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An introduction to resurgence in quantum theory

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Abstract: Notes for a course

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1 Introduction and general aspects

Most problems in theoretical physics are not solvable in closed form, and one has to resort to approximation schemes. Many of these approximations lead to formal power series in a small parameter which are generically divergent. It is crucial, both conceptually and technically, to make sense of these series. The most systematic way to do this is the theory of resurgence. So we have the first goal:

$$\boxed{\textbf{Make sense of divergent series}} \tag{1.1}$$

More precisely, the theory of resurgence allows you to do two things:

1. Discover mathematical structures hidden in perturbation theory. This leads e.g. to a window on non-perturbative effects.

2. Upgrade perturbative series to exact results.

Let us consider some examples.

Example 1.1. *Perturbation theory in quantum mechanics.* Let us consider the *quartic oscillator*. It is described by the Hamiltonian,

$$\mathsf{H} = \frac{\mathsf{p}^2}{2} + \frac{\mathsf{q}^2}{2} + \frac{g\mathsf{q}^4}{4}, \qquad g > 0, \tag{1.2}$$

where p, q are Heisenberg operators on $L^2(\mathbb{R})$ with canonical commutation relations $[\mathsf{q}, \mathsf{p}] = \mathrm{i}\hbar$. There are various rigorous results on the spectrum of this Hamiltonian, regarded as an operator on $L^2(\mathbb{R})$. Since it involves a confining potential, with

$$V(q) = \frac{q^2}{2} + \frac{gq^4}{4} \to \infty, \qquad |q| \to \infty, \tag{1.3}$$

one can show (see for example [1]) that H^{-1} is compact and positive, so that H has a discrete, positive, non-degenerate spectrum $E_n(g)$, $n = 0, 1, 2, \cdots$, with

$$0 < E_0(g) < E_1(g) < \cdots \tag{1.4}$$

The asymptotic expansion of $E_n(g)$, for small g, can be calculated by using stationary perturbation theory. For example, for the ground state energy $E_0(g)$ one finds

$$E_0(g) \sim \varphi(g), \tag{1.5}$$

where

$$\varphi(g) = \sum_{n\geq 0} a_n g^n = \frac{1}{2} + \frac{3}{4}\left(\frac{g}{4}\right) - \frac{21}{8}\left(\frac{g}{4}\right)^2 + \frac{333}{16}\left(\frac{g}{4}\right)^3 + \mathcal{O}(g^4). \tag{1.6}$$

Here, we set $\hbar = 1$. It is known that the coefficients in this series, a_n, grow factorially [2],

$$a_n \sim \left(\frac{3}{4}\right)^n (-1)^{n+1} n!, \qquad n \gg 1. \tag{1.7}$$

Therefore, perturbation theory gives a divergent series. One important question is *whether* (and *how*) one can reconstruct the *exact* $E_0(g)$ from this asymptotic series. The only thing that we know from classical asymptotics is that one can *approximate* $E_0(g)$ by using for example an optimal truncation of the asymptotic series. It may happen however that, in order to reconstruct $E_0(g)$, one needs more information than just what is contained in the perturbative series. This is in fact generally the case, and as we will see leads to the introduction of *trans-series*.

Example 1.2. *Perturbative series from saddle point integrals.* A rich source of asymptotic series is the saddle-point approximation to integrals. For example, solutions to the Airy equation

$$y''(x) = xy(x) \tag{1.8}$$

can be constructed as integrals,

$$y(x) = \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{C}} \mathrm{e}^{xw - w^3/3} \mathrm{d}w, \tag{1.9}$$

where $\mathcal{C}$ is such that the integral converges. It is easy to show that there are two saddle-points for this integral. To see this, we write

$$x = r\mathrm{e}^{\mathrm{i}\kappa} \tag{1.10}$$

and rescale the integrand

$$w = u r^{\frac{1}{2}}. \tag{1.11}$$

We find in this way

$$y(x) = \frac{r^{\frac{1}{2}}}{2\pi \mathrm{i}} \int_{\mathcal{C}} \mathrm{e}^{r^{3/2}(\mathrm{e}^{\mathrm{i}\kappa} u - u^3/3)} \mathrm{d}u. \tag{1.12}$$

We then study the integral

$$F_\kappa(\lambda) = \int_{\mathcal{C}} \mathrm{e}^{\lambda S_\kappa(u)} \mathrm{d}u \tag{1.13}$$

where

$$S_\kappa(u) = \mathrm{e}^{\mathrm{i}\kappa} u - \frac{u^3}{3}. \tag{1.14}$$

There are *two* saddle points:

$$u_0^{\mathrm{R}} = \mathrm{e}^{\mathrm{i}\kappa/2}, \qquad u_0^{\mathrm{L}} = -\mathrm{e}^{\mathrm{i}\kappa/2} \tag{1.15}$$

with

$$\begin{aligned} S_\kappa(u_0^{\mathrm{R}}) &= \frac{2}{3}\cos\left(\frac{3}{2}\kappa\right) + \mathrm{i}\frac{2}{3}\sin\left(\frac{3}{2}\kappa\right) = \frac{2}{3}\mathrm{e}^{3\mathrm{i}\kappa/2}, \\ S_\kappa(u_0^{\mathrm{L}}) &= -\frac{2}{3}\cos\left(\frac{3}{2}\kappa\right) - \mathrm{i}\frac{2}{3}\sin\left(\frac{3}{2}\kappa\right) = -\frac{2}{3}\mathrm{e}^{3\mathrm{i}\kappa/2}. \end{aligned} \tag{1.16}$$

We then find two different expansions of the form:

$$\begin{aligned} \Phi_{\mathrm{Ai}}(x) &= \frac{1}{2x^{1/4}\sqrt{\pi}} \mathrm{e}^{-2x^{3/2}/3} \varphi_1(x^{-3/2}), \\ \Phi_{\mathrm{Bi}}(x) &= \frac{1}{2x^{1/4}\sqrt{\pi}} \mathrm{e}^{2x^{3/2}/3} \varphi_2(x^{-3/2}), \end{aligned} \tag{1.17}$$

where

$$\begin{aligned} \varphi_{1,2}(z) &= \sum_{n=0}^{\infty} \frac{1}{2\pi} \left(\mp\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!} z^n \\ &= 1 \pm \frac{5}{48} z + \frac{385}{4608} z^2 \pm \frac{85085}{663552} z^3 + \cdots \end{aligned} \tag{1.18}$$

These series provide all-orders asymptotic expansions for the Airy functions $\mathrm{Ai}(x)$, $\mathrm{Bi}(x)$. For example, when $x > 0$, we have

$$\mathrm{Ai}(x) \sim \Phi_{\mathrm{Ai}}(x). \tag{1.19}$$

This means that, for a given value of x, we can obtain an approximate value of $\mathrm{Ai}(x)$ by *truncating* appropriately (optimally) the formal power series appearing in the r.h.s. The number of terms that we can keep in such an *optimal truncation* depends on the value of x (and increases with x), but there is always an unavoidable error that is made in approximating the value of the function with the truncated asymptotics.

2 Elementary resurgent technology

2.1 Borel transform

The *Borel transform* acts on formal power series as follows

$$\begin{aligned} \mathcal{B} : \mathbb{C}[[z]] &\to \mathbb{C}[[\zeta]], \\ z^n &\mapsto \zeta^n / n! \end{aligned} \tag{2.1}$$

Therefore, if we write the starting series as

$$\varphi(z) = \sum_{n \geq 0} a_n z^n \tag{2.2}$$

its Borel transfom will be given by

$$\widehat{\varphi}(\zeta) = \sum_{n \geq 0} a_n \frac{\zeta^n}{n!}. \tag{2.3}$$

Definition 2.1. We say that a formal power series $\varphi(z)$ is *Gevrey-1* if there exist two constants $M, \rho > 0$ such that

$$|a_n| < M n! \rho^n. \tag{2.4}$$

The following result is elementary:

Lemma 2.2. *If $\varphi(z)$ is a Gevrey-1 series, its Borel transform is analytic in a neigbourhood of $\zeta = 0$.*

Example 2.3. Consider the series

$$\varphi(z) = \sum_{k=0}^{\infty} \frac{\Gamma(k+b)}{\Gamma(b)} A^{-k} z^k. \tag{2.5}$$

This series is Gevrey-1. The Borel transform is given by

$$\widehat{\varphi}(\zeta) = \sum_{k=0}^{\infty} \frac{\Gamma(k+b)}{k!\Gamma(b)} A^{-k} \zeta^k = (1 - \zeta/A)^{-b}, \tag{2.6}$$

which has a singularity at $\zeta = A$. If $b = 1$, this singularity is a pole. If $0 < b < 1$, it is a branch point. The case $b = 0$ corresponds to a logarithmic singularity. More precisely, if

$$\varphi(z) = \sum_{k=0}^{\infty} \Gamma(k) A^{-k} z^k, \tag{2.7}$$

we have

$$\widehat{\varphi}(\zeta) = \sum_{k \geq 0} \frac{1}{k} A^{-k} \zeta^k = -\log\left(1 - \frac{\zeta}{A}\right). \tag{2.8}$$

□

One of the key ideas of resurgence is that *the singularities of the Borel transform contain information about additional sectors of the theory.* To find these additional sectors, we will introduce a basic category of formal power series:

Definition 2.4. *Resurgent function.* A resurgent function is a Gevrey-1 series $\varphi(z)$ whose Borel transform has the following property: on any line issuing from the origin, there is a finite set of points (the singularities of the Borel transform) such that $\widehat{\varphi}(\zeta)$ may be continued analytically along any path that follows the line, while circumventing (from above or from below) those singular points.

We will assume that our functions are resurgent. In fact, to simplify our life, we will most of the time assume that the singularities of the Borel transform are poles or logarithmic branch cuts. In this case, the resurgent function is called *simple*. The local expansion of $\widehat{\varphi}(\zeta)$ is of the form

$$\widehat{\varphi}(\zeta_\omega + \xi) = -\frac{a}{2\pi\xi} - \frac{1}{2\pi}\log(\xi)\sum_{n\geq 0}\widehat{c}_n\xi^n + \text{regular}, \tag{2.9}$$

where the series

$$\widehat{\varphi}_\omega(\xi) = \sum_{n\geq 0}\widehat{c}_n\xi^n \tag{2.10}$$

has a finite radius of convergence. We will regard $\widehat{\varphi}_\omega(\xi)$ as the Borel transform of

$$\varphi_\omega(z) = \sum_{n\geq 0} c_n z^n, \qquad c_n = n!\widehat{c}_n. \tag{2.11}$$

A more general case involves functions with branch cuts of the form

$$(\zeta_\omega - \zeta)^{-b}, \qquad 0 < b < 1. \tag{2.12}$$

In this case, we have the local expansion

$$\widehat{\varphi}(\zeta_\omega + \xi) = (-\xi)^{-b}\sum_{n\geq 0}\widehat{c}_n\xi^n + \text{regular}, \tag{2.13}$$

and we will regard $\widehat{\varphi}_\omega(\xi)$ as the Borel transform of

$$\varphi_\omega(z) = \sum_{n\geq 0} c_n z^n, \qquad c_n = \Gamma(n+1-b)\widehat{c}_n. \tag{2.14}$$

The reason for transforming these convergent series into formal power series will be understood shortly.

