$) are

$$\begin{aligned} [\phi(t, \mathbf{x}), \phi(t, \mathbf{y})] &= 0, \\ [\pi(t, \mathbf{x}), \pi(t, \mathbf{y})] &= 0, \\ [\phi(t, \mathbf{x}), \pi(t, \mathbf{y})] &= \mathrm{i}\delta(\mathbf{x} - \mathbf{y}). \end{aligned} \tag{3.21}$$

Using the canonical momentum, we can write

$$a_{\mathbf{k}} = \langle u_{\mathbf{k}}, \phi \rangle = \mathrm{i} \int \mathrm{d}^{d-1}\mathbf{x}\, (u_{\mathbf{k}}^* \partial_t \phi - \partial_t u_{\mathbf{k}}^* \phi) = \mathrm{i} \int \mathrm{d}^{d-1}\mathbf{x}\, (u_{\mathbf{k}}^* \pi - \partial_t u_{\mathbf{k}}^* \phi)\,, \tag{3.22}$$

and by taking the Hermitian conjugate we find,

$$a_{\mathbf{k}}^\dagger = -\mathrm{i} \int \mathrm{d}^{d-1}\mathbf{x}\, (u_{\mathbf{k}} \pi - \partial_t u_{\mathbf{k}} \phi)\,. \tag{3.23}$$

By using the canonical commutation relations we compute

$$\begin{aligned} [a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] &= \int \mathrm{d}^{d-1}\mathbf{x} \int \mathrm{d}^{n-1}\mathbf{y}\, [(u_{\mathbf{k}}^* \pi - \partial_t u_{\mathbf{k}}^* \phi)\,(t,\mathbf{x}), (u_{\mathbf{k}'} \pi - \partial_t u_{\mathbf{k}'} \phi)\,(t,\mathbf{y})] \\ &= \mathrm{i} \int \mathrm{d}^{d-1}\mathbf{x}\, \left(u_{\mathbf{k}}^*(t,\mathbf{x}) \partial_t u_{\mathbf{k}'}(t,\mathbf{x}) - \partial_t u_{\mathbf{k}}^*(t,\mathbf{x}) u_{\mathbf{k}'}(t,\mathbf{x})\right) \\ &= \langle u_{\mathbf{k}}, u_{\mathbf{k}'} \rangle = \delta(\mathbf{k} - \mathbf{k}'). \end{aligned} \tag{3.24}$$

The operators $a_{\mathbf{k}}$ ($a_{\mathbf{k}}^\dagger$) annihilate (create) particles of momentum $\mathbf{k}$. Notice that *positive frequency modes* are associated to *annihilation operators.*

3.2 Scalar fields in curved space

We now want to consider a scalar field in curved space. The general rule to generalize the action is to replace all occurrences of the flat metric $\eta^{\mu\nu}$, and all the standard derivatives ∂_μ by *covariant* derivatives ∇_μ. This is called the *minimal coupling to gravity.* The measure is now taken to be the one induced by the Riemannian metric, i.e $\mathrm{d}^d x\sqrt{|g|}$. We then find the action

$$S = \frac{1}{2} \int \mathrm{d}^d x \sqrt{|g|} \Big(g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - m^2 \phi^2(x) \Big), \tag{3.25}$$

which leads to the EOM

$$(\Box + m^2)\phi = 0, \tag{3.26}$$

where, now,

$$\Box\phi = g^{\mu\nu} \nabla_\mu \nabla_\nu \phi = |g|^{-1/2} \partial_\mu \left[|g|^{1/2} g^{\mu\nu} \partial_\nu \phi \right] \tag{3.27}$$

is computed with the covariant derivatives.

However, in the case of the scalar field, there is a slightly more general action which one can consider, and depending on one parameter ξ, namely

$$S = \frac{1}{2} \int \mathrm{d}^d x \sqrt{|g|} \Big(g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - (m^2 + \xi R) \phi^2(x) \Big), \tag{3.28}$$

where R is the scalar curvature. The wave equation is

$$(\Box + m^2 + \xi R)\phi = 0. \tag{3.29}$$

The term proportional to R disappears in flat space of course, so this is still a generalization of the standard action for a scalar field. What is reason to include this term, and not for example a term proportional to $R^2\phi^2$?

One reason to include such a term is *conformal invariance*. The theory of a massless scalar field is invariant if we rescale the metric as

$$g_{\mu\nu} \to \overline{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{3.30}$$

where Ω is a constant scaling factor, and if we change as well

$$\phi \to \overline{\phi} = \Omega^{(2-d)/2}\phi. \tag{3.31}$$

It is easy to see that the action and the EOM for a massless field are invariant under this transformation. This is the classical *scale invariance* of the massless scalar field theory, and $(2-d)/2$ is the classical dimension of the field. We can now promote the above global symmetry to a local one, by making Ω to be space-time dependent

$$g_{\mu\nu} \to \overline{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}. \tag{3.32}$$

This is called a *conformal transformation* of the metric. The corresponding transformation of the scalar field is

$$\phi(x) \to \overline{\phi}(x) = \Omega^{(2-d)/2}(x)\phi(x). \tag{3.33}$$

Under a conformal transformation, one has the following transformation of the scalar curvature,

$$\overline{R} = \Omega^{-2}R + 2(d-1)\Omega^{-3}g^{\mu\nu}\nabla_\mu\nabla_\nu\Omega + (d-1)(d-4)\Omega^{-4}g^{\mu\nu}\nabla_\mu\Omega\nabla_\nu\Omega. \tag{3.34}$$

Example 3.2. The metric (2.9) is a conformal transformation of the product metric (2.6) on $\mathbb{R} \times M_k$ with $a = 1$. □

Example 3.3. A metric $g_{\mu\nu}$ is *conformally flat* if it can be put in the form

$$g_{\mu\nu}(x) = \Omega^2(x)\eta_{\mu\nu} \tag{3.35}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. A necessary condition for the metric to be conformally flat is that the Weyl tensor W vanishes:

$$W = 0 \tag{3.36}$$

In dimensions ≥ 4, this condition is also sufficient. For example, the RW metric (2.9) with $k = 0$ is conformally flat. Another example, in Euclidean signature, is the static metric (2.6):

$$\mathrm{d}s^2 = \mathrm{d}t^2 + a^2\mathrm{d}\Sigma_k^2. \tag{3.37}$$

For $k = 1$ this space is conformally equivalent to flat space $\mathbb{R}^4$. To see this, introduce the variable

$$r = a\mathrm{e}^{t/a}. \tag{3.38}$$

Then,

$$\mathrm{d}r^2 = \mathrm{e}^{2t/a}\mathrm{d}t^2, \tag{3.39}$$

and

$$\mathrm{d}s^2 = \mathrm{e}^{-2t/a}\mathrm{d}r^2 + a^2\mathrm{d}\Sigma_1^2 = \mathrm{e}^{-2t/a}\left(\mathrm{d}r^2 + r^2\mathrm{d}\Sigma_1^2\right), \tag{3.40}$$

which is clearly a conformal transformation of the metric of $\mathbb{R}^4$ (in spherical coordinates). In general, one can show that the spacetime $\mathbb{R} \times \mathbb{S}^{n-1}$ is conformally equivalent to $\mathbb{R}^n$ (see for example [9]). □

It is easy to see that the action (3.28), even when $m^2 = 0$, is in general not invariant under a conformal transformation of the metric (3.32), due to the derivatives acting on Ω. However, if ξ takes the special value

$$\xi(d) = \frac{1}{4}\frac{d-2}{d-1}, \tag{3.41}$$

the action (3.28) is invariant under (3.32), if $m^2 = 0$. When governed by the action (3.28) with the above value of ξ, the scalar field is said to be *conformally coupled* to gravity. An alternative way to see this is to notice that the wave equation transforms as

$$\left(\overline{\Box} + \xi(d)\overline{R}\right)\overline{\phi} = \Omega^{-\frac{d+2}{2}}\left(\Box + \xi(d)R\right)\phi, \tag{3.42}$$

so a solution of the wave equation (3.29) in the metric g maps to a solution in the metric $\overline{g}$ if and only if $m = 0$ and $\xi = \xi(d)$.

In order to proceed with the generalization to curved space, the first step is to generalize the inner product (3.6) of complex solutions to the KG equation. This requires a spacelike hypersurface Σ with unit normal vector n_μ. We now define

$$\langle \phi_1, \phi_2 \rangle = \int_\Sigma \mathrm{d}\Sigma^\mu J_\mu \tag{3.43}$$

where J_μ is the current

$$J_\mu = \mathrm{i}\Big(\phi_1^* \partial_\mu \phi_2(x) - \partial_\mu \phi_1^* \phi_2(x)\Big). \tag{3.44}$$

Since $\phi_{1,2}$ satisfy the KG equation, the current J^μ is conserved,

$$\nabla^\mu J_\mu = \phi_1^* \Box \phi_2 - (\Box \phi_1^*)\,\phi_2 = 0 \tag{3.45}$$

One consequence of this is that the bracket does not depend on the choice of spacelike hypersurface Σ, if the fields decay sufficiently fast at spatial infinity: if we consider another spacelike hypersurface Σ', we have, by (2.26), that

$$\int_\Sigma \mathrm{d}\Sigma^\mu J_\mu - \int_{\Sigma'} \mathrm{d}\Sigma^\mu J_\mu = \int_M \sqrt{-g}\,\mathrm{d}^d x \nabla^\mu J_\mu = 0, \tag{3.46}$$

where M is the spacetime region bounded by the hypersurfaces Σ, Σ'. This generalizes the time-independence of the inner product in Minkowski space.

Let us now proceed with the canonical quantization of the scalar field. In order to do this, we need a Hamiltonian description of the theory, therefore a preferred time coordinate. We will then assume that there is a splitting

$$x^\mu = (t, x^i) \tag{3.47}$$

into a time-like coordinate t and space-like coordinates x^i, $i = 1, 2, 3$. We will choose as the hypersurface Σ to compute the inner product

$$t = \text{constant}, \tag{3.48}$$

which is space-like. The metric is of the form

$$\mathrm{d}s^2 = g_{00}\mathrm{d}t^2 + 2g_{0i}\mathrm{d}t\mathrm{d}x^i - h_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{3.49}$$

where h_{ij} is the induced metric on Σ. In terms of these coordinates, we can write the momentum conjugate to ϕ as

$$\pi = \frac{\delta \mathcal{L}}{\delta(\partial_0 \phi)} = |g|^{1/2} g^{0\mu} \partial_\mu \phi. \tag{3.50}$$

It turns out that this can be written as

$$\pi = |h|^{1/2} n^\mu \partial_\mu \phi. \tag{3.51}$$

The proof is a simple exercise with hypersurfaces. Since the normal vector to Σ is proportional to the gradient of the defining function, we have

$$n_\mu = c \frac{\partial t}{\partial x^\mu} = c \delta_{\mu 0}, \tag{3.52}$$

and from normalization we conclude that

$$c = \frac{1}{(g^{00})^{1/2}}. \tag{3.53}$$

Therefore,

$$n^\mu = \frac{g^{\mu 0}}{(g^{00})^{1/2}} \tag{3.54}$$

and

$$|g|^{1/2} g^{\mu 0} = |g g^{00}|^{1/2} n^\mu. \tag{3.55}$$

Now, g^{00} can be computed as

$$g^{00} = \text{cofactor}(g_{00})/|g|, \tag{3.56}$$

where the cofactor of g_{00} is the determinant of the matrix obtained after eliminating the row and the column to which g_{00} belongs. But this is nothing that the determinant of the induced metric, h_{ij}, and we conclude that

$$|g|^{1/2} g^{\mu 0} = |h|^{1/2} n^\mu. \tag{3.57}$$

Plugging this into (3.50) we obtain (3.51), as we wanted. The canonical commutation relations are (3.21), exactly as in Minkowski space, but π is now given by (3.51).

Given now *any* solution f to the KG equation, we can associate to it an annihilation operator $a(f)$

$$f \to a(f) \tag{3.58}$$

by

$$a(f) = \langle f, \phi \rangle. \tag{3.59}$$

Since ϕ is Hermitian, we have that

$$a^\dagger(f) = -a(f^*). \tag{3.60}$$

We can write this operator in terms of ϕ and π, as follows.

$$a(f) = \mathrm{i} \int_\Sigma \mathrm{d}^{d-1}x |h|^{1/2} n^\mu \left(f^* \partial_\mu \phi - \partial_\mu f^* \phi \right) = \mathrm{i} \int_\Sigma \mathrm{d}^{d-1}x \left(f^* \pi - |h|^{1/2} n^\mu \partial_\mu f^* \phi \right). \tag{3.61}$$

From the canonical commutation relations (3.21) we find immediately that

$$
\begin{aligned}
[a(f), a^\dagger(g)] &= \int \mathrm{d}^{d-1}\mathbf{x} \int \mathrm{d}^{n-1}\mathbf{y} \left[\left(f^\star \pi - |h|^{1/2} n^\mu \partial_\mu f^\star \phi \right)(t, \mathbf{x}), \left(g\pi - |h|^{1/2} n^\mu \partial_\mu g \phi \right)(t, \mathbf{y}) \right] \\
&= \mathrm{i} \int \mathrm{d}^{d-1}\mathbf{x}\, |h|^{1/2} n^\mu \left(f^\star(t, \mathbf{x}) \partial_\mu g(t, \mathbf{x}) - \partial_\mu f^\star(t, \mathbf{x}) g(t, \mathbf{x}) \right) \\
&= \langle f, g \rangle .
\end{aligned}
\tag{3.62}
$$

Furthermore, one finds

$$
[a(f), a(g)] = -\langle f, g^* \rangle, \qquad [a^\dagger(f), a^\dagger(g)] = -\langle f^*, g \rangle . \tag{3.63}
$$

We have now found the appropriate procedure to quantize the scalar field and construct a Hilbert space for the theory. First, we must *find a complete orthonormal basis of solutions u_i to the wave equation*, i.e. they have to satisfy the equations

$$
\langle u_i, u_j \rangle = \delta_{ij}, \qquad \langle u_i^*, u_j \rangle = 0, \qquad \langle u_i^*, u_j^* \rangle = -\delta_{ij} . \tag{3.64}
$$

The corresponding creation and annihilation operators are denoted by a_i, $a_i^\dagger$. We now expand

$$
\phi(x) = \sum_i (a_i u_i + a_i^\dagger u_i^*), \tag{3.65}
$$

where the $a_i, a_i^\dagger$ are quantum operators that satisfy the standard commutation relations

$$
[a_i, a_j^\dagger] = \delta_{ij} . \tag{3.66}
$$

One important point is that *there is no natural way to perform this mode decomposition.* One possible, “natural” choice of modes occurs when the spacetime has a Killing vector. Then, one can choose the modes in such a way that they are eigenfunctions of the Killing vector. This is indeed what one does in Minkowski space, where $\partial/\partial t$ is Killing. But in general one has to consider the possibility of choosing a completely different set of modes $\bar{u}_j$ in such a way that

$$
\phi(x) = \sum_j (\bar{a}_j \bar{u}_j + \bar{a}_j^\dagger \bar{u}_j^*). \tag{3.67}
$$

In particular, the modes $\bar{u}_i$ can be expanded in terms of the u_i

$$
\bar{u}_j = \sum_i \left(\alpha_{ji} u_i + \beta_{ji} u_i^* \right), \tag{3.68}
$$

and the coefficients can be obtained as

$$
\alpha_{ij} = \langle u_j, \bar{u}_i \rangle, \qquad \beta_{ij} = -\langle u_j^*, \bar{u}_i \rangle . \tag{3.69}
$$

We can also do it the other way around:

$$
u_i = \sum_j \left(\lambda_{ij} \bar{u}_j + \mu_{ij} \bar{u}_j^* \right), \tag{3.70}
$$

and

$$\lambda_{ij} = \langle \bar{u}_j, u_i\rangle = \langle u_i, \bar{u}_j\rangle^* = \alpha^*_{ji}, \qquad \mu_{ij} = -\langle \bar{u}^*_j, u_i\rangle = \langle u^*_i, \bar{u}_j\rangle = -\beta_{ji}, \tag{3.71}$$

therefore

$$u_i = \sum_j \big(\alpha^*_{ji}\bar{u}_j - \beta_{ji}\bar{u}^*_j\big). \tag{3.72}$$

We also have

$$\begin{aligned} \phi &= \sum_j (\bar{a}_j \bar{u}_j + \bar{a}^\dagger_j \bar{u}^*_j) \\ &= \sum_j \bar{a}_j \sum_k (\alpha_{jk} u_k + \beta_{jk} u^*_k) + \bar{a}^\dagger_j \sum_k (\alpha^*_{jk} u^*_k + \beta^*_{jk} u_k) \\ &= \sum_{j,k} (\alpha_{jk}\bar{a}_j + \beta^*_{jk}\bar{a}^\dagger_j) u_k + \sum_{j,k} (\alpha^*_{jk}\bar{a}^\dagger_j + \beta_{jk}\bar{a}_j) u^*_k, \end{aligned} \tag{3.73}$$

therefore

$$a_i = \sum_j (\alpha_{ji}\bar{a}_j + \beta^*_{ji}\bar{a}^\dagger_j), \qquad a^\dagger_i = \sum_j (\alpha^*_{ji}\bar{a}^\dagger_j + \beta_{ji}\bar{a}_j). \tag{3.74}$$

Conversely,

$$\begin{aligned} \phi &= \sum_i (a_i u_i + a^\dagger_i u^*_i) \\ &= \sum_i a_i \sum_j \big(\alpha^*_{ji}\bar{u}_j - \beta_{ji}\bar{u}^*_j\big) + a^\dagger_i \sum_j \big(\alpha_{ji}\bar{u}^*_j - \beta^*_{ji}\bar{u}_j\big) \\ &= \sum_{i,j} (\alpha^*_{ji} a_i - \beta^*_{ji} a^\dagger_i)\bar{u}_j + \sum_{i,j} (\alpha_{ji} a^\dagger_i - \beta_{ji} a_i)\bar{u}^*_j \end{aligned} \tag{3.75}$$

and one can also deduce

$$\bar{a}_j = \sum_i (\alpha^*_{ji} a_i - \beta^*_{ji} a^\dagger_i), \qquad \bar{a}^\dagger_j = \sum_i (\alpha_{ji} a^\dagger_i - \beta_{ji} a_i) \tag{3.76}$$

Such a relation between the different set of modes is called a *Bogoliubov transformation*. Imposing that the new operators have canonical and properly normalized commutation relations (which is equivalent to the orthonormality of the new set of modes), we find the following relations between the coefficients:

$$\begin{aligned} [\bar{a}_i, \bar{a}_j] = 0 &\Rightarrow \sum_k (\alpha_{ik}\beta_{jk} - \beta_{ik}\alpha_{jk}) = 0, \\ [\bar{a}_i, \bar{a}^\dagger_j] = \delta_{ij} &\Rightarrow \sum_k (\alpha^*_{ik}\alpha_{jk} - \beta^*_{ik}\beta_{jk}) = \delta_{ij}. \end{aligned} \tag{3.77}$$

The fact that the original variables satisfy the commutation relations leads to two additional relations among the Bogoliubov coefficients,

$$\begin{aligned} [a_i, a_j] = 0 &\Rightarrow \sum_k (\alpha_{ki}\beta^*_{kj} - \beta^*_{ki}\alpha_{kj}) = 0, \\ [a_i, a^\dagger_j] = \delta_{ij} &\Rightarrow \sum_k (\alpha_{ki}\alpha^*_{kj} - \beta^*_{ki}\beta_{kj}) = \delta_{ij}. \end{aligned} \tag{3.78}$$

One important consequence of the Bogoliubov transformation is that the Fock vacuum w.r.t. to one set of oscillators is a populated state w.r.t. the other set. The Fock vacua are defined as

$$a_j|0\rangle = 0, \qquad \bar{a}_j|\bar{0}\rangle = 0, \qquad \forall j. \tag{3.79}$$

One can see that *the* $|\bar{0}\rangle$ *vacuum contains* $\sum_j |\beta_{ji}|^2$ *particles of the* u_i *mode.* Let us check this.

$$\begin{aligned} n_i = \langle\bar{0}|a_i^\dagger a_i|\bar{0}\rangle &= \langle\bar{0}|\sum_j(\alpha_{ji}^*\bar{a}_j^\dagger + \beta_{ji}\bar{a}_j)\sum_k(\alpha_{ki}\bar{a}_k + \beta_{ki}^*\bar{a}_k^\dagger)|\bar{0}\rangle \\ &= \sum_{jk}\beta_{ki}^*\beta_{ji}\langle\bar{0}|\bar{a}_j\bar{a}_k^\dagger|\bar{0}\rangle = \sum_j |\beta_{ji}|^2. \end{aligned} \tag{3.80}$$

The number of quanta in the new vacuum should be finite, i.e. we require that the Bogoliubov transformation is such that

$$\sum_j |\beta_{ji}|^2 < \infty, \tag{3.81}$$

for all i.

The relation between the vacua can be easily found by using matrix notation. First of all, notice that the relations (3.77) can be written as

$$\begin{aligned} \alpha\beta^T - \beta\alpha^T &= 0, \\ \alpha\alpha^\dagger - \beta\beta^\dagger &= \mathbf{1}. \end{aligned} \tag{3.82}$$

while (3.78) can be written as

$$\begin{aligned} \alpha^T\beta^* - \beta^\dagger\alpha &= 0, \\ \alpha^\dagger\alpha - \beta^T\beta^* &= \mathbf{1}. \end{aligned} \tag{3.83}$$

This implies that the matrix $\beta^*\alpha^{-1}$ is symmetric: indeed, we have

$$(\beta^*\alpha^{-1})^T = \alpha^{-1T}\beta^\dagger, \tag{3.84}$$

but from the first relation in (3.83) we find,

$$\beta^\dagger = \alpha^T\beta^*\alpha^{-1} \Rightarrow = \alpha^{-1T}\beta^\dagger = \beta^*\alpha^{-1}, \tag{3.85}$$

and we conclude that

$$(\beta^*\alpha^{-1})^T = \beta^*\alpha^{-1}. \tag{3.86}$$

We can now write the first relation in (3.74) as

$$a = \bar{a}\alpha + \bar{a}^\dagger\beta^*, \tag{3.87}$$

and we solve

$$\bar{a} = -\bar{a}^\dagger\beta^*\alpha^{-1} + a\alpha^{-1}. \tag{3.88}$$

Since the vacuum $|0\rangle$ satisfies

$$a_i|0\rangle = 0 \tag{3.89}$$

for all i, it is annihilated by any linear combination of the a_i operators. Applying now (3.88) to this vacuum, we find

$$\bar{a}|0\rangle = -\bar{a}^\dagger\beta^*\alpha^{-1}|0\rangle. \tag{3.90}$$

Since the matrix appearing here is symmetric, we can solve this by

$$|0\rangle = N \exp\left\{ -\frac{1}{2}\bar{a}^\dagger \beta^* \alpha^{-1} \bar{a}^\dagger \right\} |\bar{0}\rangle, \tag{3.91}$$

where N is a normalization constant.

We now note that $\alpha^{-1}\beta$ is a symmetric matrix, since

$$\alpha^{-1}\beta = \beta^T \alpha^{T-1} = (\alpha^{-1}\beta)^T, \tag{3.92}$$

where in the first step we have used the first relation in (3.82). Therefore, its complex conjugate $\alpha^{*-1}\beta^*$ is also symmetric. The first relationship in (3.76) can be written, in matrix notation, as

$$\bar{a} = \alpha^* a - \beta^* a^\dagger, \tag{3.93}$$

therefore

$$a = -\alpha^{*-1}\bar{a} + \alpha^{*-1}\beta^* a^\dagger. \tag{3.94}$$

If we act on the vacuum $|\bar{0}\rangle$, which is annihilated by all operators $\bar{a}_i$, we find

$$a|\bar{0}\rangle = \alpha^{*-1}\beta^* a^\dagger |\bar{0}\rangle. \tag{3.95}$$

and we can write

$$|\bar{0}\rangle = \bar{N} \exp\left\{ \frac{1}{2} a^\dagger \alpha^{*-1} \beta^* a^\dagger \right\} |0\rangle. \tag{3.96}$$

(3.95) and (3.96) give the relationship between the vacua.

3.3 Two-point functions

Much information about a quantum field is encoded in its two-point functions. There are various types of two-point or Green functions. Particularly useful is the so-called *Wightman functions*, which are defined simply by

$$G^+(x,x') = \langle 0|\phi(x)\phi(x')|0\rangle, \qquad G^-(x,x') = \langle 0|\phi(x')\phi(x)|0\rangle. \tag{3.97}$$

These functions depend on the choice of the vacuum. If we use the mode decomposition

$$\phi(x) = \int \mathrm{d}^{d-1}\mathbf{k} \left[a_\mathbf{k} u_\mathbf{k}(x) + a_\mathbf{k}^\dagger u_\mathbf{k}^*(x) \right] \tag{3.98}$$

appropriate for a given choice of vacuum, then one has

$$G^+(x,x') = \int \mathrm{d}^{d-1}\mathbf{k}\, u_\mathbf{k}(x) u_\mathbf{k}^*(x'). \tag{3.99}$$

Since $\phi(x)$ satisfies the KG equation, we must have

$$\left(\Box_x + m^2 + \xi R\right) G^+(x,x') = 0. \tag{3.100}$$

Let us compute the Wightman function for a massless scalar field in four-dimensional Minkowski spacetime. We can use translation invariance to set $x' = 0$, and we will denote

$$G^+(x) = G^+(x,0). \tag{3.101}$$

Using the plane waves (3.3), we obtain

$$G^+(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k} \mathrm{e}^{-\mathrm{i}(kt-\mathbf{k}\cdot\mathbf{x})} \tag{3.102}$$

We can use spherical coordinates to write

$$\mathbf{k}\cdot\mathbf{x} = kr\cos\theta, \tag{3.103}$$

so that

$$G^+(x) = \int \frac{k\mathrm{d}k\sin\theta\mathrm{d}\varphi\mathrm{d}\theta}{2(2\pi)^3}\mathrm{e}^{\mathrm{i}kr\cos\theta - kt} = -\frac{\mathrm{i}}{2(2\pi)^2 r}\int_0^\infty \mathrm{d}k\left(\mathrm{e}^{-\mathrm{i}k(t-r)} - \mathrm{e}^{-\mathrm{i}k(r+t)}\right). \tag{3.104}$$

This integral is divergent and has to be regularized. We can shift $t \to t - \mathrm{i}\epsilon$, with $\epsilon > 0$, so that

$$\int_0^\infty \mathrm{d}k\mathrm{e}^{-\mathrm{i}k(t-r-\mathrm{i}\epsilon)} = -\frac{\mathrm{i}}{t-r-\mathrm{i}\epsilon} \tag{3.105}$$

This has to be understood as a distribution, since

$$\lim_{\epsilon\to 0}\frac{1}{x\pm\mathrm{i}\epsilon} = \mathrm{PP}\frac{1}{x} \mp \mathrm{i}\pi\delta(x). \tag{3.106}$$

We then find,

$$G^+(x) = \frac{1}{2(2\pi)^2 r}\left[\frac{1}{t+r-\mathrm{i}\epsilon} - \frac{1}{t-r-\mathrm{i}\epsilon}\right] = -\frac{1}{4\pi^2}\frac{1}{(t-\mathrm{i}\epsilon)^2 - r^2}. \tag{3.107}$$

We now use

$$\mathrm{PP}\frac{1}{t+r} - \mathrm{PP}\frac{1}{t-r} = -2r\mathrm{PP}\frac{1}{t^2-r^2}, \tag{3.108}$$

and

$$\delta(t^2-r^2) = \frac{1}{2r}\left\{\delta(t-r)+\delta(t+r)\right\}, \qquad \delta(t^2-r^2)\epsilon(t) = \frac{1}{2r}\left\{\delta(t-r)-\delta(t+r)\right\} \tag{3.109}$$

to write,

$$G^+(x) = -\frac{1}{4\pi^2}\mathrm{PP}\frac{1}{x^2} - \frac{\mathrm{i}}{4\pi}\delta(x^2)\epsilon(t). \tag{3.110}$$

In the following we will be interested in the value of the Wightman function for a massless field in d dimensions. To obtain this, we first go to Euclidean space, and we consider the consider the Laplacian operator $A = -\Box_x$ on Euclidean $\mathbb{R}^d$. The heat kernel of an operator A is defined as by

$$K(x,x';\tau) = \langle x|\mathrm{e}^{-\tau A}|x'\rangle. \tag{3.111}$$

In the case of the Laplacian, it is immediate to see that the heat kernel solves the heat equation

$$\Box_x K(x,x';\tau) = \frac{\partial K(x,x';\tau)}{\partial\tau}, \tag{3.112}$$

with boundary condition

$$K(x,x';0) = \delta(x-x'). \tag{3.113}$$

The Euclidean Green's function of A is then given by

$$G(x,x') = \int_0^\infty K(x,x';\tau)\mathrm{d}\tau, \tag{3.114}$$

since it is formally given by

$$G(x,x') = \langle x|\frac{1}{A}|x'\rangle. \tag{3.115}$$

On $\mathbb{R}^d$, the heat equation with the b.c. (3.113) can be solved explicitly by

$$K(x,x';\tau) = \frac{1}{(4\pi\tau)^{d/2}}\exp\left[-\frac{(x-x')^2}{4\tau}\right], \tag{3.116}$$

and the integration can be done immediately, with the result

$$G(x,x') = \frac{\Gamma(d/2-1)}{(4\pi)^{d/2}}\frac{1}{\ell^{d-2}}, \qquad \ell^2 = (x-x')^2. \tag{3.117}$$

When going to Minkowski space, we have

$$x^2 = x_d^2 + \mathbf{x}^2 \to -t^2 + \mathbf{x}^2, \tag{3.118}$$

and in addition we have to introduce the ϵ prescription $t \to t - \mathrm{i}\epsilon$, as in (3.107). We then obtain,

$$G^+(x) = \frac{\Gamma(d/2-1)}{(4\pi)^{d/2}}\frac{(-1)^{d/2-1}}{[(t-\mathrm{i}\epsilon)^2 - \mathbf{x}^2]^{d/2-1}}, \tag{3.119}$$

and we recover (3.107) for $d=4$.

3.4 Conformal vacuum

One of the lessons of the above formalism is that, in a general curved space-time, there is no canonical choice of vacuum. However, in some situations there are "natural" choices. As in Minkowski space, such choices are possible when there are special symmetries in the gravitational background. One example of this are conformally flat backgrounds. Let us assume that we have a conformally flat metric, i.e.

$$g_{\mu\nu}(x) = \Omega^2(x)\eta_{\mu\nu}. \tag{3.120}$$

Let us further assume that we have a massless, conformally coupled scalar. In this case, it is clear that the solutions of the KG equation can be obtained from the Minkowski solution $u_{\mathbf{k}}^{\mathrm{M}}(x)$ given in (3.3) simply by using the conformal transformation (3.33), i.e. there is a canonical choice of solutions of the wave equation for a massless, conformally coupled scalar given by

$$u_{\mathbf{k}}(x) = \Omega^{\frac{2-d}{2}}(x)u_{\mathbf{k}}^{\mathrm{M}}(x). \tag{3.121}$$

The choice of vacuum associated to this choice of modes is called the *conformal vacuum*. For the conformal vacuum, the two-point functions are easily computed. One finds,

$$G^{\pm}(x,x') = \Omega^{\frac{2-d}{2}}(x)G_{\mathrm{M}}^{\pm}(x,x')\Omega^{\frac{2-d}{2}}(x'), \tag{3.122}$$

where $G_{\mathrm{M}}^{\pm}(x,x')$ is the Wightman function in Minkowski space.

3.5 Cosmological particle creation: a toy model with harmonic oscillators

In order to motivate the phenomenon of cosmological particle creation, we consider a simple toy model of a (quantum) harmonic oscillator with time-dependent frequency.

We will consider a harmonic oscillator in the Heisenberg picture, where operators are time-dependent. The Lagrangian is

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2(t)x^2, \tag{3.123}$$

where we considered a time-dependent frequency $\omega(t)$. The equation of motion for the operator $x(t)$, which can be defined from the Schrödinger picture by

$$x(t) = \mathrm{e}^{\mathrm{i}Ht/\hbar}\, x\, \mathrm{e}^{-\mathrm{i}Ht/\hbar}, \tag{3.124}$$

is

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega^2(t)x^2 = 0. \tag{3.125}$$

We also have the time-dependent commutator

$$[x(t), p(t)] = \mathrm{i}, \tag{3.126}$$

or

$$[x(t), \dot{x}(t)] = \mathrm{i}. \tag{3.127}$$

The equation (3.125) is the analogue of the wave equation in QFT, and we solve it in terms of time-independent creation/annihilation operators as

$$x(t) = f(t)a + f^*(t)a^\dagger. \tag{3.128}$$

It follows that $f(t)$ satisfies

$$\ddot{f} + \omega^2(t)f = 0. \tag{3.129}$$

In terms of f, the commutation relation reads

$$\langle f, f\rangle [a, a^\dagger] = 1, \tag{3.130}$$

where

$$\langle f, g\rangle = \mathrm{i}(f^*\partial_t g - (\partial_t f^*)g). \tag{3.131}$$

We can choose f to be orthonormal

$$\langle f, f\rangle = 1. \tag{3.132}$$

We will consider a situation in which the frequency evolves between two constant values at asymptotic infinity. The boundary conditions are

$$\lim_{t\to-\infty}\omega(t) = \omega_{\text{in}}, \qquad \lim_{t\to\infty}\omega(t) = \omega_{\text{out}}. \tag{3.133}$$

We assume that the oscillator starts in the ground state

$$|0\rangle_{\text{in}} \tag{3.134}$$

and we wonder how this state will look like when expanded in terms of the states appropriate for the Hilbert space $\mathcal{H}_{\text{out}}$. We will solve this problem in terms of the QFT-like formalism we introduced above. The function $f(t)$ describing the harmonic oscillator, has, at asymptotic times, its conventional value for a harmonic oscillator of constant frequency,

$$
\begin{aligned}
f_{\rm in} &\to \sqrt{\frac{1}{2\omega_{\rm in}}}\mathrm{e}^{-\mathrm{i}\omega_{\rm in}t}, \quad t\to-\infty, \\
f_{\rm out} &\to \sqrt{\frac{1}{2\omega_{\rm out}}}\mathrm{e}^{-\mathrm{i}\omega_{\rm out}t}, \quad t\to\infty.
\end{aligned}
\tag{3.135}
$$

The analogue of (3.68) is here

$$
f_{\rm out} = \alpha f_{\rm in} + \beta f_{\rm in}^*, \tag{3.136}
$$

The second equation in (3.77) then reads

$$
|\alpha|^2 - |\beta|^2 = 1. \tag{3.137}
$$

The relation (3.91) reads in this simple case

$$
\begin{aligned}
|0\rangle_{\rm in} &= N\exp\Big[-\frac{\beta^*}{2\alpha}a^\dagger_{\rm out}a^\dagger_{\rm out}\Big]|0\rangle_{\rm out} \\
&= N\sum_{n\geq 0}\frac{(-1)^n}{n!}\Big(\frac{\beta^*}{2\alpha}\Big)^n(a^\dagger_{\rm out})^{2n}|0\rangle_{\rm out} \\
&= N\sum_{n\geq 0}(-1)^n\frac{\sqrt{(2n)!}}{n!}\Big(\frac{\beta^*}{2\alpha}\Big)^n|2n\rangle
\end{aligned}
\tag{3.138}
$$

An alternative way of writing this state uses the squeeze operator

$$
S = \exp\Big[\frac{z}{2}a^\dagger a^\dagger - \frac{\bar z}{2}aa\Big]. \tag{3.139}
$$

Since its exponent is anti-hermitian, S is unitary. Conjugating a by S yields

$$
S^\dagger a S = \cosh|z|\,a + \sinh|z|\frac{z}{|z|}\,a^\dagger. \tag{3.140}
$$

This has the form of the Bogoliubov transformation (3.76) with

$$
\alpha = \cosh|z, \qquad \beta = \sinh|z|(\bar z/|z|). \tag{3.141}
$$

With $a_{\rm out}$ in place of a in S this gives

$$
a_{\rm in} = S^\dagger a_{\rm out} S. \tag{3.142}
$$

The condition $a_{\rm in}|0_{\rm in}\rangle = 0$ thus implies $a_{\rm out}S|0_{\rm in}\rangle = 0$, so evidently

$$
|0_{\rm in}\rangle = S^\dagger|0_{\rm out}\rangle \tag{3.143}
$$

up to a constant phase factor. That is, the *in* and *out* ground states are related by the action of the squeeze operator S. Since S is unitary, the right hand side of (3.143) is manifestly normalized.

Example 3.4. Let us consider the situation in which we have a sudden change in the frequency,

$$\omega(t) = \begin{cases} \omega_{\rm in}, & t < 0, \\ \omega_{\rm out}, & t > 0. \end{cases} \tag{3.144}$$

In this case,

$$f_{\rm in}(t) = \frac{1}{\sqrt{2\omega_{\rm in}}} e^{-i\omega_{\rm in} t}, \qquad f_{\rm out}(t) = \frac{1}{\sqrt{2\omega_{\rm out}}} e^{-i\omega_{\rm out} t}. \tag{3.145}$$

This is a slightly different case, since we do not have smooth solutions defined for all t. Let us consider the solution of (3.129) which is given by $f_{\rm out}(t)$, for $t > 0$. For $t < 0$, this solution must become a linear combination of $f_{\rm in}(t)$ and $f^\star_{\rm in}(t)$, which are the two independent solutions of (3.129) in this region. The relation (3.136) is now a matching between the in and the out solutions, done at the point of sudden transition $t = 0$. Requiring matching of the functions and their derivatives, we find

$$\begin{aligned} f_{\rm out}(0) &= \alpha f_{\rm in}(0) + \beta f^\star_{\rm int}(0), \\ f'_{\rm out}(0) &= \alpha f'_{\rm in}(0) + \beta f'^\star_{\rm int}(0), \end{aligned} \tag{3.146}$$

which give

$$\begin{aligned} \alpha &= \frac{1}{2}\left(\sqrt{\frac{\omega_{\rm in}}{\omega_{\rm out}}} + \sqrt{\frac{\omega_{\rm out}}{\omega_{\rm in}}}\right), \\ \beta &= \frac{1}{2}\left(\sqrt{\frac{\omega_{\rm in}}{\omega_{\rm out}}} - \sqrt{\frac{\omega_{\rm out}}{\omega_{\rm in}}}\right). \end{aligned} \tag{3.147}$$

3.6 Cosmological particle creation

We now apply the ideas just developed to a free scalar quantum field satisfying the KG equation in a Robertson–Walker spacetime, representing an expanding universe. In this case, there is a preferred splitting of the spacetime into spatial slices and a time direction, therefore we can proceed with the canonical quantization. We will show that the modes of a scalar field are similar to the modes of a harmonic oscillator with a time-dependent frequency. If we have a Minkowski spacetime asymptotically, both at infinite past and future, we can define two vacua – one in the asymptotic past and another in the asymptotic future. As in the case of the harmonic oscillator considered above, the vacuum in the remote past will look like a squeezed state in the asymptotic future, and will contain excitations of arbitrary number. This result can be interpreted as a creation of particles due to cosmological evolution.

Let us consider the spatially flat RW metric (2.9), with $k = 0$. The d'Alembertian $\Box$ for this metric is given by

$$\Box\phi = C(\eta)^{-d/2}\partial_\eta\left(C(\eta)^{\frac{d-2}{2}}\partial_\eta\phi\right) - C^{-1}\nabla^2\phi, \tag{3.148}$$

where η is the conformal time (2.8). The spatial translation symmetry allows the spatial dependence to be separated from the time dependence, and we consider the solution to the wave-equation

$$u_{\mathbf{k}}(\eta, \mathbf{x}) = \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{(d-1)/2}} C^{\frac{2-d}{4}}(\eta)\chi_k(\eta). \tag{3.149}$$

A direct calculation leads to

$$\Box u_{\mathbf{k}} = \frac{C^{-1/2-d/4}}{(2\pi)^{(d-1)/2}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \left[\chi_k''(\eta) + k^2 \chi_k(\eta) + \frac{2-d}{4} \left(\frac{C''}{C} + \frac{d-6}{4} \left(\frac{C'}{C} \right)^2 \right) \chi_k(\eta) \right]$$
$$= \frac{C^{-1/2-d/4}}{(2\pi)^{(d-1)/2}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \left(\chi_k''(\eta) + k^2 \chi_k(\eta) - \xi(d) C R \chi_k(\eta) \right), \tag{3.150}$$

where

$$R = (d-1) \left[\frac{C''}{C^2} + \frac{d-6}{4} \frac{(C')^2}{C^3} \right] \tag{3.151}$$

is the scalar curvature of the RW metric. In terms of $a(\eta)$, we also have

$$R = (d-1) \left[2 \frac{a''}{a^3} + (d-4) \frac{(a')^2}{a^4} \right]. \tag{3.152}$$

The above equation can be derived in a simple way by noticing that the RW metric is conformally flat with conformal factor $\Omega^2 = C(\eta)$. Therefore, by applying (3.42) we obtain

$$\Box u_{\mathbf{k}} = -\xi(n) R u_{\mathbf{k}} + C^{-1/2-d/4} \left(\partial_\eta^2 - \nabla^2 \right) \left(\frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{(d-1)/2}} \chi_k(\eta) \right) \tag{3.153}$$

which is precisely (3.150). Notice that the power of $C(\eta)$ in (3.149) is dictated by conformal invariance. Putting everything together, we find that $u_{\mathbf{k}}$ solves the KG equation (3.29) if χ_k solves

$$\chi_k'' + \omega^2(\eta) \chi_k = 0, \tag{3.154}$$

where the prime stands for $\mathrm{d}/\mathrm{d}\eta$ and

$$\omega^2(\eta) = k^2 + m^2 C(\eta) + C(\eta) R(\eta) \left(\xi - \xi(d) \right), \tag{3.155}$$

where $\xi(d)$ is given in (3.41). The normalization condition for the modes is the following

$$\langle u_{\mathbf{k}}, u_{\mathbf{k}'} \rangle = \delta(\mathbf{k} - \mathbf{k}'). \tag{3.156}$$

For the definition of the scalar product, we consider the hypersurface $\eta =$ constant. The induced metric on the spatial sections is,

$$h_{ij} \mathrm{d}x^i \mathrm{d}x^j = C(\eta) \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j, \tag{3.157}$$

we find

$$\langle u_{\mathbf{k}}, u_{\mathbf{k}'} \rangle = \mathrm{i} \int \mathrm{d}^{d-1}\mathbf{x}\, C^{d/2-1} \left(u_{\mathbf{k}}^* \partial_\eta u_{\mathbf{k}'} - \partial_\eta u_{\mathbf{k}}^* u_{\mathbf{k}'} \right) = \mathrm{i} \delta(\mathbf{k} - \mathbf{k}') \left(\chi_k^* \partial_\eta \chi_k - \partial_\eta \chi_k^* \chi_k \right) \tag{3.158}$$

so that the modes are properly normalized if

$$\mathrm{i} \left(\chi_k^* \partial_\eta \chi_k - \partial_\eta \chi_k^* \chi_k \right) = 1. \tag{3.159}$$

This is a normalization condition on the Wronskian of the solutions to (3.154). The fact that the Wronskian is indepedent of η is a manifestation of the fact that the inner product does not depend on the particular hypersurface η =constant which we use.

Consider now the special case where there is no time dependence in the past and future, i.e.

$$C(\eta) \to \begin{cases} C_1, & \eta \to -\infty, \\ C_2, & \eta \to \infty. \end{cases} \tag{3.160}$$

These constants can be the same, as long as the function $C(\eta)$ does something non-trivial in between. In this case, the asymptotic past and future spacetimes are Minkowski spacetimes, and the η-dependent frequencies have limiting values

$$\omega(\eta) \to \begin{cases} \omega_{\rm in}, & \eta \to -\infty, \\ \omega_{\rm out}, & \eta \to \infty. \end{cases} \tag{3.161}$$

We can then define two natural vacua, at asymptotic past and future. These vacua are defined by considering the basis of solutions $u_{\mathbf{k}}^{\rm in,out}$ where the $\chi_k^{\rm in,out}$ satisfy the following asymptotic condition:

$$\begin{aligned} \chi_k^{\rm in}(\eta) &\to \sqrt{\frac{1}{2\omega_{\rm in}}}\exp(-{\rm i}\omega_{\rm in}\eta), \quad \eta \to -\infty, \\ \chi_k^{\rm out}(\eta) &\to \sqrt{\frac{1}{2\omega_{\rm out}}}\exp(-{\rm i}\omega_{\rm out}\eta), \quad \eta \to \infty, \end{aligned} \tag{3.162}$$

These modes define annihilation operators $a_{\mathbf{k}}^{\rm in,out}$, and the corresponding vacua are annihilated by them:

$$a_{\mathbf{k}}^{\rm in,out}|0_{\rm in,out}\rangle = 0. \tag{3.163}$$

Due to spatial homogeneity, we see that both $u_{\mathbf{k}}^{\rm in,out}$ are plane waves. This means that $\alpha_{\mathbf{k}\mathbf{k}'}$ will be proportional to $\delta(\mathbf{k}-\mathbf{k}')$, and since $\beta_{\mathbf{k}\mathbf{k}'}$ involves the product of $u_{\mathbf{k}}^{\rm in}$ with $u_{\mathbf{k}'}^{*\rm out}$, it will be proportional to $\delta(\mathbf{k}+\mathbf{k}')$:

$$\alpha_{\mathbf{k}\mathbf{k}'} = \alpha_k\delta(\mathbf{k}-\mathbf{k}'), \qquad \beta_{\mathbf{k}\mathbf{k}'} = \beta_k\delta(\mathbf{k}+\mathbf{k}'), \tag{3.164}$$

i.e. the Bogoliubov coefficients mix only modes of wave vectors $\mathbf{k}$ and $-\mathbf{k}$, and in addition they depend only upon the magnitude of the wavevector due to rotational symmetry. The Bogoliubov transformation will then take the form

$$u_{\mathbf{k}}^{\rm in} = \alpha_k u_{\mathbf{k}}^{\rm out} + \beta_k u_{-\mathbf{k}}^{*\rm out} \tag{3.165}$$

The first condition of (3.77) is automatically satisfied in this case, while the second condition implies

$$|\alpha_k|^2 - |\beta_k|^2 = 1. \tag{3.166}$$

The condition that the vacuum in the in region contains only a finite number of modes in the out Hilbert space reads,

$$\int {\rm d}^{n-1}\mathbf{k}\,|\beta_k|^2 < \infty. \tag{3.167}$$

The RW metric is conformally flat, as it is manifest in (2.9). Therefore, in the case of a massless, conformally coupled scalar, we can always construct a conformal vacuum. Indeed, in this case we have

$$\omega^2 = k^2, \tag{3.168}$$

and the solution is

$$\chi_k(\eta) = \frac{1}{\sqrt{2\omega_k}}{\rm e}^{-{\rm i}k\eta}. \tag{3.169}$$

Therefore, the solution (3.149) is precisely (3.121), since $\Omega = (C(\eta))^{1/2}$.