The key result so far is that, given a formal power series $\varphi(z)$, the expansion of its Borel transform around its singularities generates additional formal power series:

$$\varphi(z) \to \{\varphi_\omega(z)\}_{\omega\in\mathcal{S}}, \tag{2.15}$$

where $\mathcal{S}$ denotes the set of singular points.

Example 2.5. Let us consider the formal power series $\varphi_1(z)$ appearing in (1.18). In this case, the Borel transform can be computed explicitly,

$$\widehat{\varphi}(\zeta) = {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; -\frac{3\zeta}{4}\right). \tag{2.16}$$

This has a singularity at

$$\zeta_\omega = -\frac{4}{3}. \tag{2.17}$$

It is easy to see from the general theory of hypergeometric functions that this is a logarithmic singularity, and one finds

$$\widehat{\varphi}_1(\zeta) = -\frac{1}{2\pi}\log\left(\zeta + \frac{4}{3}\right){}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; \frac{3\zeta}{4}\right) + \text{regular} \tag{2.18}$$

therefore

$$\varphi_{-4/3}(z) = \varphi_2(z) = \sum_{n\geq 0} \frac{1}{2\pi} \left(\frac{3}{4}\right)^n \frac{\Gamma(n+5/6)\Gamma(n+1/6)}{n!} z^n. \tag{2.19}$$

This is the other formal power series that appears when solving the Airy equation.

□

The above procedure can be repeated with the new formal power series $\varphi_\omega(z)$, to generate yet more power series. This might lead to a finite set of functions (this is the case of the Airy functions). Generically, however, one finds an infinite set of formal power series. We will label such series by the index ω, and we will write then as $\varphi_\omega(z)$. Let us assume that they are all simple resurgent functions. Then, we have the general relation

$$\widehat{\varphi}_\omega(\zeta_{\omega'} + \xi) = -\mathsf{S}_{\omega\omega'} \frac{\log(\xi)}{2\pi} \widehat{\varphi}_{\omega'}(\xi) + \text{regular}. \tag{2.20}$$

The constants $\mathsf{S}_{\omega\omega'}$ are called *Stokes constants*.

Example 2.6. In the Airy example, the Stokes constants are simply

$$\mathsf{S}_{12} = \mathsf{S}_{21} = 1. \tag{2.21}$$

The above structure can be extended to a more general type of singularity, with e.g. a branch cut structure.

We can now understand the origin of the name "resurgence". We have seen that the singularities of the Borel transform lead to new power series. It turns out that these new series "resurge" in the original series through the behavior of the coefficients a_k when k is large. In terms of the Borel transform (which is analytic at the origin), this is essentially an old theorem of Darboux, which relates the large order behavior of the coefficients of an analytic function at the origin, to the behavior near the closest singularity (see e.g. [3]).

Let us first state the result. Let $\varphi(z)$ be a simple resurgent function as in (2.2). Let A be the singularity of the Borel transform which is closest to the origin in the complex plane (we will assume for simplicity that there is only one, although the generalization is straightforward). Let the behavior near this singularity be as in (2.9), with $\zeta_\omega = A$. We will assume for simplicity that the residue at $\xi = 0$ vanishes, i.e. $a = 0$. Then, the coefficients a_k have the following asymptotic behavior,

$$a_k \sim \frac{1}{2\pi} \sum_{n\geq 0} A^{-k+n} c_n \Gamma(k-n), \qquad k \gg 1. \tag{2.22}$$

To understand this formula better, it is convenient to write explicitly the very first terms:

$$a_k \sim \frac{1}{2\pi} A^{-k} \Gamma(k) \left\{ c_0 + \frac{c_1 A}{k-1} + \frac{c_2 A^2}{(k-1)(k-2)} + \cdots \right\}, \qquad k \gg 1. \tag{2.23}$$

The first factor in the r.h.s. gives the leading factorial asymptotics, while the second factor gives a series of corrections in $1/k$ to the leading factorial behavior. These corrections involve the coefficients c_n of the power series obtained in (2.11). One can use this asymptotic formula in two ways: as a procedure to extract the numbers A, c_n from the knowledge of the series a_k, of conversely, as a way to obtain the large order asymptotics of these coefficients once A, c_n are known.

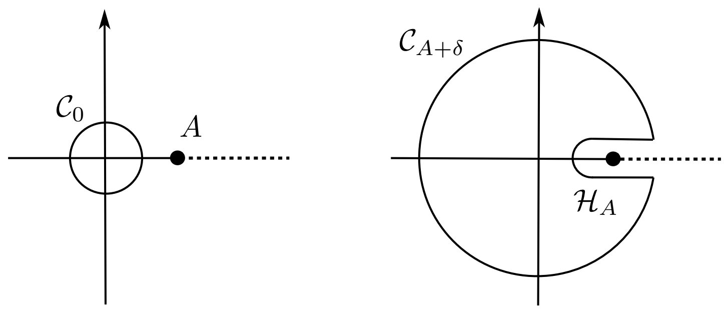

Figure 1. Contour deformation in (2.25).

Let us sketch a proof of this result. The coefficients of the Borel transform are given by the Cauchy formula

$$\frac{a_k}{k!} = \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{C}_0} \frac{\widehat{\varphi}(\zeta)}{\zeta^{k+1}} \mathrm{d}\zeta, \tag{2.24}$$

where $\mathcal{C}_0$ is a small circle around $\zeta = 0$. Let us choose a $\delta > 0$. We now enlarge the contour $\mathcal{C}_0$ to a contour $\mathcal{C}_{A+\delta} \cup \mathcal{H}_A$, where $\mathcal{C}_{A+\delta}$ is a circle of radius $A+\delta$, minus an arc, and $\mathcal{H}_A$ is a Hankel contour centered around A, see Fig. 1. By deforming the contour we find

$$\frac{a_k}{k!} = \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{C}_{A+\delta}} \frac{\widehat{\varphi}(\zeta)}{\zeta^{k+1}} \mathrm{d}\zeta + \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{H}_A} \frac{\widehat{\varphi}(\zeta)}{\zeta^{k+1}} \mathrm{d}\zeta. \tag{2.25}$$

The first integral can be estimated to be of order $\mathcal{O}\left((A+\delta)^{-k}\right)$. Since, as we will now show, the leading large k asymptotics goes like A^{-k}, and $A+\delta > A$, this is a subleading, exponentially small correction as k grows large, and it does not contribute to the leading $1/k$ asymptotics. The integral around the contour $\mathcal{H}_A$ can be evaluated by using (2.9). and we obtain

$$\frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{H}_A} \frac{\widehat{\varphi}(\zeta)}{\zeta^{k+1}} \mathrm{d}\zeta = \frac{1}{2\pi} \sum_{n \geq 0} \hat{c}_n \int_0^\delta \frac{\xi^n}{(A+\xi)^{k+1}} \mathrm{d}\xi, \tag{2.26}$$

where we have set $\zeta = A + \xi$. An easy estimate shows that, at fixed n,

$$\begin{aligned} \int_0^\delta \frac{\xi^n}{(A+\xi)^{k+1}} \mathrm{d}\xi &= \int_0^\infty \frac{\xi^n}{(A+\xi)^{k+1}} \mathrm{d}\xi + \mathcal{O}\left((A+\delta)^{-k}\right) \\ &= A^{n-k} \frac{\Gamma(k-n)\Gamma(n+1)}{\Gamma(k+1)} + \mathcal{O}\left((A+\delta)^{-k}\right). \end{aligned} \tag{2.27}$$

We conclude that the asymptotics of a_k is given by (2.22).

Remark 2.7. When the Borel transform has a branch cut as in (2.13), a similar argument leads to

$$a_k \sim \frac{1}{2\pi} \sum_{n \geq 0} A^{-k-b+n} c_n \Gamma(k+b-n). \tag{2.28}$$

□

Remark 2.8. The contribution of the residue in (2.9) to the asymptotics is simply

$$\frac{a}{2\pi} A^{-k-1}\Gamma(k+1). \tag{2.29}$$

□

The asymptotics (2.22) implies that the coefficients c_n determining the series $\varphi_A(z)$ are encoded in the large order behavior of a_k. This has multiple applications. In particular, when there is no easy method to determine $\varphi_A(z)$, one can extract it from the knowledge of the large order behavior. This has become one of the most powerful heuristic methods in resurgence.

Example 2.9. In the case of the series $\varphi_1(z)$ in (1.18), the above result leads to the asymptotics

$$\begin{aligned} a_k &\sim \frac{1}{2\pi}\left(-\frac{4}{3}\right)^{-k}(k-1)!\sum_{n\geq 0} c_n \left(-\frac{4}{3}\right)^n \frac{\Gamma(k-n)}{\Gamma(k)} \\ &\sim \frac{1}{2\pi}\left(-\frac{4}{3}\right)^{-k}(k-1)!\left(1-\frac{4}{3}\cdot\frac{5}{48}\frac{1}{k-1}+\frac{16}{9}\cdot\frac{385}{4608}\frac{1}{(k-1)(k-2)}+\cdots\right). \end{aligned} \tag{2.30}$$

Since we have a closed form for a_k, we can calculate the asymptotics directly:

$$a_k = \frac{1}{2\pi}\left(-\frac{3}{4}\right)^k \frac{\Gamma(k+\frac{5}{6})\Gamma(k+\frac{1}{6})}{k!}, \tag{2.31}$$

and one indeed verifies that

$$\frac{\Gamma(k+\frac{5}{6})\Gamma(k+\frac{1}{6})}{k!(k-1)!} \sim 1-\frac{4}{3}\cdot\frac{5}{48}\frac{1}{k-1}+\frac{16}{9}\cdot\frac{385}{4608}\frac{1}{(k-1)(k-2)}+\cdots, \qquad k \gg 1. \tag{2.32}$$

□

2.2 Borel resummation and Stokes automorphism

Definition 2.10. Let ζ_ω be a singularity of $\widehat{\varphi}(\zeta)$. A ray in the Borel plane which starts at the origin and passes through ζ_ω is called a *Stokes ray*. It is of the form $\mathrm{e}^{\mathrm{i}\theta}\mathbb{R}_+$, where $\theta = \arg(\zeta_\omega)$.

Note that a Stokes ray might pass through many singularities.

Definition 2.11. Let $\varphi(z)$ a Gevrey-1 formal power series series, $z \in \mathbb{C}$, and $\theta = \arg z$. If $\widehat{\varphi}(\zeta)$ analytically continues to an L^1-analytic function along the ray $\mathcal{C}^\theta := \mathrm{e}^{\mathrm{i}\theta}\mathbb{R}_+$ we define its Laplace transform by

$$s(\varphi)(z) = \int_0^\infty \widehat{\varphi}(z\zeta)\mathrm{e}^{-\zeta}\mathrm{d}\zeta = \frac{1}{z}\int_{\mathcal{C}^\theta}\widehat{\varphi}(\zeta)\mathrm{e}^{-\zeta/z}\mathrm{d}\zeta. \tag{2.33}$$

The function $s(\varphi)(z)$ is often called the *Borel resummation* of the formal power series φ.