3.7 An exactly solvable model of cosmological particle creation

It is interesting to work out some non-trivial models where the equation for χ_k can be solved explicitly. One such model has been considered in [3]. This is a two-dimensional model with $\xi = 0$ and

$$C(\eta) = A + B\tanh(\rho\eta). \tag{3.170}$$

In this case,

$$C(\eta) \to \begin{cases} A - B, & \eta \to -\infty, \\ A + B, & \eta \to \infty. \end{cases} \tag{3.171}$$

and

$$\omega_{\text{in}} = \left[k^2 + m^2(A - B)\right]^{1/2}, \qquad \omega_{\text{out}} = \left[k^2 + m^2(A + B)\right]^{1/2} \tag{3.172}$$

Let us define

$$\omega_{\pm} = \frac{1}{2}\left(\omega_{\text{out}} \pm \omega_{\text{in}}\right). \tag{3.173}$$

The EOM for χ_k can be solved in terms of hypergeometric functions. One finds a solution

$$\begin{aligned} \chi_k^{\text{in}}(\eta) = & \frac{1}{\sqrt{2\omega_{\text{in}}}} \exp\left[-\mathrm{i}\omega_+\eta - \mathrm{i}\frac{\omega_-}{\rho}\log\left(2\cosh(\rho\eta)\right)\right] \\ & \times\ {}_2F_1\left(1 + \mathrm{i}\frac{\omega_-}{\rho}, \mathrm{i}\frac{\omega_-}{\rho}; 1 - \mathrm{i}\frac{\omega_{\text{in}}}{\rho}; \frac{1}{2}\left(1 + \tanh(\rho\eta)\right)\right). \end{aligned} \tag{3.174}$$

As $\eta \to -\infty$, the hypergeometric function goes to 1, $\log\left(2\cosh(\rho\eta)\right) \to -\eta$, and we have

$$\chi_k^{\text{in}}(\eta) \to \frac{1}{\sqrt{2\omega_{\text{in}}}} \mathrm{e}^{-\mathrm{i}\omega_{\text{in}}\eta}, \tag{3.175}$$

as required for an "in" function. The out function turns out to be

$$\begin{aligned} \chi_k^{\text{out}}(\eta) = & \frac{1}{\sqrt{2\omega_{\text{out}}}} \exp\left[-\mathrm{i}\omega_+\eta - \mathrm{i}\frac{\omega_-}{\rho}\log\left(2\cosh(\rho\eta)\right)\right] \\ & \times\ {}_2F_1\left(1 + \mathrm{i}\frac{\omega_-}{\rho}, \mathrm{i}\frac{\omega_-}{\rho}; 1 + \mathrm{i}\frac{\omega_{\text{out}}}{\rho}; \frac{1}{2}\left(1 - \tanh(\rho\eta)\right)\right). \end{aligned} \tag{3.176}$$

As $\eta \to \infty$, the hypergeometric function goes to 1, $\log\left(2\cosh(\rho\eta)\right) \to \eta$, and we have

$$\chi_k^{\text{out}}(\eta) \to \frac{1}{\sqrt{2\omega_{\text{out}}}} \mathrm{e}^{-\mathrm{i}\omega_{\text{out}}\eta}, \tag{3.177}$$

as required for an "out" function. Since the complex conjugate of $\chi_k^{\text{out}}(\eta)$ provides another, independent solution of the original differential equation, we must have a linear relation

$$\chi_k^{\text{in}}(\eta) = \alpha_k \chi_k^{\text{out}}(\eta) + \beta_k \chi_k^{*\text{out}}(\eta). \tag{3.178}$$

To find the explicit form of such a relationship, we use the identities (A.6) and (A.7) for hypergeometric functions. Combining both identities, we find

$$\begin{aligned} {}_2F_1\left(\alpha, \beta; \gamma; z\right) = & \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}\ {}_2F_1\left(\alpha, \beta; 1 + \alpha + \beta - \gamma; 1 - z\right) \\ & + \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1 - z)^{\gamma - \alpha - \beta} z^{1-\gamma}\ {}_2F_1\left(1 - \beta, 1 - \alpha; 1 + \gamma - \alpha - \beta; 1 - z\right) \end{aligned} \tag{3.179}$$

To apply these identities to our case, we set

$$\alpha = 1 + \mathrm{i}\frac{\omega_-}{\rho}, \quad \beta = \mathrm{i}\frac{\omega_-}{\rho}, \quad \gamma = 1 + \mathrm{i}\frac{\omega_{\mathrm{in}}}{\rho} \tag{3.180}$$

and

$$z = \frac{1}{2}\left(1 + \tanh(\rho\eta)\right). \tag{3.181}$$

Therefore,

$$(1-z)^{\gamma-\alpha-\beta} z^{1-\gamma} = \left[\frac{1-z}{z}\right]^{-\mathrm{i}\omega_+/\rho} \left[z(1-z)\right]^{-\mathrm{i}\omega_-/\rho}. \tag{3.182}$$

Since

$$z(1-z) = \frac{1}{4\cosh^2(\eta\rho)}, \quad \frac{1-z}{z} = \mathrm{e}^{-2\eta\rho} \tag{3.183}$$

we find

$$\begin{aligned}
&{}_2F_1\left(1 + \mathrm{i}\frac{\omega_-}{\rho}, \mathrm{i}\frac{\omega_-}{\rho}; 1 - \mathrm{i}\frac{\omega_{\mathrm{in}}}{\rho}; \frac{1}{2}\left(1 + \tanh(\rho\eta)\right)\right) \\
&= \frac{\Gamma(1-\mathrm{i}\omega_{\mathrm{in}}/\rho)\Gamma(-\mathrm{i}\omega_{\mathrm{out}}/\rho)}{\Gamma(-\mathrm{i}\omega_+/\rho)\Gamma(1-\mathrm{i}\omega_+/\rho)}\, {}_2F_1\left(1 + \mathrm{i}\frac{\omega_-}{\rho}, \mathrm{i}\frac{\omega_-}{\rho}; 1 + \mathrm{i}\frac{\omega_{\mathrm{out}}}{\rho}; \frac{1}{2}\left(1 + \tanh(\rho\eta)\right)\right) \\
&+ \frac{\Gamma(1-\mathrm{i}\omega_{\mathrm{in}}/\rho)\Gamma(\mathrm{i}\omega_{\mathrm{out}}/\rho)}{\Gamma(1+\mathrm{i}\omega_-/\rho)\Gamma(\mathrm{i}\omega_-/\rho)} \mathrm{e}^{2\mathrm{i}\omega_+\eta} \left(2\cosh^2(\eta\rho)\right)^{2\mathrm{i}\omega_-/\rho}\, {}_2F_1\left(1 - \mathrm{i}\frac{\omega_-}{\rho}, -\mathrm{i}\frac{\omega_-}{\rho}; 1 - \mathrm{i}\frac{\omega_{\mathrm{out}}}{\rho}; \frac{1}{2}\left(1 - \tanh(\rho\eta)\right)\right)
\end{aligned} \tag{3.184}$$

from which we deduce

$$\begin{aligned}
\alpha_k &= \left(\frac{\omega_{\mathrm{out}}}{\omega_{\mathrm{in}}}\right)^{1/2} \frac{\Gamma(1-\mathrm{i}\omega_{\mathrm{in}}/\rho)\Gamma(-\mathrm{i}\omega_{\mathrm{out}}/\rho)}{\Gamma(-\mathrm{i}\omega_+/\rho)\Gamma(1-\mathrm{i}\omega_+/\rho)}, \\
\beta_k &= \left(\frac{\omega_{\mathrm{out}}}{\omega_{\mathrm{in}}}\right)^{1/2} \frac{\Gamma(1-\mathrm{i}\omega_{\mathrm{in}}/\rho)\Gamma(\mathrm{i}\omega_{\mathrm{out}}/\rho)}{\Gamma(1+\mathrm{i}\omega_-/\rho)\Gamma(\mathrm{i}\omega_-/\rho)}.
\end{aligned} \tag{3.185}$$

Using now

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sinh(\pi z)}, \qquad \Gamma(1+z) = z\Gamma(z), \tag{3.186}$$

we find

$$\begin{aligned}
|\alpha_k|^2 &= \frac{\sinh^2\left(\frac{\pi\omega_+}{\rho}\right)}{\sinh\left(\frac{\pi\omega_{\mathrm{in}}}{\rho}\right)\sinh\left(\frac{\pi\omega_{\mathrm{out}}}{\rho}\right)}, \\
|\beta_k|^2 &= \frac{\sinh^2\left(\frac{\pi\omega_-}{\rho}\right)}{\sinh\left(\frac{\pi\omega_{\mathrm{in}}}{\rho}\right)\sinh\left(\frac{\pi\omega_{\mathrm{out}}}{\rho}\right)},
\end{aligned} \tag{3.187}$$

which satisfy indeed

$$|\alpha_k|^2 - |\beta_k|^2 = 1. \tag{3.188}$$

Therefore, if a quantum field is in the vacuum $|0_{\mathrm{in}}\rangle$ in the infinite past, it will contain $|\beta_k|^2$ particles with momentum $\mathbf{k}$ in the infinite future, with respect to the out vacuum, and after the cosmological expansion is over.

4 Quantum fields in de Sitter space

de Sitter space plays an important rôle in modern physics, since it describes the period of rapid expansion typical of theories of inflation. On a more mathematical level, it is a maximally symmetric space and a solution of the Einstein's equations with a positive cosmological constant. A useful summary of results on quantum field theory in de Sitter space can be found in [25].

4.1 Geometry of de Sitter space

De Sitter space in d dimensions is described by the hyperboloid,

$$X_0^2 - \sum_{i=1}^{d} X_i^2 = -H^{-2} \tag{4.1}$$

embedded in a $(d+1)$-dimensional Minkowski space with metric

$$\mathrm{d}s^2 = \mathrm{d}X_0^2 - \sum_{i=1}^{d} \mathrm{d}X_i^2 . \tag{4.2}$$

In the same way that the two-sphere embeded in $\mathbb{R}^3$ inherits the $O(3)$ symmetry from its ambient space, de Sitter space inherits an $O(1,d)$ symmetry from the Minkowski ambient space. It can be covered by various coordinate systems:

1. *Global coordinates.* These are the coordinates τ, ω_i defined by

$$X_0 = \frac{1}{H}\sinh\left(H\tau\right), \quad X_i = \frac{1}{H}\omega_i \cosh\left(H\tau\right), \qquad i = 1, \cdots, d, \tag{4.3}$$

   where ω_i are coordinates in $\mathbb{S}^{d-1}$, i.e.

$$\sum_{i=1}^{d} \omega_i^2 = 1. \tag{4.4}$$

   In these coordinates, the metric reads

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 - \frac{1}{H^2}\cosh^2\left(H\tau\right)\mathrm{d}\Omega_{d-1}^2, \tag{4.5}$$

   where $\mathrm{d}\Omega_{d-1}^2$ is the standard metric on a $(d-1)$-dimensional sphere of unit radius. Note that, in these coordinates, de Sitter space is a RW space with a spherical section and

$$a(\tau) = \frac{1}{H}\cosh(H\tau). \tag{4.6}$$

2. *Planar coordinates.* These are the coordinates t, $\mathbf{x}$, defined by

$$\begin{aligned} X_0 &= \frac{1}{H}\sinh\left(Ht\right) + \frac{H}{2}\mathbf{x}^2 \mathrm{e}^{tH}, \\ X_d &= \frac{1}{H}\cosh\left(Ht\right) - \frac{H}{2}\mathbf{x}^2 \mathrm{e}^{tH}, \\ X_i &= x_i \mathrm{e}^{tH}, \quad i = 1, \cdots, d-1. \end{aligned} \tag{4.7}$$

These coordinates do not cover the full de Sitter space, but only the patch

$$X_0 + X_d = \frac{1}{H} \mathrm{e}^{Ht} > 0. \tag{4.8}$$

In these coordinates, the metric reads

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{e}^{2Ht} \sum_{i=1}^{d-1} \mathrm{d}x_i^2. \tag{4.9}$$

This metric can be regarded as a RW cosmology with an exponential function $a(t)$:

$$a(t) = \mathrm{e}^{Ht}. \tag{4.10}$$

3. *Static coordinates.* They are given by

$$\begin{aligned} X_0 &= \frac{1}{H}\sqrt{1 - H^2 r^2} \sinh(Ht), \\ X_d &= \frac{1}{H}\sqrt{1 - H^2 r^2} \cosh(H\tau), \\ X_i &= r\omega_i, \quad i = 1, \cdots, d-1, \end{aligned} \tag{4.11}$$

where

$$0 \le r < \frac{1}{H}. \tag{4.12}$$

They only cover the region

$$X_d > 0, \quad \sum_{i=1}^{d-1} X_i^2 < \frac{1}{H^2}. \tag{4.13}$$

In these coordinates, the metric reads

$$\mathrm{d}s^2 = (1 - H^2 r^2)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - H^2 r^2} - r^2 \mathrm{d}\Omega_{d-1}^2. \tag{4.14}$$

Notice the presence of an explicit horizon at $r = 1/H$, which makes manifest the presence of event horizons for observers in de Sitter space.

4. *Conformal coordinates.* Let us start with the planar coordinates (4.9). The conformal time is given by

$$\eta = \int_\infty^t \frac{\mathrm{d}t'}{a(t')} = -\frac{1}{H} \mathrm{e}^{-Ht}. \tag{4.15}$$

Therefore, in principle

$$-\infty < \eta < 0, \tag{4.16}$$

and

$$\eta \to -\infty \tag{4.17}$$

corresponds to the infinite past $t \to \infty$. However, one can cover almost all of de Sitter spacetime by taking $\eta \in \mathbb{R}$. In these conformal coordinates, de Sitter space is parametrized as

$$
\begin{aligned}
X_0 &= \frac{1}{2\eta}\left(\eta^2 - \mathbf{x}^2 - \frac{1}{H^2}\right), \\
X_d &= -\frac{1}{2\eta}\left(\eta^2 - \mathbf{x}^2 + \frac{1}{H^2}\right), \\
X_i &= -\frac{1}{H\eta}x_i, \quad i = 1, \cdots, d-1,
\end{aligned} \tag{4.18}
$$

and the metric is given by

$$
\mathrm{d}s^2 = \frac{1}{H^2\eta^2}\left(\mathrm{d}\eta^2 - \sum_{i=1}^{d-1}\mathrm{d}x_i^2\right). \tag{4.19}
$$

Only the submanifold

$$
X_0 + X_d = 0 \tag{4.20}
$$

is not covered by these coordinates.

When written in conformal coordinates, the metric of de Sitter (4.19), is conformally flat, and we can the use the expression (3.151) to calculate its scalar curvature,

$$
R = d(d-1)H^2. \tag{4.21}
$$

Therefore, de Sitter space is a space of constant scalar curvature.

An important quantity in the geometry of de Sitter space is the *geodesic distance* between two points. As in the case of the sphere or the hyperbolic plane in two dimensions, the distance between two points of de Sitter is closely related to the distance defined in the embedding space. Therefore, we define

$$
P(X, X') = -H^2\eta_{ab}X^aX^{b'}. \tag{4.22}
$$

Notice that, if $X = X'$ are identical, we have $P = 1$. However, if X and X' are antipodal, i.e. $X' = -X$, one has $P = H^2\eta_{ab}X^aX^b = -1$. If we plug (4.18) in (4.22), we find the explicit expression of the geodesic distance in conformal coordinates:

$$
P(X, X') = \frac{1}{2\eta\eta'}\left\{\eta^2 + (\eta')^2 - (\mathbf{x} - \mathbf{x}')^2\right\} = 1 + \frac{(\eta - \eta')^2 - (\mathbf{x} - \mathbf{x}')^2}{2\eta\eta'}. \tag{4.23}
$$

One important property of $P(X, X')$ is that it is a manifestly $O(1, d)$ invariant function on de Sitter space, since it is constructed out of the Lorentz product in $\mathbb{R}^{1,d}$. Depending on the causal relationship between X and X', we have the following behavior for $P(X, X')$:

1. If X' is in the future or past of X, so that they are joined by a timelike geodesic, $P(X, X') > 1$ and the geodesic distance is given by

$$
d(X, X') = \frac{1}{H}\cosh^{-1}(P). \tag{4.24}
$$

2. If X and X' are spacelike separated, one has $|P(X, X')| < 1$, and

$$
d(X, X') = \frac{1}{H}\cos^{-1}(P). \tag{4.25}
$$

3. Finally, if X' is in the lightcone of X, $P(X, X') = 1$ and $d(X, X') = 0$.

Notice that there are points of de Sitter space which cannot be joined by geodesics to a given point X. These are the points in the interior of the past and future light cones of $-X$, the antipodal point of X. For these points, we have that $P(X, X') < -1$. The results listed above can be obtained by an explicit analysis of geodesics in de Sitter space, which can be found in Appendix B.

4.2 Scalar fields in de Sitter space

Let us now study the wave equation in de Sitter space. If we work in conformal coordinates, quantization in de Sitter space might be regarded as a particular case of the quantization in RW cosmologies described above, since the metric (4.19) is of the form (2.9) with a flat section. The function $C(\eta)$ is given by

$$C(\eta) = \frac{1}{H^2\eta^2}, \tag{4.26}$$

and in view of (4.21) we have

$$C(\eta)R(\eta) = \frac{d(d-1)}{\eta^2}. \tag{4.27}$$

The KG equation (3.154) becomes

$$\chi_k'' + \left[k^2 + \frac{m^2}{H^2\eta^2} + \frac{d(d-1)\left(\xi - \xi(d)\right)}{\eta^2}\right]\chi_k = 0. \tag{4.28}$$

Let us perform the change of variables:

$$s = -k\eta, \tag{4.29}$$

and let us define

$$\chi_k = \sqrt{s} f(s). \tag{4.30}$$

Then, an elementary computation shows that $f(s)$ satisfies the Bessel equation,

$$s^2 f''(s) + s f'(s) + \left(s^2 - \nu^2\right) f(s) = 0, \tag{4.31}$$

where

$$\nu^2 = \frac{1}{4} + \left(\xi(d) - \xi\right) d(d-1) - \frac{m^2}{H^2} = \frac{(d-1)^2}{4} - d(d-1)\xi - \frac{m^2}{H^2}. \tag{4.32}$$

In four dimensions we have

$$\nu^2 = \frac{9}{4} - 12\xi - \frac{m^2}{H^2}. \tag{4.33}$$

We can then write $\chi_k(\eta)$ in terms of Bessel functions, for $\eta < 0$, as

$$\chi_k(\eta) = \sqrt{k|\eta|}\left\{A_k J_\nu(k|\eta|) + B_k Y_\nu(k|\eta|)\right\} \tag{4.34}$$

The normalization (3.159) leads to the condition

$$-\,\mathrm{i}k^2\eta\left(A_k B_k^* - A_k^* B_k\right) W\left[J_\nu(k|\eta|), Y_\nu(k|\eta|)\right] = 1, \tag{4.35}$$

where $W[u, v] = uv' - u'v$ is the Wronskian, and the derivative is w.r.t. the total argument $k|\eta|$. Using that (see, for example, [18], 5.9.2)

$$W\left[J_\nu(x), Y_\nu(x)\right] = \frac{2}{\pi x}, \tag{4.36}$$

we find

$$A_k B_k^* - A_k^* B_k = -\frac{\mathrm{i}\pi}{2k} \tag{4.37}$$

In prder to make a choice of vacuum, we consider the behavior of the modes in the infinite past, for

$$k|\eta| \gg 1 \tag{4.38}$$

In this case, $\omega_k \approx k$ and we can consider modes of the form

$$\frac{1}{\sqrt{2\omega_k}} \mathrm{e}^{-\mathrm{i}\omega_k \eta} \tag{4.39}$$

which look like Minkowski modes in the conformal time η. We want to find solutions of the KG which have the asymptotic behavior (4.39) in the infinite past. If we look at the asymptotics of the Bessel function, we find

$$\chi_k \sim \sqrt{\frac{2}{\pi}} \left\{ \frac{A + \mathrm{i}B}{2} \mathrm{e}^{-\mathrm{i}\lambda} + \frac{A - \mathrm{i}B}{2} \mathrm{e}^{\mathrm{i}\lambda} \right\} \tag{4.40}$$

where

$$\lambda = k|\eta| - \frac{\nu\pi}{2} - \frac{\pi}{4}. \tag{4.41}$$

Therefore, we require $A + \mathrm{i}B = 0$. Together with the condition (4.37) we find

$$|A|^2 = \frac{\pi}{4k}. \tag{4.42}$$

and we obtain the modes

$$\chi_k(\eta) = \frac{1}{2} (\pi|\eta|)^{1/2} \left(J_\nu(k|\eta|) + \mathrm{i} Y_\nu(k|\eta|) \right) = \frac{1}{2} (\pi|\eta|)^{1/2} H_\nu^{(1)}(k|\eta|), \tag{4.43}$$

where we have expressed the result in terms of the Hankel function. With this choice of wavefunctions, the corresponding vacuum is called the *Euclidean vacuum*[1]. This vacuum has the following properties:

1. It is invariant under the de Sitter symmetry group. It is clearly invariant under rotations of the spatial coordinates $\mathbf{x}$, since $\chi_k(\eta)$ only depends on the modulus of $\mathbf{k}$. It is also invariant under the dilatation

   $$\eta \to \alpha\eta, \quad \mathbf{x} \to \alpha\mathbf{x}, \qquad \alpha \in \mathbb{R}^\star. \tag{4.44}$$

   Indeed, the wavevector transforms then as

   $$\mathbf{k} \to \frac{1}{\alpha}\mathbf{k}, \tag{4.45}$$

   and the argument of the Hankel function remains invariant. Collecting the overall factor $|\eta|^{1/2}$ in $\chi_k(\eta)$, together with the factor $C^{(d-2)/4}$ coming from (3.149), we get in total a factor $|\eta|^{(d-1)/2}$. However, there is a factor of $1/V^{1/2}$ in the wavefunction $u_{\mathbf{k}}$, where $V = L^{d-1}$ is the spatial volume. This factor combines with the η factor to produce

   $$(|\eta|/L)^{(d-1)/2}, \tag{4.46}$$

   which is invariant under dilatations. We will see that the invariance of this vacuum manifests itself in the $O(1,4)$ invariance of the corresponding Wightman function.

[1] This is sometimes called the Bunch–Davies vacuum, after the paper [6], although this vacuum was studied before Bunch and Davies in various papers, like for example [24].

2. When $m = 0$, $\xi = 1/6$, this vacuum reduces to the conformal vacuum. Indeed, in this case, the index of the Hankel function is $\nu = 1/2$, and we have

$$H_\nu^{(1)}(z) = -\mathrm{i}\left(\frac{2}{\pi z}\right)^{1/2} \mathrm{e}^{\mathrm{i}z} \tag{4.47}$$

so that

$$\chi_k(\eta) = -\frac{\mathrm{i}}{\sqrt{2k}}\mathrm{e}^{-\mathrm{i}k\eta} \tag{4.48}$$

which is, up to an overall phase, the solution for a conformal vacuum (3.169).

4.3 The two-point function on de Sitter space

We would like to compute the Wightman function for the Bunch–Davies vacuum. This can be done in two ways. One way is a brute-force approach, by plugging the equation for the modes in (3.99) and performing the integration explicitly. This has been done in [6, 24]. Another, more elegant approach is to solve the KG equation for $G^+(x, x')$ by assuming $O(1, d)$ invariance of the answer. This leads in fact to a more general family of vacua [1, 19].