Let us first note that, if $s(\varphi)(z)$ exists, its asymptotic behavior for small z can be obtained by expanding the integrand and integrating term by term:

$$s(\varphi)(z) \sim \sum_{n\geq 0} a_n z^n. \tag{2.34}$$

This is the formal power series that we started with. Therefore, if we are lucky, Borel resummation produces an actual function which reproduces the original series. It is then a way to “make sense” of our original formal power series.

If we vary $\theta = \arg z$ and we do not encounter singularities of $\widehat{\varphi}$, the function $s_\theta(\varphi)(z)$ is locally analytic. However, when $\theta = \arg(z)$ crosses a Stokes ray, the Borel resummation has a discontinuity. To define this discontinuity more precisely, we introduce *lateral Borel resummations*.

Definition 2.12. Let $\mathcal{C}_\pm^\theta$ be contours starting at the origin and going slightly above (respectively, below) the Stokes ray. Then,

$$s_\pm(\varphi)(z) = \frac{1}{z}\int_{\mathcal{C}_\pm^\theta} \widehat{\varphi}(\zeta)\mathrm{e}^{-\zeta/z}\mathrm{d}\zeta. \tag{2.35}$$

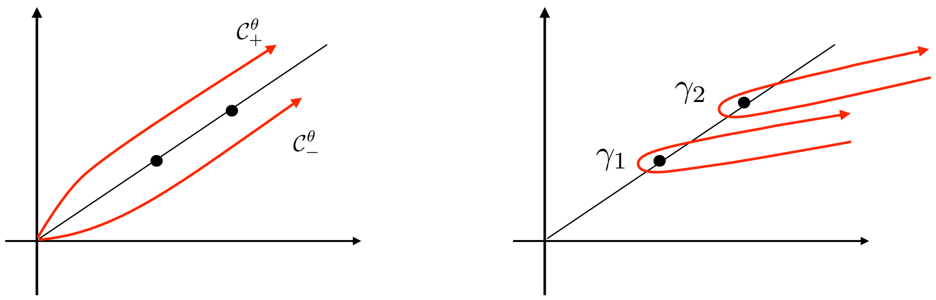

Figure 2. Contour deformation in the calculation of the discontinuity.

The discontinuity is then simply given by

$$s_+(\varphi)(z) - s_-(\varphi)(z). \tag{2.36}$$

Note that, since $s_\pm(\varphi)(z)$ have the same asymptotics for small z, given in (2.34), the discontinuity must be invisible as an asymptotic expansion. As we will now see, this difference is *exponentially small* and closely related to the local structure of the Borel transform. Indeed, let us assume that $\varphi(z)$ is a simple resurgent function, and that there is a sequence of isolated logarithmic singularities ζ_ω in the Stokes ray, where $\omega \in \Omega$. The difference between the two contours $\mathcal{C}_+^\theta - \mathcal{C}_-^\theta$ can be deformed into a sum of Hankel-like contours γ_ω around the logarithmic branch cuts. We then have, for each ω,

$$\oint_{\gamma_\omega} \widehat{\varphi}(\zeta)\mathrm{e}^{-\zeta/z}\mathrm{d}\zeta = -\frac{\mathrm{e}^{-\zeta_\omega/z}}{2\pi}\int_{\mathcal{C}_-^\theta} \left(\log(\xi) - \log(\xi) - 2\pi\mathrm{i}\right)\widehat{\varphi}_\omega(\xi)\mathrm{e}^{-\xi/z}\mathrm{d}\xi, \tag{2.37}$$

where in the first line we have written $\zeta = \zeta_\omega + \xi$. Therefore

$$\begin{aligned} s_+(\varphi)(z) - s_-(\varphi)(z) &= \frac{1}{z}\sum_{\omega\in\Omega}\oint_{\gamma_\omega} \widehat{\varphi}(\zeta)\mathrm{e}^{-\zeta/z}\mathrm{d}\zeta = \mathrm{i}\sum_{\omega\in\Omega}\frac{\mathrm{e}^{-\zeta_\omega/z}}{z}\int_{\mathcal{C}_-^\theta}\widehat{\varphi}_\omega(\xi)\mathrm{e}^{-\xi/z}\mathrm{d}\xi \\ &= \mathrm{i}\sum_{\omega\in\Omega}\mathrm{e}^{-\zeta_\omega/z}s_-(\varphi_\omega)(z). \end{aligned} \tag{2.38}$$

More generally, if there are many different power series, as in (2.20), we have the more general relationship

$$s_+(\varphi_\omega)(z) - s_-(\varphi_\omega)(z) = \mathrm{i} \sum_{\omega'} \mathsf{S}_{\omega\omega'} \mathrm{e}^{-\zeta_{\omega'}/z} s_-(\varphi_{\omega'})(z), \tag{2.39}$$

where the sum over ω' runs over the singularities in the Stokes ray.

2.3 Trans-series

One of the most important implications of resurgence is that, in order to reconstruct actual functions through Borel resummation, we need, in addition to the "obvious" perturbative series, all additional series $\varphi_\omega(z)$ appearing in the resurgent structure. This suggest the following definition

Definition 2.13. Let $\varphi_\omega(z)$ be resurgent functions. A *trans-series* is a (possibly infinite) formal linear combination of formal power series

$$\Phi(z; \boldsymbol{C}) = \sum_{\omega} C_\omega \mathrm{e}^{-\zeta_\omega/z} \varphi_\omega(z), \tag{2.40}$$

where $\boldsymbol{C} = (C_{\omega_1}, \cdots)$ is a vector of complex numbers.

Example 2.14. The need for trans-series can be seen already in the classical asymptotic theory of functions defined by ODEs. Let us suppose that we want to reconstruct the Airy function $\mathrm{Ai}(x)$ out of Borel resummations of the formal power series $\varphi_{1,2}(z)$. It turns out that the answer depends on the argument of x.

1. If $|\arg(x)| < 2\pi/3$, the Airy function is obtained as

$$\mathrm{Ai}(x) = \frac{1}{2x^{1/4}\sqrt{\pi}} \mathrm{e}^{-2x^{3/2}/3} s(\varphi_1)(z), \qquad z = x^{-3/2}. \tag{2.41}$$

2. If $|\arg(x) - \pi| < \frac{\pi}{3}$ we have instead

$$\mathrm{Ai}(x) = \frac{1}{2x^{1/4}\sqrt{\pi}} \left\{ \mathrm{e}^{-2x^{3/2}/3} s(\varphi_1)(z) + \mathrm{i}\mathrm{e}^{2x^{3/2}/3} s(\varphi_2)(z) \right\}, \qquad z = x^{-3/2}. \tag{2.42}$$

   For example, when $x < 0$, we have

$$\mathrm{Ai}(-x) = \frac{1}{2x^{1/4}\sqrt{\pi}} \left\{ \mathrm{e}^{-\pi\mathrm{i}/4 + 2\mathrm{i}x^{3/2}/3} s(\varphi_1)(\mathrm{i}x^{-3/2}) + \mathrm{e}^{\pi\mathrm{i}/4 - 2\mathrm{i}x^{3/2}/3} s(\varphi_2)(\mathrm{i}x^{-3/2}) \right\}. \tag{2.43}$$

   This leads to the well-known oscillatory behavior of the Airy function along the negative real axis,

$$\mathrm{Ai}(-x) \sim \frac{x^{-1/4}}{\sqrt{\pi}} \cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right), \qquad x \to \infty. \tag{2.44}$$

□

Example 2.15. *Non-linear ODEs.* In the Airy case, there are only two formal power series in the full resurgent structure. However, in the case of *non-linear* ODEs one needs an infinite number of formal power series to reconstruct solutions to the ODE. An example of this situation is the Painlevé II equation,

$$u''(\kappa) - 2u^3(\kappa) + 2\kappa u(\kappa) = 0, \tag{2.45}$$

which plays an important rôle in many areas of mathematical physics. There is a formal solution to PII which goes like $u(\kappa) \sim \sqrt{\kappa}$ as $\kappa \to \infty$:

$$u^{(0)}(\kappa) = \sqrt{\kappa} - \frac{1}{16\,\kappa^{\frac{5}{2}}} - \frac{73}{512\,\kappa^{\frac{11}{2}}} - \frac{10657}{8192\,\kappa^{\frac{17}{2}}} - \frac{13912277}{542888\,\kappa^{\frac{23}{2}}} + \cdots, \qquad \kappa \to \infty. \tag{2.46}$$

In the case of non-linear ODEs, the trans-series can be systematically constructed as follows. Let us consider a formal solution with the structure,

$$u(\kappa; C) = \sum_{\ell=0}^{\infty} C^{\ell} u^{(\ell)}(\kappa), \tag{2.47}$$

where C is a constant, and $u^{(0)}$ is the formal solution obtained above. By plugging this *ansatz* in the original equation, we find that each $u^{(\ell)}$ satisfies a linear ODE. For example, one finds immediately that

$$(u^{(1)})'' + 2\kappa u^{(1)} - 6(u^{(0)})^2 u^{(1)} = 0. \tag{2.48}$$

Work of Jean Écalle, Ovidiu Costin and others (see [4] for a wonderful introduction, and [5] for a in-depth treatment) shows that the $u^{(\ell)}(\kappa)$ obtained in this way are trans-series, in the sense that they can be obtained through the Borel transform of the formal series $u^{(0)}(\kappa)$. They lead to exponentially small corrections to the leading asymptotic behavior given by $u^{(0)}$, and they have the form

$$u^{(\ell)}(\kappa) = \kappa^{-\frac{3\ell}{4}} \mathrm{e}^{-\ell A \kappa^{3/2}} \epsilon^{(\ell)}(\kappa), \quad \kappa \to \infty, \tag{2.49}$$

where

$$A = \frac{4}{3}, \qquad \epsilon^{(\ell)}(\kappa) = \sum_{n=0}^{\infty} u_{\ell,n} \kappa^{-3n/2} \tag{2.50}$$

It can be shown that general solutions to Painlevé II with the asymptotics (2.46) can be written as an infinite sum of lateral Borel resummations

$$u(\kappa) = \sum_{\ell=0}^{\infty} C^{\ell} s_{+}(u^{(\ell)})(\kappa) \tag{2.51}$$

for an appropriate value of C. □

Let us make some conceptual comments on trans-series:

1. Trans-series involve at least two "small parameters": the first small parameter is the one appearing in the original, "perturbative" series. There is also an *exponentially small parameter* $\mathrm{e}^{-\zeta_\omega/z}$. In this sense, trans-series go beyond classical asymptotics by including exponentially small corrections

2. All series in the first small parameter are factorially divergent.

3. The different series appearing in the trans-series are not independent. For example, the large order behavior of the terms in the leading perturbative series are controlled by the first series in trans-series (the one corresponding to the smallest singularity, in absolute value).

We will now state a principle of *semiclassical decoding*.

Definition 2.16. (Semiclassical decoding). Let $f(z)$ be a function with the asymptotic expansion

$$f(z) \sim \varphi(z) = \sum_{n \geq 0} a_n z^n. \tag{2.52}$$

We say that $f(z)$ admits a *semiclassical decoding* if $\varphi(z)$ can be promoted to a trans-series $\Phi(z; \boldsymbol{C})$, which is lateral Borel summable, and such that

$$f(z) = s_{\pm}(\Phi)(z; \boldsymbol{C}_{\pm}) \tag{2.53}$$

for some vectors of complex constants $\boldsymbol{C}_{\pm}$.

When semiclassical decoding holds, one recovers the exact information by just considering Borel-resummed trans-series.

The simplest situation corresponds to the case in which $C = 0$, there are no singularities along the positive real axis, and the Borel resummation of the perturbative series reproduces the exact result. This is famously the case for the perturbative series (1.6) of the quartic oscillator.

An important question in quantum theory is whether well-defined functions in QM and QFT admit a semiclassical decoding. What we could call the *(weak) program of resurgence* is the conjectural statement that *every observable in QM/QFT can be written as the lateral Borel resummation of an appropriate trans-series.*

The program of semiclassical decoding was very active in QFT after the discovery of instantons, but then it suffered an important drawback in the late 70's when it was shown that asymptotically free theories in *infinite volume* do not admit a simple semiclassical decoding, and that trans-series made out of instantons are not applicable in that case. However, there are recent results indicating that more general trans-series, not based on classical solutions of the equations of motion, can be used to reconstruct the exact answer.

3 Resurgence in quantum mechanics

An important application of resurgence is to the WKB method in one-dimensional quantum mechanics, which is then upgraded to the so-called "exact WKB method". One consequence of this application is the result that energy levels and resonances in one-dimensional quantum mechanics can be "semiclassically decoded" in terms of the perturbative WKB series and exponentially small corrections to it.

3.1 The WKB method and quantization conditions

The WKB method is a systematic approximation scheme to solve the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x). \tag{3.1}$$

The result of this approximation is an expression for the wavefunction as a formal power series in $\hbar$. In particular, as we will see, in the analysis of bound state problems, the WKB method gives estimates of the energy spectrum when $\hbar$ is small as compared to the classical action of the problem. This happens when the quantum number labelling the bound state is very large. The resulting approximation often provides an accurate description of the energy spectrum, and it has the advantage that it can be formulated in terms of classical quantities.

In the WKB method, we treat $\hbar$ as a small parameter and we perform a systematic expansion of all quantities in a power series in $\hbar$, around $\hbar = 0$. However, if we write (3.1) as

$$\hbar^2 \psi''(x) + p^2(x)\psi(x) = 0, \qquad p(x) = \sqrt{2m(E - V(x))}, \tag{3.2}$$

it is clear that we can not send $\hbar \to 0$ directly, since in this limit the equation becomes algebraic. Let us however write the wavefunction as

$$\psi(x) = \exp\left[\frac{\mathrm{i}}{\hbar}\int^x Y(x')\mathrm{d}x'\right]. \tag{3.3}$$

When we do this, we transform the Schrödinger equation into a Riccati equation for $Y(x)$,

$$Y^2(x) - \mathrm{i}\hbar\frac{\mathrm{d}Y(x)}{\mathrm{d}x} = p^2(x), \tag{3.4}$$

which can be solved in power series in $\hbar$:

$$Y(x) = \sum_{k=0}^{\infty} Y_k(x)\hbar^k. \tag{3.5}$$

We will regard this as a *formal* power series, i.e. we will not address issues of convergence for the moment being. The functions $Y_k(x)$ can be computed recursively as

$$\begin{aligned} Y_0(x) &= \pm p(x), \\ Y_1(x) &= \frac{\mathrm{i}}{2}\frac{Y_0'(x)}{Y_0(x)}, \\ Y_{n+1}(x) &= \frac{1}{2Y_0(x)}\left(\mathrm{i}\frac{\mathrm{d}Y_n(x)}{\mathrm{d}x} - \sum_{k=1}^{n} Y_k(x)Y_{n+1-k}(x)\right), \qquad n \geq 1. \end{aligned} \tag{3.6}$$

The two choices of sign for $Y_0(x)$ will give the two independent solutions of the Schrödinger equation. Let us split the formal series $Y(x)$ into two series of even and odd powers of $\hbar$,

$$P(x) = \sum_{k=0}^{\infty} Y_{2k}(x)\hbar^{2k}, \qquad Y_{\rm odd}(x) = \sum_{k=0}^{\infty} Y_{2k+1}(x)\hbar^{2k+1}, \tag{3.7}$$

so that

$$Y(x) = P(x) + Y_{\rm odd}(x). \tag{3.8}$$

The Riccati equation (3.4) splits into two different equations:

$$\begin{aligned} &\text{even:} \quad Y_{\rm odd}^2(x) + P^2(x) - \mathrm{i}\hbar Y_{\rm odd}'(x) = p^2(x), \\ &\text{odd:} \quad 2Y_{\rm odd}(x)P(x) - \mathrm{i}\hbar P'(x) = 0. \end{aligned} \tag{3.9}$$

The second equation can be solved in closed form:

$$Y_{\rm odd}(x) = \frac{\mathrm{i}\hbar}{2}\frac{P'(x)}{P(x)} = \frac{\mathrm{i}\hbar}{2}\frac{\mathrm{d}}{\mathrm{d}x}\log P(x). \tag{3.10}$$

Therefore,

$$\frac{\mathrm{i}}{\hbar}\int^x Y(x')\mathrm{d}x' = -\frac{1}{2}\log P(x) + \frac{\mathrm{i}}{\hbar}\int^x P(x')\mathrm{d}x', \tag{3.11}$$

and the wavefunction reads

$$\psi(x) = \frac{1}{\sqrt{P(x)}} \exp\left(\frac{\mathrm{i}}{\hbar}\int^x P(x')\mathrm{d}x'\right). \tag{3.12}$$

We can think of $P(x)\mathrm{d}x$ as a "quantum differential" which promotes the Liouville differential form $p(x)\mathrm{d}x$ to a formal power series in $\hbar$.

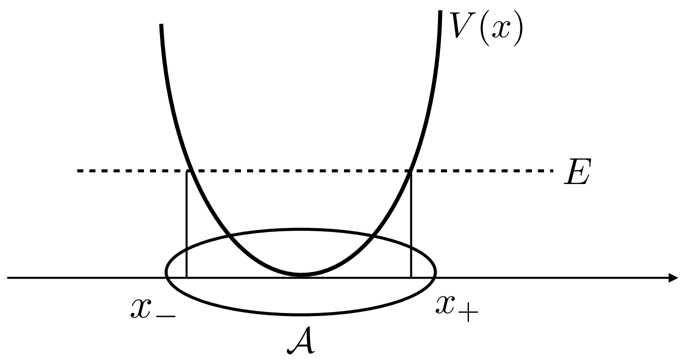

Figure 3. A contour $\mathcal{A}$ in the complex plane, encircling the turning points.

In confining potentials, leading to a discrete spectrum, approximate quantization conditions can be obtained by considering period integrals of the quantum differential. Let us suppose that we are in a situation with two turning points $x_\pm$, defined by the condition $V(x) = E$. Let us consider a cycle $\mathcal{A}$ in the complex x-plane surrounding the interval of classical motion, as shown in Fig. 3. The Bohr–Sommerfeld (BS) quantization condition says that the approximate spectrum can be obtained by solving the equation

$$\mathrm{vol}_0(E) = 2\pi\hbar\left(n + \frac{1}{2}\right), \tag{3.13}$$

where

$$\mathrm{vol}_0(E) = \oint_{\mathcal{A}} p(x)\mathrm{d}x, \tag{3.14}$$

is the classical volume in phase space. It was already clear to the founders of quantum mechanics that this quantization condition is only approximate, and it is a non-trivial problem how to improve it to obtain a more accurate treatment of the spectrum. The all-orders WKB method shows that the r.h.s. of (3.13) can be promoted to a formal power series in $\hbar^2$:

$$\mathrm{vol}(E;\hbar) = \oint_{\mathcal{A}} P(x)\mathrm{d}x = \sum_{n=0}^{\infty} \mathrm{vol}_n(E)\hbar^{2n}, \tag{3.15}$$

It turns out however that, generically, this series diverges *doubly-factorially* for a fixed value of E,

$$\mathrm{vol}_n(E) \sim (2n)!, \qquad n \gg 1. \tag{3.16}$$

One can write formally an all-orders quantization condition of the form

$$\mathrm{vol}(E;\hbar) = 2\pi\hbar\left(n+\frac{1}{2}\right) \tag{3.17}$$

which was written down by Dunham in 1929. However, since the series in the l.h.s. does not define a function, this quantization condition is not well-defined. Understanding this in detail is crucial to improve systematically the semi-classical, Bohr–Sommerfeld approximation. This procedure was pioneered by André Voros, and eventually led to "exact" quantization conditions for many QM problems.

3.2 The purely quartic oscillator

We conclude that the WKB method leads naturally to a factorially divergent series which is however physically relevant to understand the corrections to the Bohr–Sommerfeld quantization condition. Perhaps the simplest case of such a situation is the purely quartic oscillator

$$\mathsf{H} = \frac{\mathsf{p}^2}{2m} + g\mathsf{q}^4. \tag{3.18}$$

Since the potential is confining, it has an infinite tower of bound states with energies $E_k(\hbar, m, g)$. We can introduce the dimensionless parameter

$$\xi = \frac{\sqrt{2m}}{\hbar} g^{-1/4} E^{3/4} \tag{3.19}$$

Note that *there is no adjustable parameter in this problem*: the energies of the bound states have a trivial dependence on $\hbar$, g and m, given by (3.19), where ξ will take purely numerical values. The only non-trivial dependence is on k, the integer labelling the energy level, and the Bohr–Sommerfeld approximation leads to an expression for E_k which is expected to be valid for large k. Let us first work out the BS quantization condition. We have

$$\mathrm{vol}_0(\xi) = 2\hbar\xi \int_{-1}^{1} \sqrt{1-q^4}\,\mathrm{d}q = \hbar\xi b_0, \tag{3.20}$$

where

$$b_0 = \frac{\Gamma(1/4)^2}{3}\sqrt{\frac{2}{\pi}}. \tag{3.21}$$

Exercise 3.1. One can also compute the next-to leading order correction by calculating Y_2 explicitly in (3.5). One finds,

$$\mathrm{vol}_1(\xi) = \hbar\xi^{-1} b_1, \qquad b_1 = -\frac{2\sqrt{\pi}\,\Gamma\left(\frac{7}{4}\right)}{3\Gamma\left(\frac{1}{4}\right)}. \tag{3.22}$$

(See [6], Example 2.5.4 for a general, explicit expression for Y_2) □

Exercise 3.2. One can compare the numerical results for the spectrum of the pure quartic oscillator with the results obtained from the BS quantization condition (3.13), and with the result obtained from (3.17) after including the first correction (3.22). We will denote these three quantities by E_n, $E_n^{(0)}$ and $E_n^{(1)}$, respectively. The results are shown in Table 1. We observe that the BS approximation gives a reasonable estimate already for the first excited state. For the state with $n = 4$, the error is only of 0.2%. The next-to-leading approximation provides a remarkable improvement, and the error for the level $n = 4$ is reduced to one part in 10^5. □

Level	E_n	$E_n^{(0)}$	$E_n^{(1)}$
0	1.060362091	0.8671453264	0.980766290
1	3.799673029	3.751919923	3.810329951
4	16.261826019	16.233614692	16.261936744

Table 1. Comparison between the numerical values of the energies and the WKB approximations $E_n^{(0)}$, $E_n^{(1)}$ for the pure quartic oscillator. We have set $2m = g = \hbar = 1$.