If the Wightman function comes from a vacuum invariant under the de Sitter symmetry group, it should only depend on a $O(1, d)$ invariant distance between two points. This distance is nothing but the geodesic distance $P(x, x')$. Therefore, we have

$$G^+(x, x') = G^+(P(x, x')). \tag{4.49}$$

We now solve the equation (3.100), assuming that the dependence on x occurs through the geodesic distance $P(x, x')$. To start with, we list two useful properties of this function [8]:

$$\nabla^\mu P \nabla_\mu P = H^2(P^2 - 1), \qquad \nabla_\mu \nabla_\nu P = g_{\mu\nu} H^2 P. \tag{4.50}$$

These properties can be checked by direct computation in the coordinate system given by the conformal coordinates $\eta, \mathbf{x}$. To do this, one needs an explicit knowledge of the Christoffel symbols for the metric (2.9). One easily finds that the only non-zero Christoffel symbols are:

$$\Gamma^\eta_{\eta\eta} = \Gamma^i_{i\eta} = \Gamma^\eta_{ii} = \frac{1}{2}\frac{C'(\eta)}{C(\eta)}. \tag{4.51}$$

Notice that

$$\nabla_\mu \nabla_\nu P = \partial_\mu \partial_\nu P - \Gamma^\rho_{\mu\nu} \partial_\rho P. \tag{4.52}$$

Let us now consider a function $F(P)$ depending on the coordinates x through $P(x, x')$. We then have,

$$\begin{aligned} \Box F(P) &= g^{\mu\nu} \nabla_\mu \nabla_\nu F = g^{\mu\nu} \nabla_\mu \left(F'(P) \nabla_\nu P\right) \\ &= F''(P) g^{\mu\nu} \partial_\mu P \partial_\nu P + F'(P) g^{\mu\nu} \nabla_\mu \nabla_\nu P. \end{aligned} \tag{4.53}$$

Using now (4.50), we find

$$\Box F(P) = H^2(P^2 - 1)F''(P) + dH^2 P F'(P), \tag{4.54}$$

where d is the dimension of spacetime. It follows that the Wightman function for a scalar field satisfies,

$$(P^2 - 1)\partial_P^2 G + dP\partial_P G + \left(\frac{m^2}{H^2} + \xi d(d-1)\right) G = 0. \tag{4.55}$$

We now make the change of variables

$$z = \frac{1+P}{2} \tag{4.56}$$

to obtain

$$z(1-z)\partial_z^2 G + \left(\frac{d}{2} - dz\right)\partial_z G - \left(\frac{m^2}{H^2} + \xi d(d-1)\right) G = 0. \tag{4.57}$$

This is a hypergeometric equation. Indeed, if we compare to the standard form (A.3), we find that

$$\alpha + \beta = d-1, \qquad \alpha\beta = \frac{m^2}{H^2} + \xi d(d-1). \tag{4.58}$$

Therefore,

$$G(P) = c_d \, {}_2F_1\left(h_+, h_-, \frac{d}{2}; \frac{1+P}{2}\right), \tag{4.59}$$

where

$$h_\pm = \frac{1}{2}\left[d - 1 \pm \sqrt{(d-1)^2 - 4\left(\frac{m^2}{H^2} + \xi d(d-1)\right)}\right]. \tag{4.60}$$

The constant c_d can be fixed [2, 25] by using the short-distance singularity of the quantum propagator for a scalar field in a Minkowski space of dimension d, which is given by the massless propagator (3.119),

$$G^+(x,x') \approx (-1)^{d/2-1} \frac{\Gamma(d/2-1)}{4\pi^{d/2}} \frac{1}{(\ell^2)^{d/2-1}}, \qquad \ell^2 = \left(x - x'\right)^2. \tag{4.61}$$

Here, ℓ^2 is $\pm\tau^2$, where τ is the geodesic distance between two points, and the $\pm$ signs correspond to timelike (respectively, spacelike) separated events. Let us now write

$$P = 1 + \delta, \qquad \delta = \frac{(\Delta\eta)^2 - (\Delta\mathbf{x})^2}{2\eta\eta'}. \tag{4.62}$$

If two points are close, we have $|\delta| \ll 1$. It follows that the geodesic distance between two points separated by a timelike interval is

$$H\tau = \cosh^{-1}(P) = \cosh^{-1}(1+\delta) \approx \sqrt{2\delta}, \qquad \delta \to 0, \tag{4.63}$$

and we learn that

$$H^2\tau^2 = H^2\ell^2 \approx 2\delta, \tag{4.64}$$

so to investigate the short-distance behavior for timelike separations we can set

$$\delta \approx \frac{H^2\ell^2}{2}. \tag{4.65}$$

On the other hand, we have the following behavior for the hypergeometric function:

$${}_2F_1\left(h_+, h_-, \frac{d}{2}; \frac{1+P}{2}\right) \approx \frac{\Gamma(d/2)\Gamma(h_+ + h_- - d/2)}{\Gamma(h_+)\Gamma(h_-)} \left(-\frac{\delta}{2}\right)^{d/2 - h_+ - h_-}, \qquad \delta \to 0. \tag{4.66}$$

Using that

$$h_+ + h_- = d - 1, \tag{4.67}$$

we find that, in order to reproduce (4.61), we have to choose

$$c_d = H^{d-2} \frac{\Gamma(h_+)\Gamma(h_-)}{(4\pi)^{d/2}\Gamma(d/2)} \tag{4.68}$$

This determines the Wightman function in the Euclidean vacuum, in any dimension $d > 2$.

As a check of the above expression, we can verify that in the massless, conformally coupled case we reproduce the expected result (3.122). In this case, it is easy to see that

$$h_+ = \frac{d}{2}, \qquad h_- = \frac{d}{2} - 1. \tag{4.69}$$

The hypergeometric functions collapses to an elementary function,

$${}_2F_1\left(\frac{d}{2}, \frac{d}{2} - 1, \frac{d}{2}; z\right) = (1-z)^{1-d/2}, \tag{4.70}$$

and the Wightman function reads

$$\begin{aligned} G^+(x, x') &= (-1)^{d/2-1} \frac{\Gamma(d/2-1)}{4\pi^{d/2}} H^{d-2} \left(\frac{\eta\eta'}{(\Delta\eta)^2 - (\Delta\mathbf{x})^2}\right)^{d/2-1} \\ &= (H^2\eta^2)^{\frac{d-2}{4}} G^+_{\rm M}(x, x') (H^2\eta'^2)^{\frac{d-2}{4}}, \end{aligned} \tag{4.71}$$

as expected from (3.122).

Let us now look at the analytic properties of the two-point function. The hypergeometric function has a branch cut along the semi-infinite real axis, when its argument runs from 1 to ∞. This corresponds to the points where

$$P(X, X') \geq 1, \tag{4.72}$$

i.e. to the points inside or at the light-cone. The prescription for avoiding the singularity at the light-cone is the same as in Minkowski space, i.e. we change

$$(\eta - \eta')^2 \to (\eta - \eta' - {\rm i}\epsilon)^2. \tag{4.73}$$

4.4 Two-point functions in maximally symmetric spaces

We will now see how to generalize the above procedure to calculate two-point functions of scalar fields in maximally symmetric spaces [2].

The Riemann and Ricci tensors of maximally symmetric spaces of dimension d are highly constrained. For example, they satisfy (in both Euclidean and Minkoski signature)

$$R_{\mu\nu} = \frac{R}{d} g_{\mu\nu}. \tag{4.74}$$

Therefore, Einstein's equations, with a cosmological constant Λ, imply that

$$R = \frac{2d}{d-2}\Lambda. \tag{4.75}$$

The curvature is given, in terms of the radius, by

$$R = \frac{d(d-1)}{L^2} k, \tag{4.76}$$

where $k = 1$ corresponds to de Sitter and the sphere, and $k = -1$ corresponds to Anti de Sitter and the hyperbolic space. This leads to a relationship between the radius L and the cosmological constant,

$$\Lambda = k \frac{(d-1)(d-2)}{2L^2}. \tag{4.77}$$

4.4.1 The sphere

The d-dimensional sphere $\mathbb{S}^d$ is defined by the equation

$$\sum_{i=1}^{d+1} x_i^2 = R^2. \tag{4.78}$$

Let us parametrize

$$x_{d+1} = R\cos\theta, \qquad 0 \le \theta \le \pi. \tag{4.79}$$

Then, $\mathbb{S}^d$ can be regarded as a fibration of $\mathbb{S}^{d-1}$ over the interval $[0,\pi]$, since we can write it as

$$\sum_{i=1}^{d} x_i^2 = R^2\sin^2\theta. \tag{4.80}$$

The radius of the $\mathbb{S}^{d-1}$ is $R\sin\theta$. It varies with θ, and becomes zero at the endpoints of the interval. We can then write

$$x_i = R\sin\theta\omega_i, \tag{4.81}$$

where ω_i are coordinates for $\mathbb{S}^{d-1}$ and satisfy (4.4). The metric then reads,

$$\mathrm{d}s^2 = \sum_{i=1}^{d+1} \mathrm{d}x_i^2 = R^2\mathrm{d}\theta^2 + R^2\sin^2\theta\mathrm{d}\Omega_{d-1}^2, \tag{4.82}$$

where

$$\mathrm{d}\Omega_{d-1}^2 = \sum_{i=1}^{d} \mathrm{d}\omega_i^2 \tag{4.83}$$

is the metric on an $\mathbb{S}^{d-1}$ of unit radius. For example, for $\mathbb{S}^2$, this construction leads to the standard parametrization

$$\begin{aligned} x_1 &= R\sin\theta\cos\phi, \\ x_2 &= R\sin\theta\sin\phi, \\ x_3 &= R\cos\theta, \end{aligned} \tag{4.84}$$

and we have

$$\omega_1 = \cos\phi, \qquad \omega_2 = \sin\phi, \tag{4.85}$$

which parametrize an $\mathbb{S}^1$ of unit radius. Notice that, if we denote by $g_{\mathbb{S}^d}$ the determinant of metric on the sphere of dimension d, we have

$$g_{\mathbb{S}^d} = (R\sin\theta)^{2(d-1)}\, g_{\mathbb{S}^{d-1}}. \tag{4.86}$$

Let us now consider two points x, x' on $\mathbb{S}^d$. Using rotational symmetry, we can always choose our coordinates such that x is the north pole of the sphere, with coordinate $\theta = 0$. Let now x' have coordinates $\theta, \omega_i^\star$. The equations for a geodesic are obtained from the Lagrangian

$$L(\theta, \omega_i) = \dot{\theta}^2 + \sin^2(\theta)\dot{\boldsymbol{\omega}}^2, \tag{4.87}$$

where the derivatives are taken w.r.t. the proper time τ, and $\boldsymbol{\omega}$ is the d-dimensional vector with components ω_i. The geodesics satisfy in particular

$$\dot{\theta}^2 + \sin^2(\theta)\dot{\boldsymbol{\omega}}^2 = 1. \tag{4.88}$$

It is to check that the trajectory

$$\theta = \frac{\tau}{R}, \qquad \omega_i = \omega_i^\star, \tag{4.89}$$

where ω_i are constant for all $i = 1, \cdots, d$, is a geodesic. We conclude that the geodesic distance between x and x' is

$$\tau(x, x') = R\theta. \tag{4.90}$$

On a function of θ alone, the Laplacian reads, by using the Riemannian version of (3.27),

$$\Box = \frac{1}{R^2}(\sin\theta)^{1-d}\frac{\mathrm{d}}{\mathrm{d}\theta}(\sin\theta)^{d-1}\frac{\mathrm{d}}{\mathrm{d}\theta}, \tag{4.91}$$

and

$$\Box G(\tau) = G''(\tau) + \frac{d-1}{R}\cot\left(\frac{\tau}{R}\right)G'(\tau). \tag{4.92}$$

Therefore, the equation for the Green function reads in this case

$$G''(\tau) + \frac{d-1}{R}\cot\left(\frac{\tau}{R}\right)G'(\tau) - \left(m^2 + \xi\frac{d(d-1)}{R^2}\right)G(\tau) = 0. \tag{4.93}$$

Let us now perform the change of variable,

$$z = \cos^2\left(\frac{\tau}{2R}\right) = \frac{1}{2} + \frac{\cos(\tau/R)}{2}. \tag{4.94}$$

Then, it is easy to check that

$$G''(\tau) + \frac{d-1}{R}\cot\left(\frac{\tau}{R}\right)G'(\tau) = \frac{1}{R^2}\left\{z(1-z)\partial_z^2 G + \left(\frac{d}{2} - dz\right)\partial_z G\right\}, \tag{4.95}$$

and we obtain the equation

$$z(1-z)\partial_z^2 G + \left(\frac{d}{2} - dz\right)\partial_z G - \left(m^2R^2 + \xi d(d-1)\right)G = 0. \tag{4.96}$$

This is exactly the equation we found before in de Sitter space, with the only difference that H becomes R^{-1}. We conclude that

$$G(z) = c_d\ {}_2F_1\left(h_+, h_-, \frac{d}{2}; z\right), \tag{4.97}$$

where

$$h_\pm = \frac{1}{2}\left[d - 1 \pm \sqrt{(d-1)^2 - 4\left(m^2R^2 + \xi d(d-1)\right)}\right], \tag{4.98}$$

and c_d is given by the same value (4.68) (but with $1/R$ instead of H). Notice that, on the sphere, $0 \le z \le 1$. The point $z = 0$ corresponds to $\theta = \pi$, and the points x, x' are antipodal, while $z = 1$ corresponds to the single singularity at $x = x'$. As in de Sitter space, the more general solution of the original equation is

$$G = A\ {}_2F_1\left(h_+, h_-, \frac{d}{2}; \cos^2\left(\frac{\tau}{2R}\right)\right) + B\ {}_2F_1\left(h_+, h_-, \frac{d}{2}; \sin^2\left(\frac{\tau}{2R}\right)\right). \tag{4.99}$$

However, the second solution is now singular when $\theta = \pi$, i.e. when x and x' are antipodal, which does not make sense physically, therefore we set $B = 0$. The Euclidean vacuum of de Sitter space is then characterized by leading to a two-point function which is identical to the *unique* two-point function on its Euclidean section, the sphere.

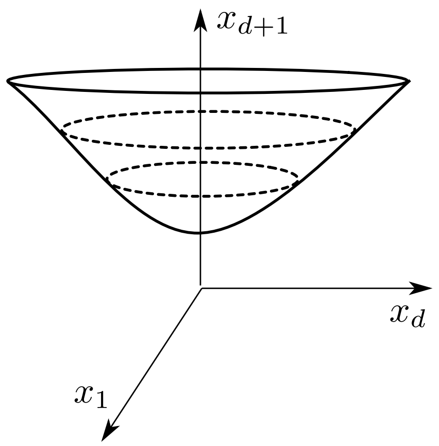

Figure 1. The hyperbolic space $\mathbb{H}^d$ can be sliced by spheres $\mathbb{S}^{d-1}$ with varying radius.

4.4.2 Hyperbolic space

The d-dimensional hyperbolic space $\mathbb{H}^d$ is defined by the equation

$$\sum_{i=1}^{d} x_i^2 - x_{d+1}^2 = -R^2 \tag{4.100}$$

and $x_{d+1} > 0$. Similarly to what we did on the sphere, we can parametrize

$$x_{d+1} = R\cosh\rho, \qquad \rho \geq 0 \tag{4.101}$$

Then, $\mathbb{H}^d$ can be regarded as a fibration of $\mathbb{S}^{d-1}$ over $[0,\infty)$, since we can write it as

$$\sum_{i=1}^{d} x_i^2 = R^2\sinh^2\rho. \tag{4.102}$$

The radius of the $\mathbb{S}^{d-1}$ is $R\sinh\rho$. It varies with ρ, and becomes zero at the endpoint $\rho = 0$. We can then write

$$x_i = R\omega_i\sinh\rho, \tag{4.103}$$

where the ω_i are coordinates on an $\mathbb{S}^{d-1}$ of radius 1. The metric then reads,

$$\mathrm{d}s^2 = \sum_{i=1}^{d+1} \mathrm{d}x_i^2 = R^2\mathrm{d}\rho^2 + R^2\sinh^2\rho\mathrm{d}\Omega_{d-1}^2, \tag{4.104}$$

Let us now consider two points x, x' on $\mathbb{H}^d$. Using rotational symmetry, we can always choose our coordinates such that x is the point $(0,\cdots,0,R)$, with coordinate $\rho = 0$. Let now x' have coordinates $\rho, \omega_i^\star$. A similar argument as we did for the sphere shows that the geodesic distance between x and x' is

$$\tau(x,x') = R\rho. \tag{4.105}$$

On a function of ρ alone, the Laplacian reads, by using the Riemannian version of (3.27),

$$\square = \frac{1}{R^2}(\sinh\rho)^{1-d}\frac{\mathrm{d}}{\mathrm{d}\rho}(\sinh\rho)^{d-1}\frac{\mathrm{d}}{\mathrm{d}\rho}, \tag{4.106}$$

and

$$\square G(\tau) = G''(\tau) + \frac{d-1}{R}\coth\left(\frac{\tau}{R}\right)G'(\tau). \tag{4.107}$$

Therefore, the equation for the Green function reads in this case

$$G''(\tau) + \frac{d-1}{R}\coth\left(\frac{\tau}{R}\right)G'(\tau) - \left(m^2 - \xi\frac{d(d-1)}{R^2}\right)G(\tau) = 0. \tag{4.108}$$

The opposite sign in front of the last term, as compared to what happened in the sphere, is due to the fact that the hyperbolic space has *negative* constant curvature. Let us now perform the change of variable,

$$z = \cosh^2\left(\frac{\tau}{2R}\right) = \frac{1}{2} + \frac{\cosh(\tau/R)}{2}. \tag{4.109}$$

Then, it is easy to check that the equation (4.108) becomes

$$z(1-z)\partial_z^2 G + \left(\frac{d}{2} - dz\right)\partial_z G + \left(m^2R^2 - \xi d(d-1)\right)G = 0. \tag{4.110}$$

We want to find solutions to this equation which decay at infinity, i.e. as $\tau \to \infty$. The above equation is a hypergeometric ODE, with the basis of solutions (see [18], eq. 9.5.9)

$$z^{-\alpha}\ {}_2F_1\left(\alpha, 1+\alpha-\gamma; 1+\alpha-\beta; z^{-1}\right) \quad \text{and} \quad z^{-\beta}\ {}_2F_1\left(\beta, 1+\beta-\gamma; 1+\beta-\alpha; z^{-1}\right) \tag{4.111}$$

where

$$\alpha = h_+, \qquad \beta = h_-, \qquad \gamma = \frac{d}{2}, \tag{4.112}$$

and

$$h_\pm = \frac{1}{2}\left[d-1 \pm \sqrt{(d-1)^2 + 4\left(m^2R^2 - \xi d(d-1)\right)}\right]. \tag{4.113}$$

The above solutions fall as $z^{-\alpha,\beta}$ when $z\to\infty$. If $m^2R^2 \geq \xi d(d-1)$ (which we will assume from now on), we have that $\beta < 0$ and the second solution diverges. Therefore, we will keep the first solution, and the Green function is then given by

$$G(x,x') = c_d z^{-\alpha}\ {}_2F_1\left(\alpha, 1+\alpha-\gamma; 1+\alpha-\beta; z^{-1}\right). \tag{4.114}$$

We can fix c_d by studying the behavior near $z \approx 1$. In that case, one has

$${}_2F_1\left(\alpha, 1+\alpha-\gamma; 1+\alpha-\beta; z^{-1}\right) \approx \frac{\Gamma(1+\alpha-\beta)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(1+\alpha-\gamma)}(z-1)^{\gamma-\alpha-\beta}. \tag{4.115}$$

On the other hand, we have

$$z \approx 1 + \frac{\tau^2}{4R^2} + \cdots, \tag{4.116}$$

for τ small. In Euclidean signature, and on $\mathbb{R}^d$, we have the leading singularity given by the massless Green function (3.117), where $\ell = \tau$, i.e.

$$\frac{\Gamma(d/2-1)}{4\pi^{d/2}}\frac{1}{\tau^{d-2}}, \tag{4.117}$$

and we deduce

$$c_d = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(h_+)\Gamma(1+h_+-d/2)}{\Gamma(1+h_+-h_-)}R^{2-d}. \tag{4.118}$$

4.4.3 Anti de Sitter space

The d-dimensional Anti de Sitter (AdS) space of dimension d can be defined as the submanifold

$$X_0^2 + X_d^2 - \sum_{i=1}^{d-1} X_i^2 = \ell^2 \tag{4.119}$$

in a space with metric

$$\mathrm{d}s^2 = \mathrm{d}X_0^2 + \mathrm{d}X_d^2 - \sum_{i=1}^{d-1} \mathrm{d}X_i^2. \tag{4.120}$$

We can define various possible coordinate systems for AdS, in the same way as we did for dS. For example, the analogue of the static coordinates in dS are

$$\begin{aligned} X_0 &= \ell\sqrt{1 + r^2/\ell^2}\sin(t/\ell)\,, \\ X_d &= \ell\sqrt{1 - +r^2/\ell^2}\cos(t/\ell)\,, \\ X_i &= r\omega_i, \quad i = 1, \cdots, d-1, \end{aligned} \tag{4.121}$$

In these coordinates, the metric of AdS reads

$$\mathrm{d}s^2 = (1 + r^2/\ell^2)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 + r^2/\ell^2} - r^2 \mathrm{d}\Omega_{d-2}^2. \tag{4.122}$$

Notice that, as $\ell \to \infty$, we recover Minkowski space. The Euclidean section of AdS is the hyperbolic space $\mathbb{H}^d$. Therefore, the results on the Green function on $\mathbb{H}^d$ can be easily used to obtain an expression for the Wightman functions on AdS after appropriate analytic continuation [2, 7].

5 Quantum fields in Rindler space: the Unruh effect

We will consider here two–dimensional *Minkowski space* in standard Minkowski coordinates but from the point of view of an accelerated observer. We will show that, *for an accelerated observer, the Minkowski vacuum appears as a thermal vacuum with a temperature proportional to the acceleration*. We will also show that this vacuum has the structure of a *thermofield double* which can be interpreted as an entangled state [16]. We will mainly follow the analysis in [4, 5].

5.1 Rindler spacetime

We first motivate Rindler spacetime by considering uniformly accelerated observers in Minkowski spacetime. Constant acceleration means that

$$a^\mu a_\mu = -a^2, \tag{5.1}$$

where

$$a^\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau}, \qquad u^\mu = \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} \tag{5.2}$$

and we recall that

$$u^\mu u_\mu = -1, \qquad a^\mu u_\mu = 0. \tag{5.3}$$

This means that

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = u^0, \quad \frac{\mathrm{d}x}{\mathrm{d}\tau} = u^1, \quad \frac{\mathrm{d}u^0}{\mathrm{d}\tau} = a^0, \quad \frac{\mathrm{d}u^1}{\mathrm{d}\tau} = a^1. \tag{5.4}$$

We can solve algebraically for the acceleration, since (5.1) and (5.3) lead to

$$\begin{aligned} -(u^0)^2 + (u^1)^2 &= -1, \\ -u^0 a^0 + u^1 a^1 &= 0, \\ -(a^0)^2 + (a^1)^2 &= a^2, \end{aligned} \tag{5.5}$$

therefore we deduce from the second one

$$a^1 = \frac{u^0 a^0}{u^1} \tag{5.6}$$

which when plugged in the last equation gives

$$(a^0)^2\left[\left(\frac{u^0}{u^1}\right)^2 - 1\right] = a^2 \Rightarrow a^0 = \frac{a}{\sqrt{\left(\frac{u^0}{u^1}\right)^2 - 1}} = \frac{au^1}{\sqrt{(u^0)^2 - (u^1)^2}} = au^1. \tag{5.7}$$

Therefore, we find

$$a^0 = au^1, \qquad a^1 = au^0. \tag{5.8}$$

We can now solve the linear differential equations

$$\frac{\mathrm{d}u^0}{\mathrm{d}\tau} = au^1, \quad \frac{\mathrm{d}u^1}{\mathrm{d}\tau} = au^0 \tag{5.9}$$

as

$$u^0 = \cosh a\tau, \quad u^1 = \sinh a\tau, \tag{5.10}$$

which clearly verify the constraint that $(u^0)^2 - (u^1)^2 = 1$. By further integrating we get,

$$t = \frac{1}{a}\sinh(a\tau), \qquad x = \frac{1}{a}\cosh(a\tau). \tag{5.11}$$

We want to find now what are the natural coordinates in Minkowski spacetime for an accelerated observer. We want the time coordinate to be its proper time, and the spatial coordinate to be characterized by the fact that the observed is at rest in it. Let us then start with Minkowski spacetime

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2. \tag{5.12}$$

It will be useful to introduce light-cone coordinates:

$$u = t - x, \quad v = t + x, \tag{5.13}$$

so that

$$\mathrm{d}s^2 = \mathrm{d}u\mathrm{d}v. \tag{5.14}$$

Let us now introduce the coordinates

$$\begin{aligned} t &= \frac{1}{\kappa}\mathrm{e}^{\kappa\xi}\sinh\kappa\eta, \\ x &= \frac{1}{\kappa}\mathrm{e}^{\kappa\xi}\cosh\kappa\eta, \end{aligned} \tag{5.15}$$

whose inverse transformation is given by

$$
\begin{aligned}
\xi &= \frac{1}{2\kappa}\log(\kappa^2(x^2-t^2)),\\
\eta &= \frac{1}{\kappa}\tanh^{-1}\frac{t}{x} = \frac{1}{2\kappa}\log\frac{x+t}{x-t}.
\end{aligned}
\tag{5.16}
$$

In these coordinates the metric reads

$$
\mathrm{d}s^2 = \mathrm{e}^{2\kappa\xi}(\mathrm{d}\eta^2 - \mathrm{d}\xi^2),
\tag{5.17}
$$

which is conformally equivalent to Minkowski. It is called the *Rindler metric*. Notice that these coordinates do not cover the whole of Minkowski spacetime, but only the wedge

$$
x > |t|.
\tag{5.18}
$$

This is the quadrant shown in Fig. 2, and we can parametrize it also by

$$
u < 0, \qquad v > 0.
\tag{5.19}
$$

The light-cone coordinates in Rindler space are

$$
u_R = \eta - \xi, \qquad v_R = \eta + \xi.
\tag{5.20}
$$

Since

$$
\begin{aligned}
t - x &= \frac{1}{\kappa}\mathrm{e}^{\kappa\xi}(\sinh\kappa\eta - \cosh\kappa\eta) = -\frac{1}{\kappa}\mathrm{e}^{-\kappa(\eta-\xi)},\\
t + x &= \frac{1}{\kappa}\mathrm{e}^{\kappa\xi}(\sinh\kappa\eta + \cosh\kappa\eta) = \frac{1}{\kappa}\mathrm{e}^{\kappa(\eta+\xi)},
\end{aligned}
\tag{5.21}
$$

we have the change of coordinates

$$
u = -\frac{1}{\kappa}\mathrm{e}^{-\kappa u_R}, \qquad v = \frac{1}{\kappa}\mathrm{e}^{\kappa v_R}.
\tag{5.22}
$$

If we now compare (5.15) to (5.11), we see that the lines of constant ξ in the metric (5.17) describe an uniformly accelerater observer with acceleration

$$
a = \kappa \mathrm{e}^{-\kappa\xi},
\tag{5.23}
$$

and for fixed ξ, η is proportional to the proper time of the observer. Therefore, Rindler space can be regarded as the description of Minkowski space by a series of uniformly accelerated observers. Near the horizon where $\xi \to -\infty$, we have

$$
a \to \infty
\tag{5.24}
$$

and the observers feel an infinite proper acceleration.