The calculations in Exercise 3.2 show that the higher order corrections to the BS quantization condition give a good asymptotic expansion for the energy levels. However, we would like to understand how to incorporate these corrections in a systematic way, in order to obtain the best possible approximation that the WKB method provides for the exact energies.

More formally, we would like to understand better the structure of the series (3.15), and in particular its resurgent structure. Where are its Borel singularities? What are the additional power series appearing at the singularities? These problems were first explored in a remarkable paper by Balian, Parisi and Voros [7] and eventually solved by Voros in [8]. Voros' results were later on generalized to arbitrary potentials in work by Écalle, Delabaere, Pham and others. It is the basis of the so-called "exact WKB method."

We will call the formal power series appearing in the r.h.s. of (3.15) a *quantum period*, more precisely the perturbative quantum period, and we will denote it by $\Pi_{\rm p}(\hbar)$. It has the structure of a formal power series in ξ^{-1}:

$$\frac{1}{\hbar}\Pi_{\rm p}(\hbar) = \sum_{n\geq 0} b_n \xi^{1-2n}. \tag{3.23}$$

Note that, if we choose our units such that $2m = g = E = 1$, $\xi = 1/\hbar$, and we can also take $\hbar$ as the natural expansion variable for the quantum period. As shown in [7], the coefficients b_n can be computed recursively to high order. A numerical determination of the Borel singularities of the Borel transform

$$\widehat{\Pi}_{\rm p}(\zeta) = \sum_{n\geq 0} \frac{b_n}{(2n)!}\zeta^{2n} \tag{3.24}$$

is easily found and shown in Fig. 4. Here we have normalized our variable to

$$z = \frac{\zeta}{b_0}. \tag{3.25}$$

In the z-variable, we find singularities at the four points

$$\frac{\pm 1 \pm {\rm i}}{2}, \tag{3.26}$$

together with subleading singularities at $z = \pm 1$. We also note that the singularity at $z = 1$ *obstructs* Borel summability, therefore we conclude that the all-orders perturbative WKB series appearing in (3.15) is not Borel summable.

What are the series appearing at these singularities? To understand this, we have to introduce a "non-perturbative" quantum period

$$\frac{1}{\hbar}\Pi_{\rm np}(\hbar) = \sum_{n\geq 0} b_n (-1)^n \xi^{1-2n}. \tag{3.27}$$

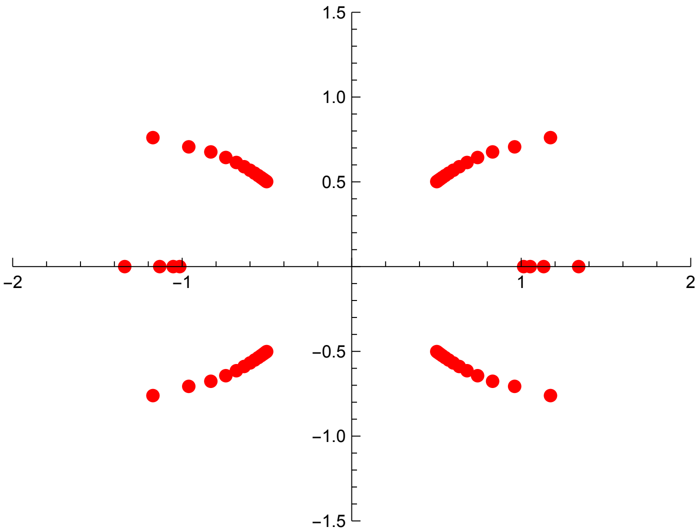

Figure 4. The very first Borel singularities of the perturbative period in the pure quartic oscillator.

Geometrically, this period is given by a period integral around the cycle $\mathcal{B}$ which encircles the complex turning points $\pm \mathrm{i}E^{1/4}$:

$$\Pi_{\rm np}(\hbar) = -\mathrm{i} \oint_{\mathcal{B}} P(x) \mathrm{d}x, \tag{3.28}$$

It can be verified that the singularities at $z = \pm 1$ lead to the formal power series

$$\exp\left(-\frac{1}{\hbar}\Pi_{\rm np}(\hbar)\right), \tag{3.29}$$

which includes the overall exponential factor

$$\exp\left(-\frac{1}{\hbar}\Pi^{(0)}_{\rm np}\right), \qquad \Pi^{(0)}_{\rm np} = b_0. \tag{3.30}$$

The four singularities in the four quadrants lead to the four series

$$\exp\left(\mp\frac{1}{2\hbar}\Pi_{\rm np}(\hbar) \mp \frac{\mathrm{i}}{2\hbar}\Pi_{\rm p}(\hbar)\right), \tag{3.31}$$

which includes as well the exponential prefactors. Note that trans-series associated to these singularities are exponentials of quantum periods. This is a general feature of the resurgent structure of the all-orders WKB method.

The above assertions can be derived with some additional technology. The key ingredient is an exact form of the connection formulae of the WKB method, also called Voros–Silverstone connection formula (see e.g. [6] for a detailed exposition). One can also test them with the tools of resurgence introduced in the previous chapter. For example, the asymptotic behavior of the coefficients b_n in the all-orders WKB series should be determined by the four series (3.31). This was already noted in [7], and further verified in [8], and it is an instructive exercise.

Exercise 3.3. *Large order behavior in the pure quartic oscillator.* First of all, we note that if a factorially divergent power series has only even terms, i.e. it is of the form

$$\varphi(z) = \sum_{m\geq 0} a_m z^m, \qquad a_{2n+1} = 0, \tag{3.32}$$

its Borel transform will have Borel singularities at pairs A and $-A$. They contribute to the larger order behavior

$$\frac{\mathsf{S}^{\pm}}{2\pi} \sum_{k\geq 0} c_k^{\pm} (\pm A)^{-m-b+k} \Gamma(m+b-k), \tag{3.33}$$

and we have the symmetry

$$c_k^- = (-1)^k c_k^+, \qquad \mathsf{S}^- = (-1)^b \mathsf{S}^+. \tag{3.34}$$

When we add both contributions we find automatically $a_{2n+1} = 0$ for odd m, and for even $m = 2n$ we find

$$a_{2n} \sim \frac{\mathsf{S}}{\pi} \sum_{k\geq 0} c_k A^{-2n-b+k} \Gamma(2n+b-k), \tag{3.35}$$

where $\mathsf{S} = \mathsf{S}^+$, $c_k = c_k^+$. In the case of the quartic oscillator, we also have to add the contributions of Borel singularities with complex conjugate actions A, $A^\star$, and since we have already considered the contribution of A and $-A$ we can assume that $\mathrm{Re}(A) > 0$. As in (3.25), it is convenient to redefine $\hbar \to b_0 \hbar$, so that the actions are

$$A = \frac{1}{\sqrt{2}} \mathrm{e}^{\pi \mathrm{i}/4}, \qquad A^\star = \frac{1}{\sqrt{2}} \mathrm{e}^{-\pi \mathrm{i}/4}. \tag{3.36}$$

The corresponding trans-series are given by (3.31). Explicitly, we have

$$\varphi_A(z) = -\hbar \, \mathrm{e}^{-\frac{A}{\hbar}} \exp\left(-\sum_{j=1}^{\infty} \left(\frac{(-1)^j}{2} + \frac{\mathrm{i}}{2} \right) \beta_j \hbar^{2j-1} \right) = -\hbar \, \mathrm{e}^{-\frac{A}{\hbar}} \sum_{k\geq 0} \gamma_k^+ \hbar^k, \tag{3.37}$$

where

$$\beta_j = b_j b_0^{2j-1}, \tag{3.38}$$

and we have an additional factor $-\hbar$ which, as we will see, is needed to reproduce the behavior of the coefficients b_j (this factor can of course be reproduced from the Voros–Silverstone connection formulae). In addition, we have the complex conjugate trans-series,

$$\varphi_{A^\star}(z) = -\hbar \, \mathrm{e}^{-\frac{A^\star}{\hbar}} \exp\left(-\sum_{j=1}^{\infty} \left(\frac{(-1)^j}{2} - \frac{\mathrm{i}}{2} \right) \beta_j \hbar^{2j-1} \right) = -\hbar \, \mathrm{e}^{-\frac{A^\star}{\hbar}} \sum_{k\geq 0} \gamma_k^- \hbar^k. \tag{3.39}$$

From the definition one easily checks that

$$\gamma_k^+ \mathrm{e}^{\pi \mathrm{i} k/4} = \gamma_k^- \mathrm{e}^{-\pi \mathrm{i} k/4}. \tag{3.40}$$

We can now use (3.35) to deduce the large order behavior of the coefficients b_j, by adding the contributions of the two trans-series:

$$\beta_n \sim -\frac{1}{\pi} \sum_{k\geq 0} \Gamma(2n-1-k) \left(\gamma_k^+ A^{-2n+1+k} + \gamma_k^- (A^\star)^{-2n+1+k} \right), \tag{3.41}$$

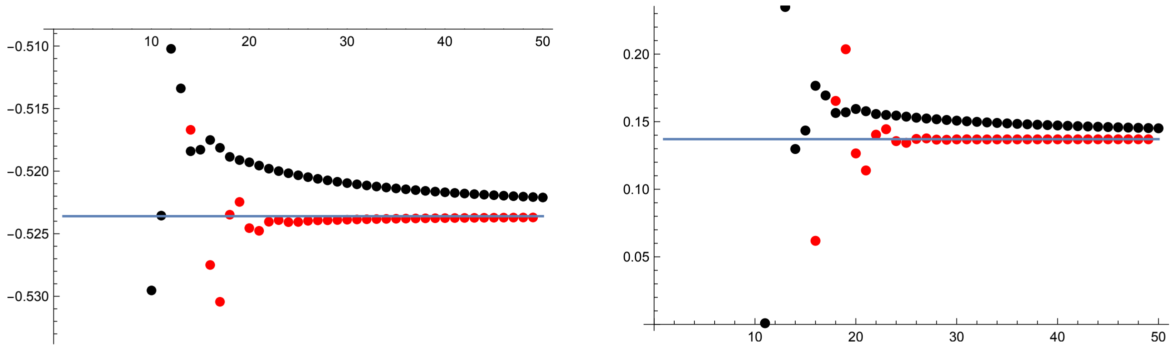

Figure 5. The sequences $\beta_n^{(1)}$ (left) and $\beta_n^{(2)}$ (right), for $n = 1, \cdots, 50$ are represented by black dots. The red dots are the sequences of their first Richardson transforms. The horizontal dashed lines are the expected values of $\alpha_{1,2}$.

where the value $b = -1$ follows from the overall power of $\hbar$ in the trans-series. By using (3.40) we conclude that

$$\beta_n \sim \frac{1}{\pi}\Gamma(2n-1)2^{n+3/2}\cos\left(\frac{\pi n}{2}+\frac{3\pi}{4}\right)\left\{1+\sum_{k\geq 1}\frac{\alpha_k}{\prod_{\ell=1}^{k}(2n-1-\ell)}\right\} \tag{3.42}$$

where

$$\alpha_k = 2^{-k/2}\gamma_k^+ e^{\pi i k/4}, \tag{3.43}$$

and we have used (3.40) again. This large order asymptotics was first derived by Balian, Parisi and Voros in their seminal article [7] by applying resurgent techniques to quantum theory.

It is instructive to test numerically the large n asymptotics, since it gives an explicit relationship between perturbative data and non-perturbative sectors. To do this, one generates a large number of coefficients β_n and tries to extract a numerical estimate for the coefficients α_k from their asymptotic behavior. This estimate is then compared to the theoretical prediction (3.42).

In this case, we have an oscillatory asymptotics since the coefficients β_n oscillate between positive and negative values. However, in this case the oscillatory term $\cos\left(\frac{\pi n}{2}+\frac{3\pi}{4}\right)$ never vanishes and one can simply produce a sequence of positive numbers by dividing

$$\beta_n^{(0)} = 2^{-n-3/2}\frac{\pi\beta_n}{\Gamma(2n-1)\cos\left(\frac{\pi n}{2}+\frac{3\pi}{4}\right)}. \tag{3.44}$$

This sequence should converge to 1. In addition, the sequences

$$\beta_n^{(1)} = (2n-2)\left(\beta_n^{(0)}-1\right), \qquad \beta_n^{(2)} = (2n-2)(2n-3)\left(\beta_n^{(0)}-1-\frac{\alpha_1}{2n-2}\right), \tag{3.45}$$

should converge to

$$\alpha_1 = -\frac{\pi}{3}, \qquad \alpha_2 = \frac{\pi^2}{72}, \tag{3.46}$$

respectively. This can be clearly seen in Fig. 5. □

There are additional singularities which are not seen in Fig. 4 (typically, numerical calculations unveil only the singularities near the origin). In fact, there are singularities at any positive integer multiple of (3.26) and at $z = \pm\ell$. One can wonder what are the formal series attached to these singularities. The answer turns out to be very simple: the ℓ-th multiple of a singularity at $\Pi^{(0)}$ leads to a formal series

$$\frac{(-1)^\ell}{\ell} \mathrm{e}^{-\frac{\ell}{\hbar}\Pi(\hbar)}, \tag{3.47}$$

up to an overall constant. One can then obtain a beautiful, simple formula for the discontinuity of the Borel resummation along the rays with singularities $\ell\Pi^{(0)}$, $\ell = 1, 2, \cdots$. In the case of the discontinuity along the positive real axis, we have for example

$$s_+(\Pi_\mathrm{p})(\hbar) - s_-(\Pi_\mathrm{p})(\hbar) = -2\mathrm{i}\hbar \log\left(1 + \mathrm{e}^{-\frac{1}{\hbar}s(\Pi_\mathrm{np})(\hbar)}\right). \tag{3.48}$$

Note that the quantum period $\Pi_\mathrm{np}(\hbar)$ is Borel summable along the positive real axis, so one can use conventional Borel resummation instead of lateral Borel resummation.

The result (3.48) was obtained by Voros in [8]. It turns out to be a particular example of a more general formula for discontinuities in the WKB method, which was found by Delabaere and Pham in [9]. We will call it the *Delabaere–Pham formula*. It has typically the form

$$s_+(\Pi)(\hbar) - s_-(\Pi)(\hbar) = \sum_i \mathsf{S}_i^\pm \log\left(1 + \mathrm{e}^{-\frac{1}{\hbar}s_\pm(\Pi_i)(\hbar)}\right), \tag{3.49}$$

where the sum in the r.h.s. is over a finite set of quantum periods and it involves a set of Stokes constants $\mathsf{S}_i^\pm$. There might be different choices of lateral resummation in the r.h.s., but these choices are compensated by different values of the Stokes constants.

The information about the set of periods and the Stokes constants depends on the particular potential of the quantum-mechanical problem, and leads to a rich theory developed in [9–11] (and partially rediscovered in [12]). This theory solves then the problem of finding the structural resurgent properties of WKB expansions in one-dimensional quantum mechanics.

A related but conceptually different question is the following: how does the set of Borel-resummed Voros symbols lead to *physical* quantities? In particular, can we use the set of Voros symbols to upgrade the Bohr–Sommerfeld quantization condition? Remember that the formal, all-orders quantization condition in this example reads

$$\Pi_\mathrm{p}(\hbar) = 2\pi\hbar\left(n + \frac{1}{2}\right). \tag{3.50}$$

The formal series appearing in the l.h.s. is factorially divergent and as we have seen in our detailed discussion, it is *not* Borel-summable. The would-be Borel-resummed quantization condition

$$s_\pm(\Pi_\mathrm{p})(\hbar) = 2\pi\hbar\left(n + \frac{1}{2}\right) \tag{3.51}$$

does not make sense, since the l.h.s. has a (small) imaginary part and it would lead to imaginary energies. One can implement the reality condition by taking the real part of the Borel resummation, namely

$$\frac{1}{2}\left(s_+(\Pi_\mathrm{p})(\hbar) + s_-(\Pi_\mathrm{p})(\hbar)\right) = 2\pi\hbar\left(n + \frac{1}{2}\right). \tag{3.52}$$

This does make sense, but it does *not* lead to the exact spectrum! There is an exponentially small contribution due to the non-perturbative quantum period. This is a remarkable example of “tunneling through a complex barrier”. The *exact* quantization condition first found by Voros in [13] and studied in [7] is

$$\frac{1}{2}\left(s_+(\Pi_{\rm p})(\hbar) + s_-(\Pi_{\rm p})(\hbar)\right) - 2(-1)^n \tan^{-1}\left({\rm e}^{-\frac{1}{2\hbar}s(\Pi_{\rm np})}\right) = 2\pi\hbar\left(n+\frac{1}{2}\right). \tag{3.53}$$

The moral of this example is that Borel-resummed perturbative series, when complemented with Borel-resummed non-perturbative corrections, can lead to exact results in quantum theory. More precisely, we have here a realization of two ideas in the theory of resurgence, formulated in this language in [14]:

1. *“Weak” resurgence.* The exact, non-perturbative quantum volume (which leads to the exact spectrum) is given by a lateral Borel resummation of a trans-series. This trans-series is given by the perturbative series, plus a series of exponentially small corrections.
2. *“Strong” resurgence.* The non-perturbative series which is needed in the exact result (in this case $\Pi_{\rm np}(\hbar)$) can be obtained from the Borel singularities of the perturbative series.

Note that, although the non-perturbative ingredient of the exact answer $\Pi_{\rm np}(\hbar)$ can be obtained from the Borel analysis of $\Pi_{\rm p}(\hbar)$, the precise trans-series appearing in the l.h.s. of (3.53) requires additional knowledge, which in this case comes ultimately from the Schrödinger equation.

3.3 Other examples and generalizations

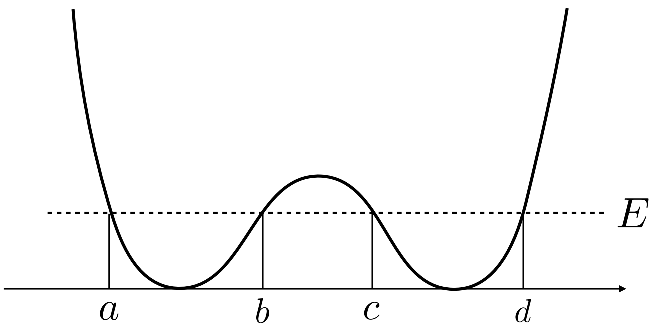

Figure 6. The double-well potential.

Another example which shares many important properties with the pure quartic oscillator is the famous double-well potential. This is a symmetric, quartic potential with two degenerate minima and a local maximum at its middle point, see Fig. 6. We can take for example

$$V(x) = \frac{g^2}{2}\left(x^2 - \frac{1}{4g^2}\right)^2, \tag{3.54}$$

where g is a coupling constant. Let us consider energies below the top of the barrier in the middle:

$$E < \frac{1}{32g^2}, \tag{3.55}$$

so there are four turning points which will be labelled a, b, c, d from left to right, see Fig. 6. Due to the symmetry of the potential, we have $a = -d$, $b = -c$. Let us suppose that we want to calculate (even approximately) the energy of the ground state of this system. The Bohr–Sommerfeld quantization condition tells us that there are *two* degenerate ground states, whose approximate energy is given by (3.13), and the $\mathcal{A}$ cycle encircles the region of allowed motion $[a, b]$.

It turns out that this is even *qualitatively wrong*. The elementary spectral theory of the Schrödinger operator tells us that the ground state is non-degenerate.

One could think that incorporating the all-order corrections in (3.17) would solve the problem. However, as in the case of the pure quartic oscillator, it is possible to show that the resulting formal power series is *not* Borel summable along the real axis of $\hbar$. We can also ask what is the structure of the Borel singularities for the quantum period

$$\Pi_{\mathcal{A}}(\hbar) = \oint_{\mathcal{A}} P(x, \hbar) \mathrm{d}x. \tag{3.56}$$

Not surprisingly, they involve a *different* period, related to tunneling across the barrier, i.e. involving the "forbidden" classical region between the points b and c. Let us not that, in this region, the momentum is imaginary, and we make a choice of branch cut such that

$$p(x) = -\mathrm{i} p_1(x), \qquad p_1(x) = \sqrt{2m(V(x) - E)} > 0. \tag{3.57}$$

We then consider a cycle $\mathcal{B}$ around the forbidden integral $[b, c]$, and define the quantum period as

$$\Pi_{\mathcal{B}}(\hbar) = 2 \int_b^c P_1(x, \hbar) \mathrm{d}x \tag{3.58}$$

where $P_1(x, \hbar)$ is obtained from $P(x, \hbar)$ by the above choice of branch cut. Then, it follows that the Borel singularities of $\Pi_{\mathcal{A}}(\hbar)$ are located at

$$\ell \Pi_{\mathcal{B}}^{(0)} = 2\ell \int_b^c p_1(x) \mathrm{d}x, \qquad \ell = 1, 2, \cdots \tag{3.59}$$

and lead to the formal power series

$$\frac{(-1)^\ell}{\ell} \mathrm{e}^{-\frac{\ell}{\hbar} \Pi_{\mathcal{B}}(\hbar)}. \tag{3.60}$$

In addition, we have another incarnation of the Delabaere–Pham formula

$$s_+(\Pi_{\mathcal{A}})(\hbar) - s_-(\Pi_{\mathcal{A}})(\hbar) = -\mathrm{i}\hbar \log\left(1 + \mathrm{e}^{-s(\Pi_{\mathcal{B}})(\hbar)/\hbar}\right). \tag{3.61}$$

We can ask now what is the analogue of the exact quantization condition (3.53). In this case, it reads

$$1 + \exp\left(\pm \frac{\mathrm{i}}{\hbar} s_\pm(\Pi_{\mathcal{A}})(\hbar)\right) = \pm \mathrm{i}\epsilon \exp\left(-\frac{1}{2\hbar} s(\Pi_{\mathcal{B}})(\hbar)\right). \tag{3.62}$$

Note that, if we neglect the r.h.s. of (3.62), which is exponentially small, we recover the all-orders quantization condition (3.17). Interestingly, the correction given by the r.h.s. removes the degeneracy of the states. Each solution to (3.17) gives rise to two different energy levels, with n even and odd, respectively. This is an example in which the conventional perturbative series in $\hbar$ misses an important qualitative effect, which is only recovered after including the exponentially small effect.