We can also cover the L wedge of Minkowski space, defined by

$$
x < |t|,
\tag{5.25}
$$

with the coordinates ξ, η now related to t, x by

$$
\begin{aligned}
t &= -\frac{1}{\kappa}\mathrm{e}^{\kappa\xi}\sinh\kappa\eta,\\
x &= -\frac{1}{\kappa}\mathrm{e}^{\kappa\xi}\cosh\kappa\eta.
\end{aligned}
\tag{5.26}
$$

Notice that the events happening in the L wedge are causally disconnected from the worldlines of a Rindler observer, and the line $u = 0$ behaves as a event horizon. This will be relevant for our discussion of black holes.

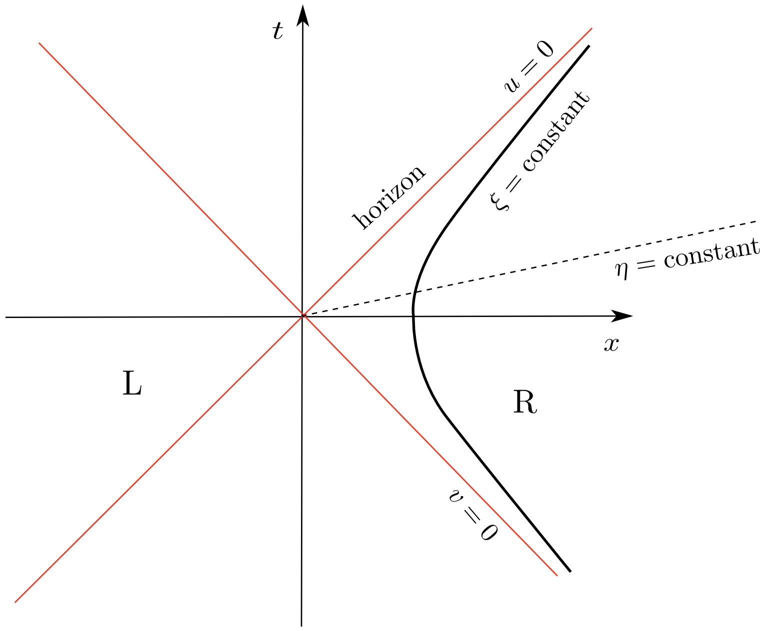

Figure 2. Rindler spacetime.

5.2 Bogoliubov transformation

Let us now consider the quantization of a massless scalar field in two-dimensional Minkowski spacetime. The standard modes are

$$\begin{aligned} \xi_\omega(u) &= \frac{1}{\sqrt{4\pi\omega}} \mathrm{e}^{-\mathrm{i}\omega u}, \\ \tilde{\xi}_\omega(v) &= \frac{1}{\sqrt{4\pi\omega}} \mathrm{e}^{-\mathrm{i}\omega v}, \end{aligned} \tag{5.27}$$

where $\omega > 0$. Notice that $\xi_\omega(u)$ are right-moving waves (i.e. $k > 0$) while $\tilde{\xi}_\omega(v)$ are left-moving waves (i.e. $k < 0$). Both satisfy (3.11) and are positive-frequency modes.

Let us now consider the Rindler coordinates given in (5.16). These cover the quadrant or wedge $x > |t|$, which is limited by $u = 0, v = 0$. The resulting metric is conformally equivalent to Minkowski in η, ξ coordinates, and since the wave equation in two dimensions is conformally invariant, we can consider solutions of the form

$$\begin{aligned} \phi_{\omega,R} &= \frac{\theta(-u)}{\sqrt{4\pi\omega}} \mathrm{e}^{-\mathrm{i}\omega u_R}, \\ \tilde{\phi}_{\omega,R} &= \frac{\theta(v)}{\sqrt{4\pi\omega}} \mathrm{e}^{-\mathrm{i}\omega v_R}, \end{aligned} \tag{5.28}$$

where $\omega > 0$. The Heaviside functions impose that these modes are only nonzero in the R wedge.

In the light-cone coordinates (5.22) we have

$$\begin{aligned}\phi_{\omega,R} &= \frac{\theta(-u)}{\sqrt{4\pi\omega}}(-\kappa u)^{\mathrm{i}\omega/\kappa},\\ \tilde{\phi}_{\omega,R} &= \frac{\theta(v)}{\sqrt{4\pi\omega}}(\kappa v)^{-\mathrm{i}\omega/\kappa},\end{aligned} \tag{5.29}$$

Finally, we note that these modes are eigenfunctions of the Killing vector $\mathcal{L}_{\partial_\eta}$:

$$\mathcal{L}_{\partial_\eta}\phi_{\omega,R} = -\mathrm{i}\omega\phi_{\omega,R}, \tag{5.30}$$

(similarly for $\tilde{\phi}_{\omega,R}$).

We now consider the L wedge of Rindler spacetime, described in Minkowski coordinates by

$$u > 0, \qquad v < 0. \tag{5.31}$$

The Rindler parameters are again (ξ, η), but they are now related to Minkowski variables by the transformation (5.26). The corresponding light–like coordinates will be denoted by u_L, v_L, and the modes for quantum fields there are given by

$$\begin{aligned}\phi_{\omega,L} &= \frac{\theta(u)}{\sqrt{4\pi\omega}}\mathrm{e}^{\mathrm{i}\omega u_L} = \frac{\theta(u)}{\sqrt{4\pi\omega}}(\kappa u)^{-\mathrm{i}\omega/\kappa},\\ \tilde{\phi}_{\omega,L} &= \frac{\theta(-v)}{\sqrt{4\pi\omega}}\mathrm{e}^{\mathrm{i}\omega v_L} = \frac{\theta(-v)}{\sqrt{4\pi\omega}}(-\kappa v)^{\mathrm{i}\omega/\kappa}.\end{aligned} \tag{5.32}$$

We will now restrict to the right–moving modes of the fields. Notice that right movers and left movers do not mix under Bogoliubov transformations, since for example $\phi_{\omega,R}$ only depends on u. The general formulae of the previous section give the following relation between the Minkowski modes and the Rindler modes in the R region:

$$\theta(-u)\xi_\omega(u) = \int \mathrm{d}\lambda\,[\alpha^R_{\omega\lambda}\phi_{\lambda,R}(u_R) + \beta^R_{\omega\lambda}\phi^*_{\lambda,R}(u_R)]. \tag{5.33}$$

To calculate the coefficients of the Bogoliubov transformation, we work in the coordinates (η, ξ), and we take Σ to be the hypersurface $\xi = \text{constant}$. The normal vector is

$$n^\mu = \mathrm{e}^{-\kappa\xi}(1, 0), \tag{5.34}$$

and (3.69) gives,

$$\begin{aligned}\alpha^R_{\omega\lambda} &= \mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}u_R\,\left[\partial_{u_R}\phi^*_{\lambda,R}(u_R)\xi_\omega(u(u_R)) - \phi^*_{\lambda,R}(u_R)\partial_{u_R}\xi_\omega(u(u_R))\right]\\ &= \mathrm{i}\int_{-\infty}^{0}\mathrm{d}u\,\left[\partial_u\phi^*_{\lambda,R}(u)\xi_\omega(u) - \phi^*_{\lambda,R}(u)\partial_u\xi_\omega(u)\right],\\ \beta^R_{\omega\lambda} &= -\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}u_R\,\left[\partial_{u_R}\phi_{\lambda,R}(u_R)\xi_\omega(u(u_R)) - \phi_{\lambda,R}(u_R)\partial_{u_R}\xi_\omega(u(u_R))\right]\\ &= -\mathrm{i}\int_{-\infty}^{0}\mathrm{d}u\,\left[\partial_u\phi_{\lambda,R}(u)\xi_\omega(u) - \phi_{\lambda,R}(u)\partial_u\xi_\omega(u)\right],\end{aligned} \tag{5.35}$$

where we changed the integration variable from ξ to u_R and to u. We now compute them explicitly

$$\alpha^R_{\omega\lambda} = -\frac{\mathrm{i}}{4\pi(\omega\lambda)^{\frac{1}{2}}}\int_{-\infty}^{0}\mathrm{d}u\,\mathrm{e}^{-\mathrm{i}\omega u}\Big(\mathrm{i}\lambda(-\kappa u)^{-\mathrm{i}\lambda/\kappa-1}+\mathrm{i}\omega(-\kappa u)^{-\mathrm{i}\lambda/\kappa}\Big) \tag{5.36}$$

Let us first consider the integral of the first summand. After changing $u \to -u$ and making the change of variables $z = -\mathrm{i}\omega u$ we find,

$$\lambda\int_{-\infty}^{0}\mathrm{d}u\,\mathrm{e}^{-\mathrm{i}\omega u}(-\kappa u)^{-\mathrm{i}\lambda/\kappa-1} = \frac{\lambda}{\kappa}\left(\frac{\mathrm{i}\kappa}{\omega}\right)^{-\mathrm{i}\lambda/\kappa}\int_{0}^{\infty}\mathrm{d}z\,\mathrm{e}^{-z}z^{-\mathrm{i}\lambda/\kappa-1} = \frac{\lambda}{\kappa}\left(\frac{\mathrm{i}\kappa}{\omega}\right)^{-\mathrm{i}\lambda/\kappa}\Gamma(-\mathrm{i}\lambda/\kappa). \tag{5.37}$$

The second summand gives the same contribution, after using $\Gamma(z+1) = z\Gamma(z)$, and we finally obtain

$$\alpha^R_{\omega\lambda} = \frac{1}{2\pi\kappa}\sqrt{\frac{\lambda}{\omega}}\left(\frac{\kappa}{\omega}\right)^{-\mathrm{i}\lambda/\kappa}\Gamma(-\mathrm{i}\lambda/\kappa)\mathrm{e}^{\pi\lambda/2\kappa}, \tag{5.38}$$

where we have written $\mathrm{i} = \mathrm{e}^{\mathrm{i}\pi/2}$. The calculation of $\beta^R_{\omega\lambda}$ is very similar, and one finds

$$\beta^R_{\omega\lambda} = \frac{1}{2\pi\kappa}\sqrt{\frac{\lambda}{\omega}}\left(\frac{\kappa}{\omega}\right)^{\mathrm{i}\lambda/\kappa}\Gamma(\mathrm{i}\lambda/\kappa)\mathrm{e}^{-\pi\lambda/2\kappa}. \tag{5.39}$$

Similarly, we can compute the relation between the Minkowski basis and the Rindler basis $\phi_{\omega,L}$:

$$\theta(u)\xi_\omega(u) = \int\mathrm{d}\lambda\,[\alpha^L_{\omega\lambda}\phi_{\lambda,L}(u_L) + \beta^L_{\omega\lambda}\phi^*_{\lambda,L}(u_L)], \tag{5.40}$$

and one simply has

$$\alpha^L_{\omega\lambda} = \alpha^{R*}_{\omega\lambda}, \qquad \beta^L_{\omega\lambda} = \beta^{R*}_{\omega\lambda}. \tag{5.41}$$

The relation between creation/annihilation operators is the following:

$$\begin{aligned} b_{\lambda R} &= \int_0^\infty \mathrm{d}\omega\left[\alpha^R_{\omega\lambda}\bar{a}_\omega + \beta^{R*}_{\omega\lambda}\bar{a}^\dagger_\omega\right], \\ b_{\lambda L} &= \int_0^\infty \mathrm{d}\omega\left[\alpha^L_{\omega\lambda}\bar{a}_\omega + \beta^{L*}_{\omega\lambda}\bar{a}^\dagger_\omega\right], \\ \bar{a}_\omega &= \int_0^\infty \mathrm{d}\lambda\Big[\alpha^{R*}_{\omega\lambda}b_{\lambda R} + \alpha^{L*}_{\omega\lambda}b_{\lambda L} - \beta^{R*}_{\omega\lambda}b^\dagger_{\lambda R} - \beta^{L*}_{\omega\lambda}b^\dagger_{\lambda L}\Big]. \end{aligned} \tag{5.42}$$

The Minkowski vacuum is defined by

$$\bar{a}_\omega|0_{\mathrm{M}}\rangle = 0, \tag{5.43}$$

while the "right" Rindler vacuum is defined by

$$b_{\lambda R}|0_{\mathrm{R}}\rangle = 0. \tag{5.44}$$

We can now compute the number of particles that an observer in the R wedge of the Rindler spacetime will associate to the Minkowski vacuum:

$$\langle 0_M|b^\dagger_{\lambda R}b_{\lambda' R}|0_M\rangle = \int_0^\infty \mathrm{d}\omega\beta^R_{\omega\lambda}\beta^{R*}_{\omega\lambda'}. \tag{5.45}$$

Substituting the explicit values, we find

$$\langle 0_M | b^\dagger_{\lambda R} b_{\lambda' R} | 0_M \rangle = \frac{1}{4\pi^2\kappa^2} \mathrm{e}^{-\pi(\lambda+\lambda')/2\kappa} \sqrt{\lambda\lambda'} \kappa^{\mathrm{i}(\lambda-\lambda')/\kappa} \Gamma(\mathrm{i}\lambda/\kappa)\Gamma(-\mathrm{i}\lambda'/\kappa) \int_0^\infty \mathrm{d}\omega\, \omega^{\mathrm{i}(\lambda'-\lambda)/\kappa - 1}. \tag{5.46}$$

The integral can be done after the change of variables $\omega = \exp(\kappa x)$, and gives

$$\int_0^\infty \mathrm{d}\omega\, \omega^{\mathrm{i}(\lambda'-\lambda)/\kappa-1} = \kappa \int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}(\lambda-\lambda')x} \mathrm{d}x = 2\pi\kappa\delta(\lambda-\lambda'). \tag{5.47}$$

After using

$$|\Gamma(\mathrm{i}x)|^2 = \frac{\pi}{x \sinh \pi x}, \tag{5.48}$$

we finally obtain

$$\langle 0_M | b^\dagger_{\lambda R} b_{\lambda' R} | 0_M \rangle = \frac{\delta(\lambda-\lambda')}{\mathrm{e}^{2\pi\lambda/\kappa} - 1}, \tag{5.49}$$

which is a thermal distribution of scalar particles at temperature

$$T_0 = \frac{\kappa}{2\pi} \qquad \text{or} \qquad \beta = \frac{2\pi}{\kappa}. \tag{5.50}$$

This is *not* the temperature measured by an accelerated Rindler observer, which is given by Tolman's law

$$T = (g_{00})^{-\frac{1}{2}} T_0. \tag{5.51}$$

where g_{00} is the metric component in the reference frame where the observer has constant spatial coordinate. This is the (ξ, η) frame of (5.17), where $g_{00} = \mathrm{e}^{2\kappa\xi}$, and we find

$$T = \frac{\kappa}{2\pi \mathrm{e}^{\kappa\xi}} = \frac{2\pi}{a}, \tag{5.52}$$

where a is the proper acceleration (5.23). In particular, the temperature becomes infinite as we approach the horizon.

Remark 5.1. Tolman's law can be deduced by noticing that the conserved energy E_0 for an observer at rest in a stationary gravitational field is given by

$$E_0 = E\sqrt{g_{00}}, \tag{5.53}$$

where E is the energy as measured by the observer. In the case of a particle of mass m, we have $E = m$. For a weak gravitational field, this reproduces for E_0 the standard potential energy of a particle at rest in a gravitational field. Since the equilibrium temperature T_0 is defined w.r.t. the conserved energy as

$$\frac{1}{T_0} = \frac{\partial S}{\partial E_0} \tag{5.54}$$

where S is the entropy, we conclude that the relation between the equilibrium temperature and the temperature measured by the observer in a gravitational field is

$$T_0 = \sqrt{g_{00}}\, T. \tag{5.55}$$

5.3 The Unruh–Israel density matrix and the thermofield double

Let us now relate the Minkowski vacuum, defined by

$$\bar{a}_\omega|\bar{0}\rangle = 0, \qquad \omega > 0, \tag{5.56}$$

to the Rindler vacuum, defined by

$$b_{\lambda R}|0\rangle = b_{\lambda L}|0\rangle = 0. \tag{5.57}$$

It follows that the Rindler vacuum can be regarded as

$$|0\rangle = |0_R\rangle|0_L\rangle. \tag{5.58}$$

The relation between both vacua is very easy, and it can be written as

$$|\bar{0}\rangle = N\,\exp\left\{\int_0^\infty \mathrm{d}\lambda\, c_\lambda b^\dagger_{\lambda L} b^\dagger_{\lambda R}\right\}|0\rangle. \tag{5.59}$$

where c_λ is a constant to be determined. To see this and find the value of c_λ, notice that (5.56) leads to two conditions

$$\alpha^{R*}_{\omega\lambda} c_\lambda = \beta^{L*}_{\omega\lambda}, \qquad \alpha^{L*}_{\omega\lambda} c_\lambda = \beta^{R*}_{\omega\lambda}, \qquad \forall\omega > 0. \tag{5.60}$$

Both conditions lead to the same solution, namely

$$c_\lambda = \frac{\beta^R_{\omega\lambda}}{\alpha^{R*}_{\omega\lambda}} = \mathrm{e}^{-\frac{\pi\lambda}{a}} = \mathrm{e}^{-\frac{\beta\lambda}{2}}. \tag{5.61}$$

The above state can be written as

$$|\bar{0}\rangle = N\prod_\lambda \sum_{n_\lambda=0}^\infty \mathrm{e}^{-\beta\lambda n_\lambda/2}|n_{\lambda R}\rangle|n_{\lambda L}\rangle, \tag{5.62}$$

where we have discretized λ for convenience. Such states are called *thermofield doubles* (TFD), and were obtained in this context by Israel [16].

Thermofield doubles can be interpreted as follows. They describe a *pure* but *entangled* state in the Hilbert space

$$\mathcal{H}_L \otimes \mathcal{H}_R. \tag{5.63}$$

Consider now an observer that cannot measure states in the space $\mathcal{H}_L$, and wants to compute the vev of an operator defined on $\mathcal{H}_R$, $\mathcal{O}_R$. Then, one finds that

$$\langle\bar{0}|\mathcal{O}_R|\bar{0}\rangle = N^2\prod_\lambda \sum_{n_\lambda=0}^\infty \mathrm{e}^{-\beta\lambda n_\lambda}\langle n_{\lambda R}|\mathcal{O}_R|n_{\lambda R}\rangle, \tag{5.64}$$

This can be interpreted as

$$\langle\bar{0}|\mathcal{O}_R|\bar{0}\rangle = \mathrm{Tr}\,(\mathcal{O}_R\rho_R), \tag{5.65}$$

where

$$\rho = N\prod_\lambda \sum_{n_\lambda=0}^\infty \mathrm{e}^{-\beta E_\lambda}|n_{\lambda R}\rangle\langle n_{\lambda R}|, \qquad E_\lambda = \lambda n_\lambda. \tag{5.66}$$

This is nothing but a *thermal* density matrix for a scalar field at temperature $T = 1/\beta$.

Another way to interpret this is to start with the pure entangled state (5.62) in $\mathcal{H}_L \otimes \mathcal{H}_R$, with corresponding density matrix ρ (with zero Von Neumann entropy). The density matrix ρ_R can be obtained from ρ by tracing over $\mathcal{H}_L$,

$$\rho_R = \mathrm{Tr}_{\mathcal{H}_L}\, \rho. \tag{5.67}$$

The resulting density corresponds to a *mixed thermal state* and will have a non–zero Von Neumann entropy due to entanglement (the so–called entanglement entropy).

6 Hawking radiation

We will now show, by exploiting the analogy with the Unruh effect, that an eternal black hole is a thermal background, and we will calculate its temperature.

6.1 Schwarzschild black holes

The Schwarzschild metric reads

$$\mathrm{d}s^2 = \left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - \frac{2M}{r}} - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2). \tag{6.1}$$

The tortoise coordinate r^* is defined in such a way that massless falling observers follow the equation

$$t = r^* + \text{constant}. \tag{6.2}$$

For such observers, one has

$$\mathrm{d}s^2 = 0 \Rightarrow \mathrm{d}t = \frac{1}{1 - \frac{2M}{r}}\mathrm{d}r \equiv \mathrm{d}r^*, \tag{6.3}$$

therefore

$$r^* = r + 2M \log\left(\frac{r - 2M}{2M}\right) + C, \tag{6.4}$$

This is the tortoise or Regge–Wheeler coordinate r^*. For definiteness, we will set $C = 0$. In these coordinates, the Schwarzschild metric reads

$$\mathrm{d}s^2 = \left(1 - \frac{2M}{r}\right)\left(\mathrm{d}t^2 - (\mathrm{d}r^*)^2\right) - r^2\mathrm{d}\Omega_2^2. \tag{6.5}$$

This means that, in the two-dimensional submanifold defined by $\mathrm{d}\Omega_2 = 0$, the black hole metric is a conformal transformation of the Minkowski metric.

The Eddington–Finkelstein null coordinates are defined as

$$v = t + r^*, \qquad u = t - r^*. \tag{6.6}$$

Notice that

$$r \to 2M \Leftrightarrow u \to \infty,\ v \to -\infty. \tag{6.7}$$

In these coordinates, the tortoise coordinate reads

$$r^* = \frac{1}{2}(v - u), \tag{6.8}$$

and the metric reads

$$\mathrm{d}s^2 = \frac{2M\mathrm{e}^{-r/2M}}{r}\mathrm{e}^{(v-u)/(4M)}\mathrm{d}u\mathrm{d}v - r^2\mathrm{d}\Omega_2^2. \tag{6.9}$$

This metric is still singular as $r \to 2M$ (due to the factor $\mathrm{e}^{(v-u)/4}$). To remove this factor, we define

$$U = -4M\mathrm{e}^{-u/4M}, \qquad V = 4M\mathrm{e}^{v/4M}, \tag{6.10}$$

leading to

$$\mathrm{d}s^2 = \frac{2M\mathrm{e}^{-r/2M}}{r}\mathrm{d}U\mathrm{d}V - r^2\mathrm{d}\Omega_2^2. \tag{6.11}$$

We note that the transformation (6.10) is identical to the transformation (5.22) between null coordinates for Rindler and Minkowski, where $\kappa = (4M)^{-1}$. This observation will be crucial in what follows.