4 Resurgence in quantum field theory

The difficulties faced by the resurgence program in QFT should not be underestimated. To see this, it is useful to compare with the situation in one-dimensional quantum mechanics. An important source of insights about resurgent structures in quantum mechanics was the availability of *long perturbative series*. Perhaps the most important early papers on resurgence in quantum mechanics are the work by Bender and Wu on the perturbative series of the anharmonic oscillator [15] and the work by Balian, Parisi and Voros [7, 8] on the WKB expansion of the pure quartic oscillator. In both cases, long perturbative series of about 50-60 terms made it possible to check many conjectures and clarify many ideas. We are very far from having such long perturbative series in realistic theories like QED and QCD. In addition, in one-dimensional quantum mechanics we have a fantastic non-perturbative tool available: the Schrödinger equation. We don't have such an easy tool in QFT.

In view of this, it seems reasonable to consider toy models of QFT. We will start with a rather drastic simplification, which has however some interesting consequences.

4.1 A toy model for the path integral

4.1.1 The quartic integral

In quantum field theory many quantities can be represented as path integrals, of the form

$$Z = \int \mathcal{D}\,\phi(x)\, \mathrm{e}^{-S(\phi(x))/\hbar} \tag{4.1}$$

The construction of formal saddle-point expansions of this path integral is well-understood in many theories. Saddle-points correspond to *classical* solutions, i.e. solutions of the classical EOM

$$\frac{\delta S(\phi)}{\delta \phi} = 0. \tag{4.2}$$

The expansion around the trivial solution $\phi = 0$ leads to conventional perturbation theory. Many of our ideas about QFT are based on the intuition that path integrals are complicated versions of conventional integrals. We can then consider a very simple integral to start thinking about resurgence in QFT. This integral can be regarded as a zero-dimensional reduction of the $\lambda\phi^4$ theory:

$$Z(g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}z\, \mathrm{e}^{-S(z)}, \qquad S(z) = \frac{z^2}{2} + \frac{gz^4}{4}. \tag{4.3}$$

This integral has an asymptotic expansion at small g which is obtained, as in field theory, by expanding the exponent of the quartic monomial, and integrating term by term. One finds in this way an all-orders asymptotic expansion

$$Z(g) \sim \varphi(g) \tag{4.4}$$

where

$$\varphi(g) = \sum_{k=0}^{\infty} a_k g^k, \tag{4.5}$$

and

$$a_k = \frac{(-4)^{-k}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}z \frac{z^{4k}}{k!} \mathrm{e}^{-z^2/2} = (-4)^{-k} \frac{(4k-1)!!}{k!}, \tag{4.6}$$

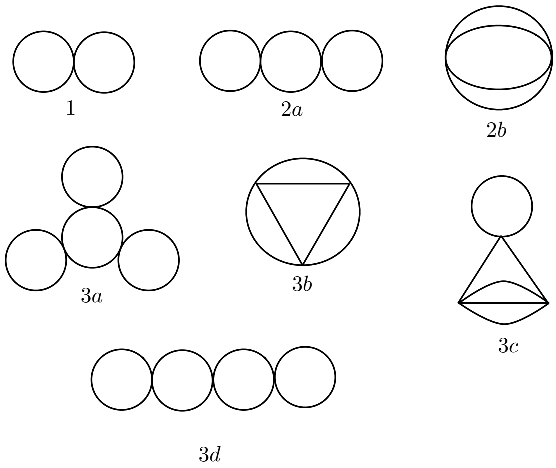

Figure 7. Connected Feynman diagrams contributing to $\log \varphi(g)$.

This is one of the rare series for which the Borel transform can be computed in closed form, and one finds

$$\widehat{\varphi}(\zeta) = \frac{2K(k)}{\pi(1+4\zeta)^{1/4}}, \qquad k^2 = \frac{1}{2} - \frac{1}{2\sqrt{1+4\zeta}}. \tag{4.7}$$

where $K(k)$ is the elliptic integral of the first kind. This function has a branch point at $\zeta = A = -1/4$, of the logarithmic type, and one can extract the formal power series $\widehat{\varphi}_{-1/4}(\xi)$:

$$\widehat{\varphi}_{-1/4}(\xi) = \frac{\mathrm{i}}{\sqrt{2}} \sum_{k\geq 0} (-1)^k \frac{a_k}{k!} \xi^k \tag{4.8}$$

and $\mathsf{S} = -2\mathrm{i}$ (the choice of normalization of $\widehat{\varphi}_{-1/4}(\xi)$ will be justified in a moment). Therefore, up to an overall factor, $\varphi_{-1/4}(z)$ is the original power series up to an alternating sign. This is the same phenomenon that we found in the Airy function, and one can also find this second power series by expanding around the non-trivial saddle-points of $Z(g)$. These are obtained from

$$z + gz^3 = 0 \Rightarrow z^2 = -\frac{1}{g}. \tag{4.9}$$

Therefore we have two nontrivial saddle-points $z_{1,2}$

$$z_{1,2} = \pm \frac{\mathrm{i}}{\sqrt{g}} \tag{4.10}$$

where we assumed $g > 0$. Expanding around z_1, we find

$$S(z) = -\frac{1}{4g} - y^2 + \mathrm{i}g^{1/2}y^3 + \frac{g}{4}y^4, \qquad y = z - z_1. \tag{4.11}$$

This gives a Gaussian with the wrong sign, so we have to perform a rotation $y \to \mathrm{i}y$. The resulting expansion is

$$Z(g) \sim \mathrm{e}^{\frac{1}{4g}} \varphi_{-1/4}(g), \tag{4.12}$$

and gives the trans-series associated to the Borel singularity at $\zeta = -1/4$.

Example 4.1. One can extract from these considerations the all orders asymptotics for a_n, similarly to what was done in the example 2.9:

$$\begin{aligned} a_k &\sim \frac{1}{\sqrt{2}\pi}\left(-\frac{1}{4}\right)^{-k}(k-1)!\sum_{n\geq 0} a_n(-1)^n\left(-\frac{1}{4}\right)^n\frac{\Gamma(k-n)}{\Gamma(k)} \\ &\sim \frac{1}{\sqrt{2}\pi}\left(-\frac{1}{4}\right)^{-k}(k-1)!\left(1-\frac{1}{4}\cdot\frac{3}{4}\frac{1}{k-1}+\frac{1}{16}\cdot\frac{105}{32}\frac{1}{(k-1)(k-2)}+\cdots\right). \end{aligned} \tag{4.13}$$

The factorial growth has a simple diagrammatic interpretation: the series $\varphi(g)$ can be calculated by zero-dimensional Feynman diagrams. After taking the logarithm,

$$\log\varphi(g) = -\frac{3g}{4}+3g^2-\frac{99g^3}{4}+\cdots \tag{4.14}$$

the relevant diagrams are the usual connected diagrams with four legs, which we show in Fig. 7 up to three vertices. It is an easy exercise to verify that indeed the coefficients in (4.14) come from their multiplicities, which we have tabulated for convenience in Table 2[1]. The number of connected diagrams grows indeed factorially with the number of vertices (in fact, its large order asymptotics is given by the r.h.s. of (4.13)).

diagram	1	2a	2b	3a	3b	3c	3d
multiplicity	3	36	12	288	288	576	432

Table 2. Multiplicities of the Feynman diagrams in Fig. 7.

4.1.2 The large N version

It is instructive to consider an N-component version of the above model, given by (see e.g. [16])

$$Z_N(g) = \frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} \mathrm{d}\boldsymbol{z}\,\mathrm{e}^{-S(\boldsymbol{z})}, \qquad S(\boldsymbol{z}) = \frac{\boldsymbol{z}^2}{2}+\frac{g}{4}(\boldsymbol{z}^2)^2. \tag{4.15}$$

The perturbative expansion of this model is slightly more complicated, but it can be still written in closed form by going to spherical coordinates:

$$\varphi_N(g) = \sum_{k\geq 0} a_k(N)g^k, \qquad a_k(N) = (-1)^k\frac{\Gamma(2k+N/2)}{k!\Gamma(N/2)}. \tag{4.16}$$

From the diagrammatic point of view, the complication is due to the fact that the multiplicities depend on N. For example, the diagram 1 has multiplicity $N(N+2)$, the diagram $2a$ has multiplicity $4N(N+2)$, while the diagram $2b$ has multiplicity $4N(N+2)^2$, and one finds

$$\frac{1}{N}\log\varphi_N(g) = -\frac{N+2}{4}g+\frac{(N+2)(N+3)}{4}g^2+\cdots, \tag{4.17}$$

in agreement with (4.16). For fixed N, this series grows again factorially,

$$a_k(N)\sim(-4)^k\Gamma\left(k+\frac{N-3}{2}\right), \qquad N\gg 1, \tag{4.18}$$

[1] We define the multiplicity of a diagram with k vertices as the total number of contractions divided by $k!$.

and the Borel singularity is located at the same position, namely $\zeta = -1/4$. This corresponds to a non-trivial saddle-point configuration with action

$$S = -\frac{1}{4g}. \tag{4.19}$$

The EOM for such configuration is the N-component version of (4.9),

$$\boldsymbol{z}^2 = -\frac{1}{g} \tag{4.20}$$

Note that in this case there is a "moduli space" of instantons with $O(N)$ symmetry, i.e. a sphere $\mathbb{S}^{N-1}$. A careful treatment of this family of instantons (taking into account zero modes) leads to the trans-series associated to the singularity at $\zeta = -1/4$.

There is however another possible treatment of this model, in which one takes the limit $N \to \infty$, $g \to 0$ but keeping fixed the 't Hooft parameter

$$\lambda = gN. \tag{4.21}$$

In this limit, the free energy of the theory organizes as follows,

$$\frac{1}{N}\log \varphi_N(g) = \sum_{\ell\geq 0} f_\ell(\lambda) N^{-\ell}, \tag{4.22}$$

where

$$f_0(\lambda) = -\frac{\lambda}{4} + \frac{\lambda^2}{4} - \frac{5\lambda^3}{12} + \frac{7\lambda^4}{8} - \frac{21\lambda^5}{10} + \mathcal{O}\left(\lambda^6\right). \tag{4.23}$$

The first two terms in this power series can be read from (4.17). However, one can obtain a closed formula for the resummed series by using large N tricks,

$$f_0(\lambda) = \frac{1}{2}\log\left(\frac{1}{2}\left(\sqrt{1+4\lambda}-1\right)+1\right) - \frac{\left(\sqrt{1+4\lambda}-1\right)^2}{16\lambda}. \tag{4.24}$$

This is an *analytic* function at $\lambda = 0$ (in fact, all the functions $f_\ell(\lambda)$ appearing in the large N expansion (4.22) are analytic, and they have the same radius of convergence). Diagramatically, this is due to the fact that selecting connected diagrams with a given power of N tames the factorial growth of the multiplicities to just exponential growth. Note that, when we use λ as our expansion variable, the Borel singularity is at $\zeta = -N/4$, therefore it goes away at infinity when $N \to \infty$. This is expected due to the analyticity of the free energy in the large N limit. Equivalently, the instanton contributions goes like $\exp(N/4\lambda)$, which vanishes when $N \to \infty$ if $\lambda < 0$.