A last transformation

$$T = \frac{1}{2}(U+V), \qquad X = \frac{1}{2}(U-V) \tag{6.12}$$

leads to the Kruskal–Szekeres coordinates

$$\mathrm{d}s^2 = \frac{2M\mathrm{e}^{-r/2M}}{r}(\mathrm{d}T^2 - \mathrm{d}X^2) - r^2\mathrm{d}\Omega_2^2. \tag{6.13}$$

The explicit change of variables between (T, X) and (t, r) is

$$X^2 - T^2 = \mathrm{e}^{r/2M}\left(\frac{r}{2M} - 1\right), \qquad \frac{T+X}{T-X} = \mathrm{e}^{t/2M}. \tag{6.14}$$

In the coordinates U, V, the horizon $r = 2M$ is at $U = 0$ or $V = 0$.

6.2 Field quantization

To simplify our analysis of Hawking radiation, we will consider a two-dimensional slice of the Schwarzschild metric (6.1),

$$\mathrm{d}s^2 = \left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - \frac{2M}{r}}. \tag{6.15}$$

The problem is very similar to the one found in Rindler space. We will consider two coordinate systems, Eddington–Filkenstein and Kruskal–Szekeres, which are similar respectively to Rindler and Minkowski. In the Eddington–Filkenstein coordinates we can consider the modes

$$\begin{aligned}\phi_\omega &= \frac{1}{\sqrt{4\pi\omega}}\mathrm{e}^{-\mathrm{i}\omega u},\\ \tilde{\phi}_\omega &= \frac{1}{\sqrt{4\pi\omega}}\mathrm{e}^{-\mathrm{i}\omega v},\end{aligned} \tag{6.16}$$

where u, v are the Eddington–Filkenstein null coordinates (6.6). The vacuum associated to these coordinates is called the *Boulware vacuum* and it is defined by

$$b_\omega|0_\mathrm{B}\rangle = 0. \tag{6.17}$$

The Eddington–Filkenstein coordinates are appropriate for an observer very far from the black hole. Indeed, in this limit, the metric looks like

$$\mathrm{d}s^2 \to \mathrm{d}u\mathrm{d}v \tag{6.18}$$

which is Minkowski space in the u, v coordinates. The Boulware vacuum contains no particles, from the point of a view of a distant observer. Since the Eddington–Filkenstein coordinates do not cover the whole of spacetime, as it happened with Rindler coordinates, this vacuum is similar to the Rindler vacuum of an accelerated observer.

Similarly, we have the modes associated to the Kruskal–Szekeres coordinates, which are given by

$$\begin{aligned} \xi_\omega(U) &= \frac{1}{\sqrt{4\pi\omega}}\mathrm{e}^{-\mathrm{i}\omega U}, \\ \tilde{\xi}_\omega(V) &= \frac{1}{\sqrt{4\pi\omega}}\mathrm{e}^{-\mathrm{i}\omega V}, \end{aligned} \tag{6.19}$$

and the corresponding Kruskal vacuum,

$$a_\omega|0_\mathrm{K}\rangle = 0. \tag{6.20}$$

In Kruskal–Szekeres coordinates, the metric (6.13) near the horizon looks like

$$\mathrm{d}s^2 \to \mathrm{d}U\mathrm{d}V, \tag{6.21}$$

so the Kruskal vacuum is the appropriate one for an observer sitting next to the black hole horizon. On the other hand, since Kruskal–Szekeres coordinates cover the whole of spacetime, they correspond to the Minkowski vacuum that we studied in the quantization of a scalar field in Rindler space.

We can now ask the following question: if a Kruskal observer is in the vacuum state, what does the Boulware observer see? Since the relation between both systems is the same as we found before in the case of the Unruh effect, the calculation of the Bogoliubov coefficients will be the same we did in the Unruh case, with the only difference that now $\kappa = (4M)^{-1}$. We conclude that the Boulware observer sees a thermal spectrum with temperature

$$T = \frac{\kappa}{2\pi} = \frac{1}{8\pi M}. \tag{6.22}$$

Far from the black hole, the factor g_{00} in Eddington–Filkenstein coordinates goes to 1, and $T_0 = T$, so (6.22) is the physical temperature observed by an observer at infinity.

7 Particle detectors

One way to make precise the concept of particle in curved spacetime is to use a particle detector. In this context, this type of detectors are also called *Unruh detectors*, and were studied in [12, 26].

Let us then consider an observer equipped with a detector in the presence of a scalar field ϕ. This detector will have some internal energy states and can interact with the scalar field by exchanging energy, i.e. by emitting or absorbing scalar particles. We can model this Unruh detector by coupling the scalar field $\phi(x(\tau))$ along the worldline $x(\tau)$ of the observer, to some operator $m(\tau)$ acting on the internal detector states:

$$g\int_{-\infty}^{\infty} \mathrm{d}\tau\; m(\tau)\phi(x(\tau)), \tag{7.1}$$

where g is the strength of the coupling and τ is the proper time along the observer's worldline. Let H_0 denote the detector Hamiltonian, with energy eigenstates $|E_j\rangle$,

$$H_0|E_j\rangle = E_j|E_j\rangle, \tag{7.2}$$

and let m_{ij} be the matrix elements of the operator $m(\tau)$ at $\tau = 0$:

$$m_{ij} \equiv \langle E_i|m(0)|E_j\rangle. \tag{7.3}$$

We will calculate the transition amplitude from a state $|0\rangle \otimes |E_i\rangle$ in the tensor product of the scalar field and detector Hilbert spaces to the state $\langle E_j|\otimes\langle\psi|$, where $\langle\psi|$ is any state of the scalar field. To first order in perturbation theory in g, the amplitude is

$$g\int_{-\infty}^{\infty} \mathrm{d}\tau \; \langle E_j| \otimes \langle\psi|m(\tau)\phi(x(\tau))|0\rangle \otimes |E_i\rangle. \tag{7.4}$$

Using the Heisenberg equation of motion for the detector,

$$m(\tau) = \mathrm{e}^{\mathrm{i}H_0\tau}m(0)\mathrm{e}^{-\mathrm{i}H_0\tau}, \tag{7.5}$$

this can be written as

$$gm_{ji}\int_{-\infty}^{\infty} \mathrm{d}\tau \; \mathrm{e}^{\mathrm{i}(E_j-E_i)\tau}\langle\psi|\phi(x(\tau))|0\rangle. \tag{7.6}$$

Since we are only interested in the probability for the detector to make the transition from E_i to E_j, we should square the amplitude (7.6) and sum over the final state $|\psi\rangle$ of the scalar field, which will not be measured. Using the resolution of identity

$$\sum_\psi |\psi\rangle\langle\psi| = 1, \tag{7.7}$$

we find the probability

$$P(E_i \to E_j) = g^2|m_{ij}|^2\int_{-\infty}^{\infty} \mathrm{d}\tau\,\mathrm{d}\tau' \; \mathrm{e}^{-\mathrm{i}(E_j-E_i)(\tau'-\tau)}G^+(x(\tau'),x(\tau)), \tag{7.8}$$

where $G^+(x(\tau'),x(\tau))$ is the Wightman function. Notice that the prefactor in (7.8) depend on the details of the detector, so it is useful to extract the piece which depends only on the scalar field and the worldline trajectory. For this reason, we define the *detector response function* as

$$\mathcal{F}(E) = \int_{-\infty}^{\infty} \mathrm{d}\tau\,\mathrm{d}\tau' \; \mathrm{e}^{-\mathrm{i}E(\tau-\tau')}G^+(x(\tau),x(\tau')). \tag{7.9}$$

In the case that $G^+(x(\tau'),x(\tau))$ only depends on the $\Delta\tau = \tau - \tau'$, we can change variables to $\Delta\tau$ and

$$\bar{\tau} = \frac{\tau+\tau'}{2} \tag{7.10}$$

The detector response function per unit of proper time is then defined by removing the volume factor coming from the integration over $\bar{\tau}$,

$$f(E) = \int_{-\infty}^{\infty} \mathrm{d}\Delta\tau\,\mathrm{e}^{-\mathrm{i}E\Delta\tau}G^+(\Delta\tau). \tag{7.11}$$

Let us now study two different situations for particle detection. First, we consider a stationary observer in Minkowski spacetime. Then, the Wightman function is given by (3.107), and we find

$$G^{+}(\Delta\tau) = -\frac{1}{4\pi^2}\frac{1}{(\Delta\tau - \mathrm{i}\epsilon)^2}. \tag{7.12}$$

If we plug this in the formula for (7.11), we can calculate the integral by residues. Since $E > 0$ we should close the contour for $\Delta\tau$ in the lower half-plane, since in this case the integral at the half-circle at infinity goes to zero due to the damping factor $\mathrm{e}^{-\mathrm{i}E\Delta\tau}$ (Jordan's lemma) and we conclude that

$$f(E) = 0, \tag{7.13}$$

so there is no particle detection.

Let us now consider an observer in four-dimensional de Sitter space, moving along a geodesic where $\mathbf{x}$ = constant, and moving in the conformal vacuum. The Wightman function is given by the $d = 4$ specialization of (4.71), i.e.

$$G^{+}(x, x') = -\frac{H^2}{4\pi^2}\frac{\eta\eta'}{(\Delta\eta - \mathrm{i}\epsilon)^2 - (\Delta\mathbf{x})^2}. \tag{7.14}$$

The proper time $\Delta\tau$ is given by the specialization of (B.37) for $\omega = 0$,

$$\cosh(H\Delta\tau) = \frac{(\Delta\eta)^2}{2\eta\eta'} + \frac{1}{2}. \tag{7.15}$$

and

$$\frac{(\Delta\eta)^2}{2\eta\eta'} = 2\sinh^2\left(\frac{H\tau}{2}\right). \tag{7.16}$$

Equivalently, we can look at (4.9) and realize that for an observer with $\mathrm{d}x_i = 0$, the cosmic time t is the proper time, therefore we can use the parametrization (4.15) (this is more useful to keep track of the iϵ). We then find,

$$G^{+}(x, x') = -\frac{H^2}{16\pi^2}\frac{1}{\left(\sinh\left(\frac{H\Delta\tau}{2}\right) - \mathrm{i}\epsilon\right)^2} \tag{7.17}$$

We can now proceed with the calculation of $f(E)$. We simply have to calculate the integral

$$f(E) = -\frac{1}{16\pi^2}\int_{-\infty}^{\infty}\mathrm{d}z\frac{\mathrm{e}^{-\mathrm{i}Ez}}{\left(\sinh\left(\frac{Hz}{2}\right) - \mathrm{i}\epsilon\right)^2}. \tag{7.18}$$

Notice that there is no pole at $z = 0$ due to the ϵ prescription, which moves the pole to

$$z = \frac{2}{H}\mathrm{i}\epsilon > 0 \tag{7.19}$$

which is in the upper half-plane. Like before, to perform the integral we close the contour in the lower half-plane. Only the nontrivial poles of the sinh are relevant for the calculation, which are located at

$$\frac{Hz}{2} = -n\pi\mathrm{i}, \qquad n = 1, 2, \cdots. \tag{7.20}$$

These are double poles, and the residue is easily calculated by expanding the exponential function. We find,

$$-\frac{4\mathrm{i}E}{H^2}\mathrm{e}^{-2n\pi E/H} \tag{7.21}$$

Therefore,

$$f(E) = 2\pi\mathrm{i}\cdot\frac{H^2}{16\pi^2}\cdot\left(-\frac{4\mathrm{i}E}{H^2}\right)\sum_{n\geq 1}\mathrm{e}^{-2n\pi E/H} = \frac{E}{2\pi}\frac{1}{\mathrm{e}^{2\pi E/H}-1}. \tag{7.22}$$

This corresponds to a thermal spectrum with temperature

$$T = \frac{H}{2\pi}. \tag{7.23}$$

Therefore, an observer at rest in an exponentially expanding universe will measure a thermal bath whose temperature is proportional to the inverse radius H. This was first noted by Gibbons and Hawking in [12].

One way to see the thermal nature of many of these examples is to note that they exhibit a periodicity $t \to t - \mathrm{i}\beta$. For example, in the de Sitter case (7.17), we find a periodicity

$$\tau \to \tau - \frac{2\pi\mathrm{i}}{H}. \tag{7.24}$$

This is typical of thermal correlation functions. One way to understand this, which we will use in the next Chapter, is to consider the canonical partition function in Quantum Mechanics,

$$Z(\beta) = \mathrm{Tr}\,\mathrm{e}^{-\beta\widehat{H}} = \int \mathrm{d}x\,\langle x|\mathrm{e}^{-\beta\widehat{H}}|x\rangle, \qquad \beta = \frac{1}{T}, \tag{7.25}$$

where $\widehat{H}$ is the quantum Hamiltonian and we set $k_B = 1$. The integrand in this expression can be interpreted in terms of the quantum propagator for a particle,

$$K(x,t;x',t') = \langle x|\mathrm{e}^{-\mathrm{i}(t-t')\widehat{H}}|x'\rangle, \tag{7.26}$$

which computes the probability amplitude that a particle described by the Hamiltonian $\widehat{H}$ and starting at (t', x') arrives to (t, x). In the integrand of (7.25), we have such a propagator but for a *periodic* trajectory and in *imaginary* time. Indeed, the inverse temperature is related to the time interval as

$$\Delta t = -\mathrm{i}\beta, \tag{7.27}$$

where

$$\Delta t = t - t'. \tag{7.28}$$

The periodicity property (7.24) can be interpreted by saying that, indeed we have a thermal system at inverse temperature

$$\beta = \frac{2\pi}{H}. \tag{7.29}$$

As we will see, this observation gives a powerful method to compute the temperature and entropy of gravitational backgrounds.

8 Euclidean quantum gravity and thermodynamics

8.1 Review of the path integral formulation

The QM propagator is defined by

$$K(q_f, q_0; t_f, t_0) = \langle q_f|\hat{U}(t_f, t_0)|q_0\rangle, \tag{8.1}$$

where

$$\hat{U}(t_f, t_0) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}(t_f - t_0)} \tag{8.2}$$

and $\hat{H}$ is the Hamiltonian. Let us now divide the interval (t_0, t_f) into $N + 1$ small intervals

$$(t_0, t_1), \quad (t_1, t_2), \quad \cdots \quad (t_N, t_f) \tag{8.3}$$

and let us denote

$$\Delta t_k = t_{k+1} - t_k. \tag{8.4}$$

Therefore, we have

$$\hat{U}(t_f, t_0) = \hat{U}(t_f, t_N) \cdots \hat{U}(t_1, t_0) = \prod_{k=0}^{N} \hat{U}(t_{k+1}, t_k). \tag{8.5}$$

Note that we have defined $t_{N+1} = t_f$. We can now introduce N resolutions of the identity,

$$\int \mathrm{d}q_i |q_i\rangle\langle q_i| = 1, \tag{8.6}$$

to obtain

$$K(q_f, q_0; t_f, t_0) = \int \mathrm{d}q_N \mathrm{d}q_{N-1} \cdots \mathrm{d}q_1 \prod_{k=0}^{N} K(q_{k+1}, q_k; t_{k+1}, t_k). \tag{8.7}$$

We are interested in taking the limit $N \to \infty$, $\Delta t_k \to 0$, as in the calculation of Riemann integrals. In this limit, we expect

$$K(q_{k+1}, q_k; t_{k+1}, t_k) = \langle q_{k+1}|\mathrm{e}^{-\frac{\mathrm{i}\Delta t_k}{\hbar}\hat{H}}|q_k\rangle \approx \langle q_{k+1}|1 - \frac{\mathrm{i}\Delta t_k}{\hbar}\hat{H}|q_k\rangle. \tag{8.8}$$

We calculate now

$$\langle q_{k+1}|\hat{H}|q_k\rangle = \int \mathrm{d}p_k \langle q_{k+1}|p_k\rangle\langle p_k|\hat{H}|q_k\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int \mathrm{d}p_k \mathrm{e}^{\frac{\mathrm{i}}{\hbar}p_k q_{k+1}} \langle p_k|\hat{H}|q_k\rangle. \tag{8.9}$$

If we now assume that

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}), \tag{8.10}$$

we have

$$\begin{aligned}\langle p_k|\hat{H}|q_k\rangle &= \left(\frac{p_k^2}{2m} + V(q_k)\right)\langle p_k|q_k\rangle = \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{p_k^2}{2m} + V(q_k)\right)\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}p_k q_k} \\ &= \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{p_k^2}{2m} + V(q_k)\right)\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}p_k q_k} H(p_k, q_k).\end{aligned} \tag{8.11}$$

We finally obtain

$$\langle q_{k+1}|\hat{H}|q_k\rangle = \int \frac{\mathrm{d}p_k}{2\pi\hbar} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}p_k(q_{k+1}-q_k)} H(p_k, q_k), \tag{8.12}$$

and by writing that

$$\langle q_{k+1}|q_k\rangle = \int \frac{\mathrm{d}p_k}{2\pi\hbar} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}p_k(q_{k+1}-q_k)}, \tag{8.13}$$

we find,

$$\begin{aligned} \langle q_{k+1}|1 - \frac{\mathrm{i}\Delta t_k}{\hbar}\hat{H}|q_k\rangle &= \int \frac{\mathrm{d}p_k}{2\pi\hbar} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}p_k(q_{k+1}-q_k)} \left(1 - \frac{\mathrm{i}\Delta t_k}{\hbar} H(p_k, q_k)\right) \\ &\approx \int \frac{\mathrm{d}p_k}{2\pi\hbar} \exp\left[\frac{\mathrm{i}\Delta t_k}{\hbar}\left(p_k \frac{q_{k+1}-q_k}{\Delta_k} - H(p_k, q_k)\right)\right]. \end{aligned} \tag{8.14}$$

Therefore, we find

$$K(q_f, q_0; t_f, t_0) \approx \int \left[\prod_{k=1}^{N} \frac{\mathrm{d}p_k \mathrm{d}q_k}{2\pi\hbar}\right] \frac{\mathrm{d}p_0}{2\pi\hbar} \exp\left[\sum_{k=0}^{N} \frac{\mathrm{i}\Delta t_k}{\hbar}\left(p_k \frac{q_{k+1}-q_k}{\Delta t_k} - H(p_k, q_k)\right)\right]. \tag{8.15}$$

In the limit $N \to 0$, $\Delta t_k \to 0$, we interpret the exponent as a discretization of the functional

$$\int_{t_0}^{t_f} (p\dot{q} - H(p, q))\, \mathrm{d}t, \tag{8.16}$$

and the integral measure as an integration over all possible paths $q(t)$, $p(t)$ going from $q(t_0) = q_0$ to $q(t_f) = q_f$, and we write the following *path integral* representation of the propagator,

$$K(q_f, q_0; t_f, t_0) = \int \mathcal{D}q(t)\mathcal{D}p(t)\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t_f}(p\dot{q}-H(p,q))\mathrm{d}t}. \tag{8.17}$$

We can write this is yet another form by integrating over the momenta. Indeed, the integral over p_k in

$$K(q_{k+1}, q_k; t_{k+1}, t_k) \approx \int \frac{\mathrm{d}p_k}{2\pi\hbar} \exp\left[\frac{\mathrm{i}\Delta t_k}{\hbar}\left(p_k \frac{q_{k+1}-q_k}{\Delta t_k} - \frac{p_k^2}{2m} - V(q_k)\right)\right] \tag{8.18}$$

is a Gaussian. We can use that

$$\int \mathrm{d}x \exp\left[-\frac{ax^2}{2} + bx\right] = \sqrt{\frac{2\pi}{a}} \exp\left[\frac{b^2}{2a}\right]. \tag{8.19}$$

In this case,

$$a = \frac{\mathrm{i}\Delta t_k^2}{2m\hbar} \tag{8.20}$$

is imaginary, but the integral can still be computed by rotating appropriately the integration contour. One finds,

$$K(q_{k+1}, q_k; t_{k+1}, t_k) = \sqrt{\frac{m}{2\pi\mathrm{i}\hbar\Delta t_k}} \exp\left[\frac{\mathrm{i}\Delta t_k}{\hbar}\left(\frac{m}{2}\left(\frac{q_{k+1}-q_k}{\Delta t_k}\right)^2 - V(q_k)\right)\right], \tag{8.21}$$

so that

$$
\begin{aligned}
&K(q_f, q_0; t_f, t_0) \\
&\approx \int \left[\sqrt{\frac{m}{2\pi \mathrm{i}\hbar\Delta t_0}} \prod_{k=1}^{N} \sqrt{\frac{m}{2\pi \mathrm{i}\hbar\Delta t_k}} \mathrm{d}q_k \right] \exp\left[\sum_{k=0}^{N} \frac{\mathrm{i}\Delta t_k}{\hbar} \left(\frac{m}{2} \left(\frac{q_{k+1} - q_k}{\Delta t_k} \right)^2 - V(q_k) \right) \right].
\end{aligned} \tag{8.22}
$$

In the limit $N \to \infty$, $\Delta t_k \to 0$, we interpret the exponent as the classical action

$$
S(q(t)) = \int_{t_0}^{t_f} L(q(t))\mathrm{d}t \tag{8.23}
$$

of the mechanical system, and the integration is done again over all possible paths $q(t)$ with the appropriate boundary conditions. In this way, we obtain the following path integral form for the propagator,

$$
K(q_f, q_0; t_f, t_0) = \int \mathcal{D}q(t) \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S(q(t))}. \tag{8.24}
$$

It turns out that the same formalism can be used to obtain the canonical density matrix. We define the (un-normalized) density matrix as

$$
\hat{\rho} = \mathrm{e}^{-\beta \hat{H}} \tag{8.25}
$$

This is formally very similar to the propagator (they are both determined by conservation of the energy, after all). We can then compute

$$
\langle q_f | \hat{\rho} | q_i \rangle = \rho(q_f, q_i; \beta) \tag{8.26}
$$

in a very similar way. We regard β as an interval of Euclidean time, and we split into N intervals $\Delta\tau_k$. The analogue of formula (8.18) above is

$$
\rho(q_{k+1}, q_k; \tau_{k+1}, \tau_k) \approx \int \frac{\mathrm{d}p_k}{2\pi\hbar} \exp\left[\frac{\mathrm{i}\Delta\tau_k}{\hbar} p_k \frac{q_{k+1} - q_k}{\Delta\tau_k} - \frac{\Delta\tau_k}{\hbar} \left(\frac{p_k^2}{2m} - V(q_k) \right) \right], \tag{8.27}
$$

which can be integrated over p_k, this time with a *bone fide* Gaussian. One finds,

$$
\rho(q_{k+1}, q_k; \tau_{k+1}, \tau_k) = \sqrt{\frac{m}{2\pi\hbar\Delta\tau_k}} \exp\left[-\frac{\Delta\tau_k}{\hbar} \left(\frac{m}{2} \left(\frac{q_{k+1} - q_k}{\Delta\tau_k} \right)^2 - \frac{\Delta\tau_k}{\hbar} V(q_k) \right) \right], \tag{8.28}
$$

and due to the change of relative sign between the kinetic and the potential terms, we find a path integral representation for the density matrix

$$
\rho(q_f, q_0; \beta) = \int \mathcal{D}q(t) \mathrm{e}^{-\frac{1}{\hbar} S_E(q(t))} \tag{8.29}
$$

in terms of the *Euclidean* action:

$$
S_E = \int_0^{\beta} \left(\frac{m}{2} \dot{q}^2 + V(q) \right) \mathrm{d}\tau, \tag{8.30}
$$

8.2 Euclidean continuation and thermal properties

We have seen that thermal correlation functions are characterized by periodicity in time, with an imaginary period. This is equivalent to periodicity in *Euclidean* time τ (this should not be confused with the proper time of previous sections),

$$t = -\mathrm{i}\tau, \tag{8.31}$$

and with a real period. The inverse of this period should then be identified with the temperature. We have also seen that quantum fields in some curved backgrounds display the same periodicity, which is an indication of their thermal nature. As shown by Gibbons and Hawking in [11], this periodicity property (and therefore the thermal nature of these backgrounds) can be seen by looking directly at the Euclidean continuation of the metric. This gives an elegant and quick derivation of the Hawking temperature for a large class of black holes.

Let us look for example at the Euclidean continuation of the Euclidean solution (6.1). We obtain then

$$\mathrm{d}s^2 = \left(1 - \frac{2M}{r}\right)\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{1 - \frac{2M}{r}} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2). \tag{8.32}$$

This is singular at the horizon $r = 2M$, but the singularity can be removed if we interpret it as the apparent singularity at $\rho = 0$ that is seen in polar coordinates

$$\mathrm{d}s^2 = \rho^2\mathrm{d}\theta^2 + \mathrm{d}\rho^2. \tag{8.33}$$

However, in order to do this, we will have to interpret τ as an *angular* variable, and then it will have a period. We will do the calculation for a general case in which the metric reads

$$\mathrm{d}s^2 = V(r)\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{V(r)} + r^2\mathrm{d}\Omega^2 \tag{8.34}$$

and the horizon is located at $r = r_+$, i.e. r_+ satisfies the equation

$$V(r_+) = 0. \tag{8.35}$$

Let us introduce the variable

$$\rho = \frac{1}{c}(V(r))^\gamma, \qquad \gamma > 0 \tag{8.36}$$

which will be regarded as a radial coordinate which vanishes at the horizon. The parameters c, γ are constants to be fixed by the requirement that, near $\rho = 0$ (i.e. near the horizon), the metric in the (ρ, τ) plane looks like

$$\mathrm{d}\rho^2 + \rho^2\mathrm{d}\tau^2. \tag{8.37}$$

We have

$$\mathrm{d}r = \frac{c}{\gamma}(c\rho)^{\frac{1-\gamma}{\gamma}}\frac{\mathrm{d}\rho}{V'(r)}. \tag{8.38}$$

Therefore,

$$\frac{\mathrm{d}r^2}{V(r)} = \frac{c^2}{\gamma^2}(c\rho)^{\frac{1-2\gamma}{\gamma}}\frac{\mathrm{d}\rho^2}{(V'(r))^2}. \tag{8.39}$$

Near $r = r_+$, this is

$$\frac{c^2}{\gamma^2}(c\rho)^{\frac{1-2\gamma}{\gamma}}\frac{\mathrm{d}\rho^2}{(V'(r_+))^2} + \cdots = \mathrm{d}\rho^2 + \cdots, \tag{8.40}$$

and if we want this to go like $\mathrm{d}\rho^2$ near $\rho = 0$, we must have

$$\gamma = \frac{1}{2}, \qquad c = \frac{1}{2}\left|V'(r_+)\right| \tag{8.41}$$

If we now look at the part of the metric involving $\mathrm{d}\tau$ we obtain

$$V(r)\mathrm{d}\tau^2 = c^2\rho^2\mathrm{d}\tau^2 \tag{8.42}$$

To avoid singularities, $c\tau$ must be an angular coordinate, i.e.

$$\theta = c\tau \tag{8.43}$$

must be periodically identified with period 2π. This means that τ is periodically identified with period $\beta = 2\pi/c$, i.e.

$$\beta = \frac{4\pi}{|V'(r_+)|}. \tag{8.44}$$

This gives the inverse temperature of the black hole described by the metric (8.34). Notice that

$$\rho = 2(r - r_+) + \mathcal{O}\left((r - r_+)^2\right). \tag{8.45}$$

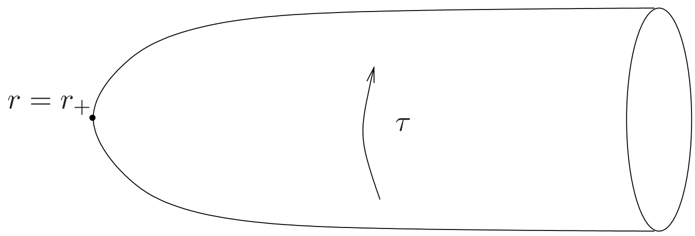

Figure 3. The geometry of the Euclidean Schwarzschild black hole in the r, τ coordinates looks like a cigar, since the circumference of the τ circle shrinks as we approach $r = r_+$.

In the Schwarzschild case one has

$$V(r) = 1 - \frac{2M}{r}, \qquad V'(r_+) = \frac{1}{2M}. \tag{8.46}$$

Therefore,

$$c = \frac{1}{4M}, \qquad \rho = 4M\left(1 - \frac{2M}{r}\right)^{1/2} \tag{8.47}$$

and

$$\beta = 8\pi M, \tag{8.48}$$

which agrees with the result found in (6.22).

The manifold obtained by imposing periodicity in the Euclidean time of the Schwarzschild metric is called the *Euclidean section of the Schwarzschild solution*. Notice that in this solution one has

$$\rho \geq 0 \Rightarrow r \geq 2M. \tag{8.49}$$

The metric in the (ρ, τ) variables can be regarded as a *cigar*, since the $\mathbb{S}^1$ corresponding to τ has radius

$$\left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} \tag{8.50}$$

which becomes zero at $r = 2M$, see Fig. 3.

We can also find the right periodicity (therefore the temperature) by using Kruskal–Skezeres coordinates, as in [11]. Consider the Euclidean continuation of the metric (6.13), which is obtained by

$$\zeta = \mathrm{i}T. \tag{8.51}$$

The change of coordinates (6.14) becomes now

$$\frac{X - \mathrm{i}\zeta}{X + \mathrm{i}\zeta} = \mathrm{e}^{-\mathrm{i}\tau/2M} \tag{8.52}$$

which makes manifest that the r.h.s. is a phase. in terms of the complex variable

$$w = X + \mathrm{i}\zeta, \tag{8.53}$$

we see that (8.52) can be written as

$$\overline{w} w^{-1} = \mathrm{e}^{-\mathrm{i}\tau/2M}. \tag{8.54}$$

We have an obvious periodicity $w \to \mathrm{e}^{2\pi \mathrm{i}} w$, which leads to

$$\overline{w} w^{-1} \to \overline{w} w^{-1} \exp(-4\pi \mathrm{i}). \tag{8.55}$$

Requiring the same periodicity in the l.h.s. and in the r.h.s. of (8.54) we find

$$\frac{\tau + \tau_0}{2M} = \frac{\tau}{2M} + 4\pi, \tag{8.56}$$

which leads again to the period $8\pi M$.

8.3 Black hole entropy and the Euclidean action

In the path-integral formulation of statistical mechanics, the canonical partition function is computed as a path integral of the form

$$Z(\beta) = \int [\mathcal{D}x(\tau)]\, \mathrm{e}^{-S_E(x(\tau))}, \tag{8.57}$$

where S_E is the Euclidean action, and we integrate over all trajectories such that

$$x(0) = x(\beta), \tag{8.58}$$

where

$$\beta = \frac{1}{T}. \tag{8.59}$$

Here, τ is the Euclidean time, and we have considered for simplicity a thermal system with a single particle describing a path $x(\tau)$. The periodicity condition (8.58) can be interpreted by saying that the particle lives in a circle (sometimes called the thermal circle) $\mathbb{S}^1_\beta$, where β is the length of the circle. From the canonical partition function $Z(\beta)$ one can compute the thermal free energy

$$F = -T \log Z, \tag{8.60}$$

and the entropy S follows from the relationship

$$F = E - TS. \tag{8.61}$$

In [11], it was proposed that the entropy of a gravitational system can be computed, by using a similar formulation. One considers the Euclidean gravitational action $S_E[g]$ and defines formally a thermal path integral for gravity,

$$Z = \int [\mathcal{D}g] \, \mathrm{e}^{-S_E[g]}, \tag{8.62}$$

where the path integral is over all metrics with a thermal circle $\mathbb{S}^1_\beta$, and some fixed asymptotic properties at infinity. Since Einstein gravity does not lead to a renormalizable quantum field theory, the above proposal should be taken with care. But we can regard gravity as an effective theory, and the Einstein action as the effective action at low energies. Therefore, it is still in principle possible to calculate the entropy of the gravitational system by using Einstein's theory at the classical level, in the same way in which one can use the pion Lagrangian (which is not renormalizable) to study thermal properties of QCD at low temperature. Therefore, we can compute the thermal partition function as

$$Z(\beta) \approx \mathrm{e}^{-S_E[g_{\mathrm{cl}}]}, \tag{8.63}$$

where g_{cl} is the metric which satisfies the EOM and has thermal periodicity in the Euclidean time direction. This strategy to analyze properties of quantum gravity is called *Euclidean quantum gravity* and it was developed by S. Hawking, G. Gibbons and others in the 1970 and 1980s, see [14] for an early review and [13] for a collection of papers.

When is the approximation (8.63) trustable? In the gravitational action, the rôle of $\hbar$ is played by Newton's constant,

$$G_N \sim \ell_p^{d-2}, \tag{8.64}$$

where ℓ_p is Planck's length. We expect (8.63) to be valid when this constant (or equivalently, ℓ_p) is small as compared to the scales of the problem. In other words, if there is an intrinsic length scale in the problem ℓ, then the semiclassical approximation (8.63) will be valid as long as

$$\ell_p \ll \ell. \tag{8.65}$$

The first thing we need in order to compute (8.63) is the Euclidean version of the Einstein–Hilbert action, including the boundary term. The Euclidean version of the metric is in general complex, due to the cross term g_{0i}, but in the absence of such a term (i.e. for static metrics), we find a metric which, with our conventions, has signature $(-, \cdots, -)$. We will take the Euclidean metric to be the one obtained in this way but with all the signs changed, so in the static case this is a conventional Riemannian metric. We then find, in going to Euclidean signature,

$$\mathrm{i} \int_M \mathrm{d}t \mathrm{d}^{d-1}x \sqrt{-g} \, (R - 2\Lambda) \to \int_M \mathrm{d}\tau \mathrm{d}^{d-1}x \sqrt{g} \, (R - 2\Lambda) \, , \tag{8.66}$$

where in the r.h.s. we consider the Euclidean metric and the curvature associated to it. Notice that there is no further change of sign in R, since with our conventions de Sitter space and its Euclidean continuation (the sphere) have both positive curvature. Therefore, the Euclidean action is

$$S_{\text{bulk}} = -\frac{1}{16\pi G_N}\int_M \mathrm{d}^d x\,\sqrt{g}\,(R-2\Lambda) \tag{8.67}$$

where R is the curvature scalar and Λ is the cosmological constant. The Wick rotation for the boundary term is very similar,

$$S_{\text{bdry}} = -\frac{1}{8\pi G_N}\int_{\partial M} K\,\sqrt{h}\mathrm{d}^{d-1}y \tag{8.68}$$

Here, h is the induced metric on the boundary by the Euclidean metric, and K is the trace of the extrinsic curvature (2.35). The total Euclidean action is then

$$S_E = -\frac{1}{16\pi G_N}\int_M \mathrm{d}^d x\,\sqrt{g}\,(R-2\Lambda) - \frac{1}{8\pi G_N}\int_{\partial M} K\,\sqrt{h}\mathrm{d}^{d-1}y + S_{\text{ct}} \tag{8.69}$$

where S_{ct} is a subtraction term or counterterm. This counterterm is included since the Euclidean action is typically divergent due to the behavior at large distances, i.e. when $r\to\infty$. The counterterm removes these IR divergences and leads to a finite value for the action. In the case of asymptotically flat spacetimes, we can use the Euclidean version of (2.69). For asymptotically AdS spaces, with a characteristic radius ℓ, there is a natural choice of counterterms given by (see for example [10])

$$S_{\text{ct}} = \frac{1}{8\pi G_N}\int_{\partial M} \mathrm{d}^{d-1}y\sqrt{h}\left[\frac{d-2}{\ell} + \frac{\ell}{2(d-3)}\mathcal{R} + \cdots\right], \tag{8.70}$$

where $\mathcal{R}$ is the scalar curvature on the boundary ∂M, computed for the induced metric h. The dots denote higher order counterterms (in the Riemann tensor) which are needed for higher dimensional spaces.

Example 8.1. Let us evaluate the gravitational action for the Euclidean version of the metric in example 2.1:

$$\mathrm{d}s^2 = V(r)\mathrm{d}\tau^2 + V(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_n^2. \tag{8.71}$$

This metric describes an asymptotically flat space and we will assume that it satisfies Einstein's equations in the vacuum with zero cosmological constant: $R_{\mu\nu}=0$. The Euclidean time variable τ is periodic with period β. In order to avoid divergences, we put a cutoff at large r, so we work on a manifold M whose boundary ∂M is the hypersurface of constant r given by the cutoff. Since by assumption the scalar curvature vanishes, $R=0$, there is no bulk contribution to the action. Let us now evaluate the boundary contribution. The extrinsic curvature is given by the Euclidean version of (2.49). The volume element is

$$\sqrt{h} = V^{1/2}(r) r^n \sqrt{g_{\mathbb{S}^n}}, \tag{8.72}$$

where $g_{\mathbb{S}^n}$ is the metric on the n-sphere, with radius one. We find

$$8\pi G_N S_{\text{bry}} = -\mathrm{vol}(\mathbb{S}^n)\beta r^n\left\{\frac{n}{r}V(r) + \frac{1}{2}V'(r)\right\}, \tag{8.73}$$

since

$$\int_{\mathbb{S}^1_\beta}\mathrm{d}\tau = \beta \tag{8.74}$$

is the length of the periodic, Euclidean time direction. The counterterm (2.69) can be evaluated as follows. The boundary r =constant, when embedded in flat space, has extrinsic curvature

$$K^{(0)} = \frac{n}{r}, \tag{8.75}$$

which is just the expression (2.49) when $V(r) = 1$. The counterterm (2.69) is then

$$8\pi G_N S_{\text{ct}} = \text{vol}(\mathbb{S}^n)\beta r^n \left\{ \frac{n}{r} V^{1/2}(r) \right\} \tag{8.76}$$

The total action is then

$$8\pi G_N S = -\text{vol}(\mathbb{S}^n)\beta r^n \left\{ \frac{n}{r} V^{1/2}(r) \left(V^{1/2}(r) - 1 \right) + \frac{1}{2} V'(r) \right\}. \tag{8.77}$$

As an important particular case, let us consider $n = 2$, and

$$V(r) = 1 - \frac{2M}{r} \tag{8.78}$$

which corresponds to a Schwarzschild black hole of mass M in four dimensions. It is easy to see that the boundary action diverges as $r \to \infty$. However, after adding the counterterm, we find

$$\begin{aligned} 8\pi G_N S_E &= -4\pi\beta r^2 \left\{ \frac{2}{r} \left(1 - \frac{M}{r} + \mathcal{O}(r^{-2}) \right) \left(-\frac{M}{r} + \mathcal{O}(r^{-2}) \right) + \frac{M}{r^2} \right\} \\ &= 4\pi\beta M + \mathcal{O}(1/r), \end{aligned} \tag{8.79}$$

which is finite as $r \to \infty$ and gives

$$S_E = \frac{M\beta}{2G_N}. \tag{8.80}$$

The above calculation allows us to compute the entropy of the Schwarzschild black hole from the Euclidean action. From (8.63) we find (we set $G_N = 1$)

$$-\log Z \approx \frac{M\beta}{2}. \tag{8.81}$$

where we used (8.48). The energy is the mass of the black hole,

$$E = M, \tag{8.82}$$

and the entropy is then given by

$$S = \beta E + \log Z \approx \frac{M\beta}{2} = 4\pi M^2 = \frac{1}{4} A, \tag{8.83}$$

where $A = 4\pi(2M)^2$ is the are of the black hole horizon. This is the famous Bekenstein–Hawking formula for the entropy of the black hole. Although we have derived it just for the Schwarzschild case, it can be shown that the entropy of a generic black hole is one-quarter of the area of the horizon, on quite general grounds.

The above derivation is a semiclassical derivation, and it is only valid when the length scale of the problem (the Schwarzschild radius) is large as compared to the Planck length, i.e. when

$$A \gg \ell_p^2. \tag{8.84}$$

In general, we expect the above formula to be the first term in an asymptotic expansion for large areas. It is indeed known that, in general, the Bekenstein–Hawking formula has subleading corrections (like for example $\log A$ corrections, which can be computed by using classical gravity).

8.4 Temperature and entropy of de Sitter space

De Sitter space, when analyzed from the point of view of Euclidean quantum gravity, has both a temperature and an entropy. The temperature is due to the presence of a horizon, as it is made explicit in static coordinates (4.14): there is a horizon at

$$r_+ = \frac{1}{H}, \tag{8.85}$$

and the general formula (8.44) gives

$$\beta = \frac{2\pi}{H}, \tag{8.86}$$

which agrees with the result obtained in (7.23) by using particle detectors.

Let us now derive the thermodynamic properties of de Sitter by using the Euclidean path integral. The Euclidean rotation of dS gives a closed space, the d-dimensional sphere $\mathbb{S}^d$, therefore there is no boundary contribution in this case. By using the relationship (4.75) between R and Λ, and between Λ and the radius $L = 1/H$ (4.77) for $k = 1$, we find that the bulk Euclidean action is given by

$$S_{\text{bulk}} = -\frac{\Lambda}{4\pi(d-2)G_N}\int \sqrt{g}\mathrm{d}^d x = -\frac{(d-1)H^2}{8\pi G_N}\mathrm{vol}(\mathbb{S}^d). \tag{8.87}$$

In the four-dimensional case, $d = 4$, we have

$$\mathrm{vol}(\mathbb{S}^4) = \frac{8\pi^2}{3H^4}. \tag{8.88}$$

Then,

$$S_E = -\frac{\pi}{H^2 G_N}. \tag{8.89}$$

In this case, $E = 0$, since this is empty de Sitter, and the entropy is given by

$$S = \log Z = \frac{\pi}{H^2 G_N} = \frac{3\pi}{\Lambda G_N}. \tag{8.90}$$

The microscopic interpretation of this result remains a deep challenge for modern theoretical physics.

8.5 Black holes in AdS. The Hawking–Page transition

The AdS black hole in AdS_{n+1} is given by the metric

$$\mathrm{d}s^2 = V(r)\mathrm{d}t^2 - V^{-1}(r)\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2_{n-1} \tag{8.91}$$

where

$$V(r) = 1 + \frac{r^2}{\ell^2} - \frac{M}{r^{n-2}}. \tag{8.92}$$

It is clearly asymptotically AdS, and it reduces to the AdS metric (4.122) for $M = 0$. The horizon is defined as the largest solution of

$$V(r_+) = 0. \tag{8.93}$$

Let us calculate the temperature of this black hole by going to the Euclidean theory $t = -\mathrm{i}\tau$. We use (8.44). Since

$$V'(r_+) = \frac{2r_+}{\ell^2} + \frac{(n-2)M}{r_+^{n-3}}, \tag{8.94}$$

and

$$\frac{M}{r_+^{n-2}} = 1 + \frac{r_+^2}{\ell^2} \tag{8.95}$$

we deduce

$$V'(r_+) = \frac{nr_+}{\ell^2} + \frac{n-2}{r_+} \tag{8.96}$$

and

$$\beta = \frac{4\pi\ell^2 r_+}{nr_+^2 + (n-2)\ell^2}. \tag{8.97}$$

Therefore, the temperature of the black hole is

$$T = n\frac{r_+}{4\pi\ell^2} + \frac{n-2}{4\pi r_+}. \tag{8.98}$$

In the case of $n = 4$ (i.e. AdS_5), the dependence of the temperature with r_+ is shown in Fig. 4.

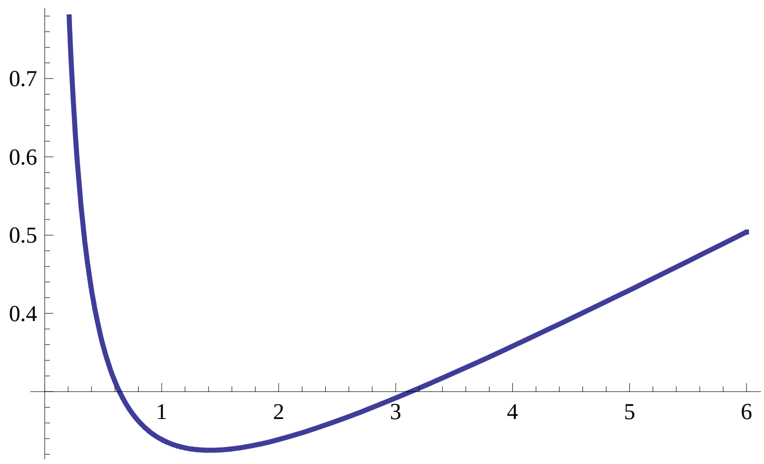

Figure 4. The curve (8.98) representing the temperature of the AdS_5 black hole as a function of the horizon radius.

We see that this curve has a minimum, which for generic n is immediately found to be at

$$r_+^* = \sqrt{\frac{n-2}{n}}\,\ell. \tag{8.99}$$

This corresponds to a minimum temperature given by

$$T_{\min} = \frac{\sqrt{n(n-2)}}{2\pi\ell}. \tag{8.100}$$

Below this temperature, black holes do not contribute to the thermodynamics of the problem. For any $T > T_{\rm min}$, there are *two* different black hole solutions, as one can see graphically in Fig. 4: any horizontal line above $T_{\rm min}$ will intersect the curve at two different points, $r_+^{\rm S,B}$, with

$$r_+^{\rm S} < r_+^{\rm B}. \tag{8.101}$$

These are called the *small* and the *big* black holes, respectively. In terms of the dimensionless parameter

$$t = 4\pi\ell T \tag{8.102}$$

we find

$$r_+^{\rm S,B} = \frac{t}{2n} \pm \frac{1}{2n}\sqrt{t^2 + 8n - 4n^2}. \tag{8.103}$$

We want to calculate now the Euclidean action of such a black hole. Like before, there is a divergence associated to the behavior at large r. One way to regulate this divergence is to compute the difference between the action of a black hole of horizon r_+, and AdS space (this is the method originally used in the calculation in [15, 27]). To do this, we have to consider a "cutoff" at a large value of the radial distance, r. However, in order to perform this comparison, we need to fix a period for the Euclidean time in AdS space. This is in principle arbitrary, since there is no intrinsic temperature associated to AdS. We will set this period, β', in such a way that the geometry of the hypersurface at the cutoff radius r is the same for both manifolds. One way to do this is to require that the length of the thermal circle, measured with the metric evaluated at r, is the same. This gives

$$\beta'\sqrt{r^2/\ell^2 + 1} = \beta\sqrt{r^2/\ell^2 + 1 - M/r^{n-2}} \tag{8.104}$$

This fixes the value of β'. Let us now compute the Euclidean bulk actions. As we known from (8.87), this action is proportional to the volume of spacetime,

$$S_{\rm bulk} = \frac{n}{8\pi G_N \ell^2}\int_X {\rm d}^{n+1}x\,\sqrt{g} = \frac{n}{8\pi G_N \ell^2}{\rm vol}(X). \tag{8.105}$$

Both volumes are infinite, and we have to regularize them by doing the integral up to the cutoff r. We then have to calculate

$$\lim_{r\to\infty}\left({\rm vol}(X_2;r) - {\rm vol}(X_1;r)\right). \tag{8.106}$$

where

$${\rm vol}(X_2;r) = \int_0^\beta {\rm d}\tau\int_{r_+}^r {\rm d}r'\,(r')^{n-1}\int_{\mathbb{S}^{n-1}}{\rm d}\Omega_{n-1} = \frac{\beta}{n}\left(r^n - r_+^n\right){\rm vol}(\mathbb{S}^{n-1}) \tag{8.107}$$

and

$${\rm vol}(X_1;r) = \int_0^{\beta'} {\rm d}\tau\int_0^r {\rm d}r'\,(r')^{n-1}\int_{\mathbb{S}^{n-1}}{\rm d}\Omega_{n-1} = \frac{\beta'}{n}r^n{\rm vol}(\mathbb{S}^{n-1}) \tag{8.108}$$

We then find

$${\rm vol}(X_2;r) - {\rm vol}(X_1;r) = -\frac{\beta}{n}r_+^n{\rm vol}(\mathbb{S}^{n-1}) + \frac{\beta-\beta'}{n}r^n{\rm vol}(\mathbb{S}^{n-1}) \tag{8.109}$$

We have that

$$\beta-\beta'=\beta\left[1-\sqrt{1-\frac{M\ell^2}{r^n(1+\ell^2/r^2)}}\right]=\frac{\beta}{2}\frac{M\ell^2}{r^n}\left(1+\mathcal{O}(r^{-1})\right). \tag{8.110}$$

Therefore, the limit above gives

$$\mathrm{vol}(\mathbb{S}^{n-1})\frac{\beta}{2n}\left(M\ell^2-2r_+^n\right) \tag{8.111}$$

which can be written in terms of r_+, since

$$M\ell^2=r_+^n+\ell^2r_+^{n-2}, \tag{8.112}$$

i.e. the limit is

$$\mathrm{vol}(\mathbb{S}^{n-1})\frac{\beta}{2n}\left(\ell^2-r_+^2\right)r_+^{n-2}. \tag{8.113}$$

We should now consider the value of the boundary term. The boundary contribution was computed in (8.73),

$$8\pi G_N S_{\mathrm{bdry}}=-\mathrm{vol}(\mathbb{S}^{n-1})\mathrm{vol}(\tau)r^{n-1}\left\{\frac{n-1}{r}V(r)+\frac{1}{2}V'(r)\right\}. \tag{8.114}$$

Therefore, we find

$$\begin{aligned}8\pi G_N\left(S_{\mathrm{bdry}}(X_2)-S_{\mathrm{bdry}}(X_1)\right)&=\beta r^{n-1}\left\{\frac{n-1}{r}\left(1-\frac{M}{r^{n-2}}+\frac{r^2}{\ell^2}\right)+\frac{(n-2)M}{2r^2}+\frac{r}{\ell^2}\right\}\\&-\beta' r^{n-1}\left\{\frac{n-1}{r}\left(1+\frac{r^2}{\ell^2}\right)+\frac{r}{\ell^2}\right\}\\&=(\beta-\beta')r^{n-1}\left\{\frac{n-1}{r}\left(1+\frac{r^2}{\ell^2}\right)+\frac{r}{\ell^2}\right\}-\frac{\beta nM}{2}\end{aligned} \tag{8.115}$$

and in the limit $r\to\infty$ this vanishes. We finally obtain, after plugging the value for β,

$$S(X_2)-S(X_1)=\frac{\mathrm{vol}(\mathbb{S}^{n-1})(\ell^2-r_+^2)r_+^{n-1}}{4G_N(nr_+^2+(n-2)\ell^2)}. \tag{8.116}$$

The free energy is

$$F=-T\log Z=\frac{\mathrm{vol}(\mathbb{S}^{n-1})}{16\pi G_N}\frac{r_+^{n-2}}{\ell^2}(\ell^2-r_+^2). \tag{8.117}$$

We have to take into account that this is rather

$$F=F_{\mathrm{BH}}-F_{\mathrm{AdS}} \tag{8.118}$$

i.e. the difference between the free energy of the AdS black hole, and the free energy of empty AdS space.

We can now study stability issues. We first notice that, if

$$r_+>\ell \tag{8.119}$$

i.e. if the horizon radius is bigger than the curvature radius of AdS, then $F < 0$ and therefore the big black hole is *thermodynamically stable* as compared to AdS space. This temperature marks the so-called *Hawking–Page transition*, and we will denote it by T_{HP}. It can be easily calculated by setting $r_+^{\text{B}} = \ell$ in (8.98). One obtains,

$$T_{\text{HP}} = \frac{n-1}{2\pi\ell}. \tag{8.120}$$

Notice that

$$T_{\text{min}} < T_{\text{HP}} \tag{8.121}$$

and at the Hawking–Page temperature one also has that

$$r_+^{\text{S}} = \left(1 - \frac{2}{n}\right)\ell, \tag{8.122}$$

therefore the free energy of the small black hole remains always positive. The free energies of the small and the big black hole, for $n = 4$, $\ell = 1$ are plotted in Fig. 5.

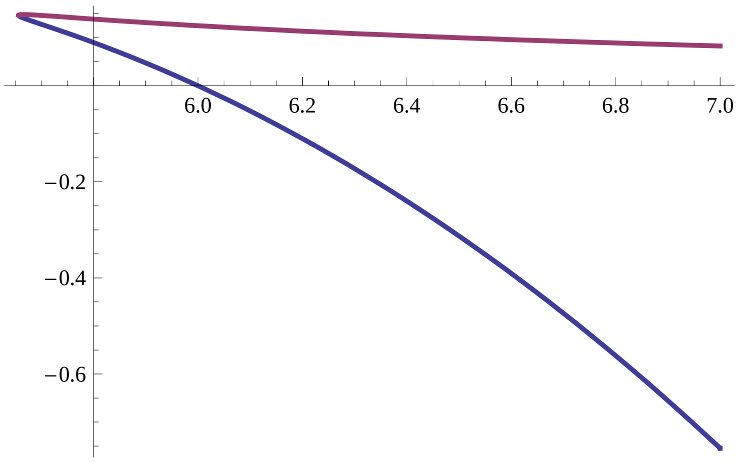

Figure 5. The free energies of the small (top) and the big (bottom) black holes as a function of the t, with $n = 4$. We removed from F the overall, r_+-independent constant. The Hawking–Page phase transition takes place here at $t = 6$, where the free energy of the big black hole changes sign.

The *phase structure* that one finds, summarizing the above discussion, is the following:

1. For $T < T_{\text{min}}$, black holes do not contribute to the canonical ensemble, and the only phase is pure AdS space.
2. If $T_{\text{min}} \leq T \leq T_{\text{HP}}$, there are two stable local minima (AdS and the BBH) and one unstable minimum (SBH). AdS is thermodynamically stable.
3. For $T \geq T_{\text{HP}}$, the structure of local extrema is the same, but now the BBH is thermodynamically stable.

It is also possible to regulate the gravitational action with the counterterms appropriate to AdS. Let us do this calculation, in a slightly more general context. We consider the Euclidean

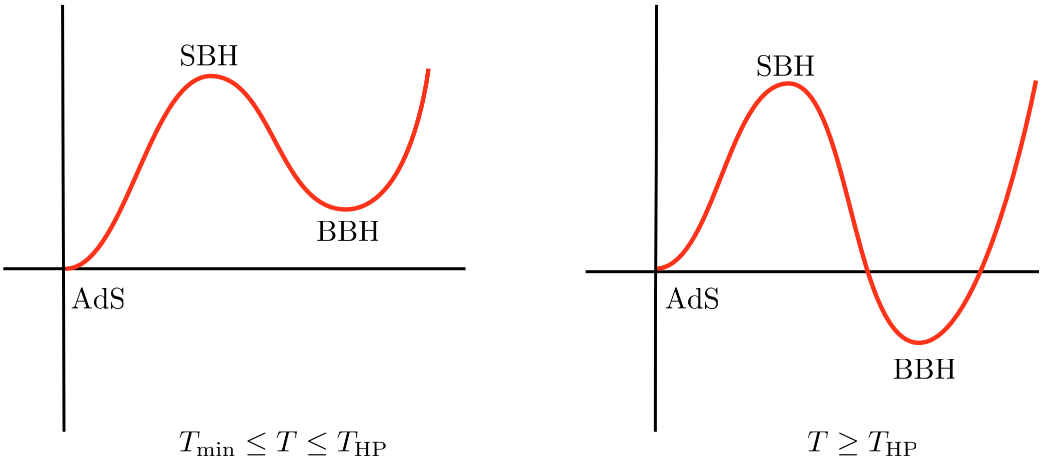

Figure 6. The Hawking–Page phase transition in terms of an effective potential.

version of the metric (2.40), but describing now an asymptotically AdS space with $k = 1$, and we change $n \to n - 1$ to agree with the above conventions. The bulk contribution is different, since it now involves the cosmological constant. It can be calculated from the general expression (8.87)

$$8\pi G_N S_{\rm bulk} = \frac{d-1}{\ell^2} \int_{r_+}^{r} r^{n-1} {\rm d}r \int {\rm d}\tau \int_{\mathbb{S}^{n-1}} \sqrt{g_{\mathbb{S}^{n-1}}} {\rm d}^{n-1}x = \frac{1}{\ell^2} \left(r^n - r_+^n \right) {\rm vol}(\mathbb{S}^{n-1}) {\rm vol}(\tau), \tag{8.123}$$

where we integrated from r_+ to r. When the spacetime describes the AdS black hole, r_+ is the location of the horizon. Notice that as $\ell \to \infty$ we obtain the vanishing result in the bulk for an asymptotically flat space. The boundary term is written down in (8.114). The counterterm can be evaluated as follows. The scalar curvature of the induced metric

$$ {\rm d}s_h^2 = V(r) {\rm d}\tau^2 + r^2 {\rm d}\Omega_{n-1}^2 \tag{8.124}$$

is just the scalar curvature of the $(n-1)$-sphere of radius r, after the appropriate conformal transformation,

$$\mathcal{R} = \frac{(n-1)(n-2)}{r^2} \tag{8.125}$$

The counterterm integral is then

$$8\pi G_N S_{\rm ct} = {\rm vol}(\mathbb{S}^{n-1}) {\rm vol}(\tau) r^{n-1} \frac{n-1}{\ell} \left\{ 1 + \frac{1}{2r^2} \right\} V^{1/2}(r) \tag{8.126}$$

and we have assumed that $n \geq 2$ (for $n = 1$ the last term in the bracket does not contribute). When everything is taken into account we find

$$\begin{aligned} 8\pi G_N S = \frac{{\rm vol}(\mathbb{S}^{n-1}) {\rm vol}(\tau)}{\ell^2} \Bigg[& -r_+^n \\ & + r^{n-1} \left\{ r - \frac{\ell^2}{2} V'(r) - \frac{(n-1)\ell^2}{r} V(r) + (n-1)\ell V^{1/2}(r) \left(1 + \frac{\ell^2}{2r^2} \right) \right\} \Bigg]. \end{aligned} \tag{8.127}$$

As a particular case of this, let us consider the AdS-Schwarzschild black hole with $V(r)$ given in (8.92), and where r_+ is the largest solution of $V(r_+) = 0$. We find, for $n = 2, 3$ (i.e. AdS_3 and AdS_4) the following result

$$8\pi G_N S = \mathrm{vol}(\mathbb{S}^{n-1})\beta \left[-\frac{r_+^n}{\ell^2} + \frac{M\ell^2}{2} \right], \tag{8.128}$$

while for $n = 4$ (i.e. AdS_5), we find

$$8\pi G_N S = \mathrm{vol}(\mathbb{S}^{n-1})\beta \left[-\frac{r_+^4}{\ell^2} + \frac{M\ell^2}{2} + \frac{3\ell^4}{8} \right]. \tag{8.129}$$

A Hypergeometric functions

The hypergeometric function is defined by

$${}_2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma_n)} \frac{z^n}{n!}, \tag{A.1}$$

where

$$(\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)}. \tag{A.2}$$

This function is a solution to the ODE,

$$z(1-z)\partial_z^2 F + [\gamma - (\alpha + \beta + 1) z] \partial_z F - \alpha\beta F = 0 \tag{A.3}$$

If $\mathrm{Re}(\gamma - \alpha - \beta) > 0$, we have

$${}_2F_1(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}. \tag{A.4}$$

If $\mathrm{Re}(\gamma - \alpha - \beta) < 0$, we have a singularity at $z = 1$ given by

$${}_2F_1(\alpha, \beta; \gamma; z) \approx \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)} (1-z)^{\gamma - \alpha - \beta} \tag{A.5}$$

We also have the following identities for the analytic continuation of hypergeometric functions:

$$\begin{aligned} {}_2F_1(\alpha, \beta; \gamma; z) = &\frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} \, {}_2F_1(\alpha, \beta; 1 + \alpha + \beta - \gamma; 1 - z) \\ &+ \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)} (1-z)^{\gamma - \alpha - \beta} \, {}_2F_1(\gamma - \alpha, \gamma - \beta; 1 + \gamma - \alpha - \beta; 1 - z) \end{aligned} \tag{A.6}$$

and

$${}_2F_1(\alpha, \beta; \gamma; z) = (1-z)^{\gamma - \alpha - \beta} \, {}_2F_1(\gamma - \alpha, \gamma - \beta; \gamma; z) \tag{A.7}$$

B Geodesics in (A)dS

B.1 Geodesics in dS

The geodesics of de Sitter can be obtained by minimizing the distance in the embedding $(d+1)$-dimensional Minkowski space, subject to the constraint (4.1). The Lagrangian is given by

$$L = \frac{1}{2}\dot{X}^2 + \lambda\left(X^2 + H^{-2}\right), \tag{B.1}$$

where the derivative is w.r.t. the proper time. The corresponding EOM are

$$-\ddot{X}^a + 2\lambda X^a = 0, \qquad a = 1, \cdots, d. \tag{B.2}$$

As a first step we would like to determine λ. We have the two following equations

$$\begin{aligned} \ddot{X}^a X_a &= 2\lambda X^2 = -2\lambda H^{-2}, \\ X^a X_a &= -H^{-2}. \end{aligned} \tag{B.3}$$

By differentiating twice the second equation we get

$$\ddot{X}^a X_a + \dot{X}^2 = 0 \Rightarrow \ddot{X}^a X_a = -\epsilon, \tag{B.4}$$

where $\epsilon = \dot{X}^2 = \pm 1$ for timelike/spacelike geodesics, respectively. This gives

$$\lambda = \epsilon H^2/2. \tag{B.5}$$

The EOM reads

$$\ddot{X}^a = \epsilon H^2 X^a. \tag{B.6}$$

Consider $\epsilon = 1$. The general solution of the EOM reads

$$X^a = \frac{m^a}{H}\mathrm{e}^{H\tau} + \frac{n^a}{H}\mathrm{e}^{-H\tau}, \tag{B.7}$$

where m^a, n^a are constant vectors. By differentiating one finds

$$\dot{X}^a \dot{X}_a = m^2 \mathrm{e}^{2H\tau} + n^2 \mathrm{e}^{-2H\tau} - 2mn = \epsilon = 1. \tag{B.8}$$

It follows that

$$m^2 = 0 = n^2, \quad 2mn = -1. \tag{B.9}$$

Let us take two points on the geodesic at τ_1 and τ_2. Then we have

$$\begin{aligned} P(X_1, X_2)/H^2 = -X(\tau_1)\cdot X(\tau_2) &= -\frac{mn}{H^2}\left(\mathrm{e}^{H(\tau_1-\tau_2)} + \mathrm{e}^{H(-\tau_1+\tau_2)}\right) \\ &= \frac{1}{H^2}\cosh(Hd(\tau_1,\tau_2)) \end{aligned} \tag{B.10}$$

where $P(X_1, X_2)$ is defined in (4.22) and

$$d(\tau_1, \tau_2) = \tau_1 - \tau_2 \tag{B.11}$$

is the geodesic distance between the two points. It follows that

$$P(X_1, X_2) = \cosh(Hd(\tau_1, \tau_2)) \tag{B.12}$$

which is (4.24).

Consider now $\epsilon = -1$. The general solution for the EOM reads now

$$X^a = \frac{m^a}{H}\cos(H\tau) + \frac{n^a}{H}\sin(H\tau), \tag{B.13}$$

where m^a, n^a are constant vectors. By differentiating one finds

$$\dot{X}^a \dot{X}_a = m^2 \sin^2(H\tau) + n^2\cos^2(H\tau) - 2mn\cos(H\tau)\sin(H\tau) = \epsilon = -1. \tag{B.14}$$

It follows that

$$m^2 = n^2 = -1, \quad 2mn = 0. \tag{B.15}$$

Let us take two points on the geodesic at τ_1 and τ_2. Then we have

$$\begin{aligned} P(X_1, X_2) =& - H^2 X(\tau_1)\cdot X(\tau_2) = -m^2\cos(H\tau_1)\cos(H\tau_2) - n^2\sin(H\tau_1)\sin(H\tau_2) \\ =& \cos(H(\tau_1 - \tau_2)). \end{aligned} \tag{B.16}$$

It follows that, in the spacelike case,

$$P(X_1, X_2) = \cos(Hd(\tau_1, \tau_2)), \tag{B.17}$$

which is (4.25).

We can also derive the same results from a more general analysis of geodesics in a flat RW cosmology. Let τ be the proper time along a geodesic. We have then the following equations,

$$\begin{aligned} \ddot{x}^\mu + \Gamma^\mu_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma = 0, \\ g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \epsilon, \end{aligned} \tag{B.18}$$

where $\epsilon = \pm 1$ for timelike (respectively, spacelike) geodesics, and the dot denotes derivative w.r.t. τ. In a flat RW cosmology, the first equation gives, for the space-like coordinates,

$$\ddot{x}^i + \frac{C'}{C}\dot{\eta}\dot{x}^i = 0, \quad i = 1, \cdots, d-1. \tag{B.19}$$

which can be integrated once to obtain

$$\dot{x}^i = \frac{v^i}{C(\eta)}, \tag{B.20}$$

where $\widehat{v}^i$ are constants. The second equation in (B.18) gives

$$\dot{\eta}^2 - \dot{\mathbf{x}}^2 = \frac{\epsilon}{C(\eta)}. \tag{B.21}$$

We can now use (B.20) to obtain

$$\dot{\eta}^2 = \frac{1}{C(\eta)}\left(\frac{v^2}{C(\eta)} + \epsilon\right), \tag{B.22}$$

where $v^2 = \mathbf{v}^2$, which can be integrated to obtain

$$\tau = \int_{\eta_1}^{\eta_2} \frac{C(\eta)\mathrm{d}\eta}{\sqrt{v^2 + \epsilon C(\eta)}}. \tag{B.23}$$

The integration constant v can be determined as follows. Let

$$d\omega^2 = \dot{\mathbf{x}}^2 d\tau^2 \tag{B.24}$$

be the line element in flat $d-1$ dimensional space. Using (B.22) we find

$$\omega = v \int_{\eta_2}^{\eta_1} \frac{d\eta}{\sqrt{v^2 + \epsilon C(\eta)}} = \sqrt{(\mathbf{x}_2 - \mathbf{x}_1)^2}, \tag{B.25}$$

which determines v in terms of the endpoints, and together with (B.23) determines the geodesic distance between the two points. Notice that, in this derivation, we have not used the second order differential equation for η coming from the first equation in (B.18). It is easy to see that this equation can be obtained by taking a derivative of (B.21) w.r.t. τ and using (B.20), therefore it is not an independent equation.

We are now ready to apply this to de Sitter space, where

$$C(\eta) = \frac{1}{H^2 \eta^2}, \tag{B.26}$$

as follows from (4.19). The integral (B.25) is immediate

$$\omega = \sqrt{\eta_2^2 + \epsilon/(v^2 H^2)} - \sqrt{\eta_1^2 + \epsilon/(v^2 H^2)}. \tag{B.27}$$

Let us now integrate (B.23), assuming that $\epsilon = -1$ (this is the case of spacelike separation). We use the integral

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = -\frac{1}{a} \text{arcsin}\left(\frac{a}{x}\right) \tag{B.28}$$

to calculate

$$H\tau = \frac{1}{Hv} \int_{\eta_1}^{\eta_2} \frac{d\eta}{\eta\sqrt{\eta^2 - 1/(v^2 H^2)}} = \text{arcsin}\left(\frac{1}{vH\eta_2}\right) - \text{arcsin}\left(\frac{1}{vH\eta_1}\right). \tag{B.29}$$

We use now

$$\cos(u_1 - u_2) = \sqrt{1 - \sin^2(u_1)}\sqrt{1 - \sin^2(u_2)} + \sin(u_1)\sin(u_2) \tag{B.30}$$

to write

$$\cos(H\tau) = \sqrt{1 - (vH\eta_1)^{-2}}\sqrt{1 - (vH\eta_2)^{-2}} + \frac{1}{v^2 H^2 \eta_1 \eta_2}. \tag{B.31}$$

On the other hand, from (B.27) we find

$$\omega^2 = \eta_1^2 + \eta_2^2 - \frac{2}{H^2 v^2} - 2\eta_1\eta_2\sqrt{1 - (vH\eta_1)^{-2}}\sqrt{1 - (vH\eta_2)^{-2}}, \tag{B.32}$$

and we conclude that

$$\cos(H\tau) = \frac{\eta_1^2 + \eta_2^2}{2\eta_1\eta_2} - \frac{\omega^2}{2\eta_1\eta_2} = P(X_1, X_2), \tag{B.33}$$

which is what we wanted to show.

A similar computation can be done for $\epsilon = 1$ (i.e. timelike separation). In this case, one finds

$$H\tau = \text{arcsinh}\left(\frac{1}{vH\eta_2}\right) - \text{arcsinh}\left(\frac{1}{vH\eta_1}\right), \tag{B.34}$$

and by using that

$$\cosh(u_1 - u_2) = \sqrt{1 + \sinh^2(u_1)}\sqrt{1 + \sinh^2(u_2)} + \sinh(u_1)\sinh(u_2) \tag{B.35}$$

together with

$$\omega^2 = \eta_1^2 + \eta_2^2 + \frac{2}{H^2 v^2} - 2\eta_1\eta_2\sqrt{1 + (vH\eta_1)^{-2}}\sqrt{1 + (vH\eta_2)^{-2}}, \tag{B.36}$$

we conclude that

$$\cosh(H\tau) = \frac{\eta_1^2 + \eta_2^2}{2\eta_1\eta_2} - \frac{\omega^2}{2\eta_1\eta_2} = P(X_1, X_2), \tag{B.37}$$

again the expected expression.

B.2 Geodesics in AdS

The geodesics of AdS can be also obtained by minimizing the distance in the embedding space, subject to the constraint (4.119). The corresponding Lagrangian is

$$L = \frac{1}{2}\dot{X}^2 + \lambda\left(X^2 - \ell^2\right). \tag{B.38}$$

The procedure is almost identical to the one in dS. The EOM is the same:

$$-\ddot{X}^a + 2\lambda X^a = 0, \tag{B.39}$$

and we have the two following equations

$$\begin{aligned} \ddot{X}^a X_a &= 2\lambda X^2 = 2\lambda\ell^2, \\ X^a X_a &= \ell^2, \end{aligned} \tag{B.40}$$

where

$$X^a X_a = X_0^2 + X_d^2 + \sum_{i=1}^{d-1} X_i^2. \tag{B.41}$$

By differentiating twice the second equation we get

$$\ddot{X}^a X_a + \dot{X}^2 = 0 \to \ddot{X}^a X_a = -\epsilon, \tag{B.42}$$

where $\epsilon = \pm 1$ for timelike/spacelike geodesics. We find

$$\lambda = -\frac{\epsilon}{2\ell^2}, \tag{B.43}$$

and the EOM reads

$$\ddot{X}^a = -\epsilon\ell^{-2}X^a. \tag{B.44}$$

This is exactly like in dS, but we have an exchange timelike $\leftrightarrow$ spacelike. Therefore, for $\epsilon = 1$ we have

$$\ell^{-2}X(\tau_1)\cdot X(\tau_2) = \cos(d(\tau_1, \tau_2)/\ell), \tag{B.45}$$

while for $\epsilon = -1$ we have

$$\ell^{-2}X(\tau_1)\cdot X(\tau_2) = \cosh((\tau_1 - \tau_2)/\ell) = \cosh(d(\tau_1, \tau_2)/\ell). \tag{B.46}$$

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