Example 4.2. *Hubbard–Stratonovich trick.* Let us go back to the integral (4.15) and let us introduce a new field σ, in such a way that

$$\mathrm{e}^{-\frac{\lambda}{4N}(\boldsymbol{z}^2)^2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}\sigma\, \mathrm{e}^{-\frac{\sigma^2}{2}+\mathrm{i}\sqrt{\frac{\lambda}{2N}}\sigma\, \boldsymbol{z}^2}. \tag{4.25}$$

Inserting this in (4.15) we find that the dependence on $\boldsymbol{z}$ is Gaussian and we can integrate it immediately. This gives

$$Z_N(g) = \sqrt{\frac{N}{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}\sigma\, \mathrm{e}^{-NK(\sigma)}, \tag{4.26}$$

where

$$K(\sigma) = \frac{\sigma^2}{2} + \frac{1}{2}\log\left(1 - \mathrm{i}\sqrt{2\lambda}\sigma\right), \tag{4.27}$$

To obtain the large N limit of (4.26) we just perform a saddle-point analysis. We find two critical points

$$\sigma_c^{\pm} = \frac{\mathrm{i}}{2\sqrt{2\lambda}}\left(-1 \pm \sqrt{1+4\lambda}\right). \tag{4.28}$$

It turns out that the critical point which reproduces perturbation theory is σ_c^+. We then find,

$$f_0(\lambda) = -K(\sigma_c^+) \tag{4.29}$$

which leads to (4.24).

4.1.3 Lessons

There are various lessons (or, more precisely, expectations) that can be obtained from this zero-dimensional example:

1. The perturbative series is factorially divergent. The factorial growth can be attributed to the factorial growth of the number of Feynman diagrams contributing to order n.
2. This factorial growth leads to a Borel singularity, which in turns leads to a trans-series. This trans-series is obtained from a non-trivial saddle point of the integral (or "instanton"), and the location of the Borel singularity is the classical action of the instanton.
3. The model can be upgraded to an N-flavour model with a 't Hooft parameter λ. By restricting ourselves to the leading, large N contribution, the factorial growth is tamed to an exponential growth. Correspondingly, the Borel singularity in the λ plane moves to infinity when $N \to \infty$.

This model suggests that perturbative series in QFT are divergent due to instanton configurations, which will be suppressed in an appropriate large N limit.

4.2 Enter renormalons

In the 1970s there was a lot of activity trying to extend the above expectations to realistic QFTs (see for example [17] for a selection of papers by Lipatov and others in that direction). However, it was soon noted in [18–21] that these expectations do not hold.

More precisely, it was found that generically, perturbative series in QFT are indeed factorially divergent. However, the factorial growth is not only due to the growth in the number of Feynman diagrams: there are special diagrams at n loops (which we will call "renormalon diagrams") which grow like $n!$ due to the integration over momenta. When the theories admit a large N limit, it is often the case that the surviving diagrams at large N are of the renormalon type. Therefore, a typical signal of theories with renormalons is that the large N limit of physical quantities has a factorially divergent expansion in an appropriate 't Hooft parameter.

The location of Borel singularities due to instantons is expected to be at the classical actions of the instanton configurations, and this is a general characterization. One can then ask what is the possible location of singularities due to renormalons. Based on various arguments, Parisi and 't Hooft conjectured in [19, 21] that, in an asymptotically free theory, these are given by

$$\zeta = \frac{\ell}{2\beta_0}, \qquad \ell \in \mathbb{Z}_{\neq 0}. \tag{4.30}$$

Note that not all these singularities are necessarily realized. Additional arguments for this conjecture were put forward for observables which admit an OPE, see e.g. [22] for a review. The conjecture (4.30) is remarkable, since it gives a universal prediction for possible renormalon singularities, independent of the observable, and which involves only the first (scheme-independent) coefficient of the beta function.

One consequence of the Parisi–'t Hooft conjecture (4.30) is that non-perturbative corrections to perturbative series in asymptotically free theories are given by *integer* powers of Λ, and for this reason they are sometimes called *power corrections*.

4.3 An example of renormalons in non-relativistic field theory

The first example of renormalons we will consider is somewhat unorthodox, since it does not occur in a relativistic quantum field theory, but in a condensed matter model. However, it has the advantage that one can avoid many of the complications of typical renormalon examples in quantum field theory. In addition, in this model, renormalons give direct information on the underlying non-perturbative physics.

We will consider a system of N non-relativistic fermions in one dimension interacting through a delta function interaction. The Hamiltonian of the system is

$$H_N = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} - 2c \sum_{1 \leq i < j \leq N} \delta(x_i - x_j), \tag{4.31}$$

i.e. we set $\hbar = 2m = 1$. We will mostly consider the attractive case. In our conventions, this corresponds to a positive coupling constant: $c > 0$. Let $E_0(N, L)$ be the ground state energy for the N-particle system on a circle of length L. In the thermodynamic limit of fixed density n,

$$N \to \infty, \qquad L \to \infty, \qquad n = \frac{N}{L} \text{ fixed}, \tag{4.32}$$

we can define the ground state energy density as

$$E = \lim_{L \to \infty} \frac{E_0(N, L)}{L}. \tag{4.33}$$

We want to find the dependence on this energy on the coupling constant c, or rather on the dimensionless coupling

$$\gamma = \frac{c}{n}. \tag{4.34}$$

Note that weak coupling, $\gamma \ll 1$, corresponds here to *high* density, in contrast to what happens in the three dimensional Fermi gas with a delta function interaction. The ground state energy density can be written as a formal power series in γ of the form

$$e_{\rm p}(\gamma) = \sum_{n \geq 1} e_n \gamma^n, \tag{4.35}$$

where we have introduced the dimensionless energy function

$$e(\gamma) = \frac{E(\gamma)}{n^3} \tag{4.36}$$

The first term in (4.35) corresponds to the free Fermi gas

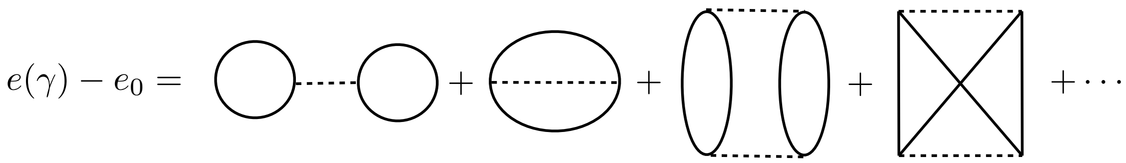

Figure 8. Feynman diagrams contributing to the ground state energy density of the Gaudin–Yang model at the very first orders. The first and second diagram are respectively the Hartree and Fock contributions.

$$e_0 = \frac{\pi^2}{12}, \tag{4.37}$$

while the coefficients e_n can be in principle computed with conventional many-body perturbation theory, see Fig. 8. However, the above system is integrable and is known in this context as the Gaudin–Yang model [23, 24]. By using a method due to D. Volin [25, 26], we were able to obtain many coefficients of the ground state energy (60 in closed analytic form and 168 numerically). One finds, for the very first orders,

$$\begin{aligned} e_{\rm p}(\gamma) = {} & \frac{\pi^2}{12} - \frac{\gamma}{2} - \frac{\gamma^2}{12} - \frac{\zeta(3)}{\pi^4}\gamma^3 - \frac{3\zeta(3)}{2\pi^6}\gamma^4 - \frac{3\zeta(3)}{\pi^8}\gamma^5 - \frac{5(5\zeta(3)+3\zeta(5))}{4\pi^{10}}\gamma^6 \\ & - \frac{3\left(12\zeta(3)^2 + 35\zeta(3) + 75\zeta(5)\right)}{8\pi^{12}}\gamma^7 - \frac{63\left(12\zeta(3)^2 + 7\zeta(3) + 35\zeta(5) + 12\zeta(7)\right)}{16\pi^{14}}\gamma^8 \\ & - \frac{3\left(404\zeta(3)^2 + 240\zeta(5)\zeta(3) + 77\zeta(3) + 735\zeta(5) + 882\zeta(7)\right)}{4\pi^{16}}\gamma^9 + \mathcal{O}\left(\gamma^{10}\right). \end{aligned} \tag{4.38}$$

One can check that this series is *factorially divergent*. More precisely, one has

$$e_k \sim -\frac{1}{\pi}(\pi^2)^{-k+1}\Gamma(k-1), \qquad k \gg 1. \tag{4.39}$$

This indicates that there is a singularity in the Borel plane at

$$\zeta = \pi^2, \tag{4.40}$$

and equivalently, a non-perturbative effect of order

$$\exp\left(-\frac{\pi^2}{\gamma}\right). \tag{4.41}$$

This was established numerically in [27, 28], and then analytically in [29]. More precisely, we expect that the energy is given by a (resummed) trans-series of the form

$$e(\gamma) \sim \sum_{n\geq 0} e_n \gamma^n + \sum_{\ell \geq 1} C_\ell \, {\rm e}^{-\ell\pi^2/\gamma}\gamma^{b_\ell} \sum_{n \geq 0} e_n^{(\ell)}\gamma^n. \tag{4.42}$$

There are two different situations. In the *repulsive* case, the series is Borel summable and we expect

$$e(\gamma) = s\left(e_{\rm p}\right)(\gamma). \tag{4.43}$$

In the *attractive* case $c > 0$ the perturbative series is *not* Borel summable. One can show that

$$C_\pm = \pm {\rm i}, \tag{4.44}$$

where the choice of sign depends on the choice of lateral resummation for the series. Since the first trans-series is purely imaginary, one could think that it is there just to cancel the ambiguity in the Borel resummation of the perturbative part, and that one could simply have

$$e(\gamma) = \mathrm{Re}\left(s\left(e_{\mathrm{p}}\right)(\gamma)\right). \tag{4.45}$$

However, this can be checked to be false by using the exact answer for $e(\gamma)$ from the Bethe ansatz. One finds,

$$e(\gamma) - \mathrm{Re}\left(s\left(e_{\mathrm{p}}\right)(\gamma)\right) \sim \exp\left(-\frac{2\pi^2}{\gamma}\right) \tag{4.46}$$

which indicates that $C_2 \neq 0$.

It has been argued that the perturbative series in fermionic systems is more "tame" than in the bosonic case, and that in $d = 1$ space dimension, the series does not even grow factorially [30]. These arguments were based on instanton-like estimates, and the above model is an explicit counter-example to these expectations.

The reason that instanton expectations in this model are not fulfilled is that the asymptotic growth of this series is a renormalon effect, as argued in [28]. One way to argue for this is to consider, as we mentioned above, a version of the model in which the fermions come in κ flavors. This version is also integrable [31], for any κ, but one can instead analyze the model in the large κ limit. To do this, we introduce an appropriate scaled coupling constant or 't Hooft parameter,

$$\lambda = \left(\frac{\kappa}{2}\right)^2 \gamma. \tag{4.47}$$

It plays the rôle of a 't Hooft parameter and is kept fixed in the large κ limit. We also define the rescaled ground energy density as

$$e(\lambda;\kappa) = \frac{1}{4}\frac{E/\kappa}{(n/\kappa)^3}. \tag{4.48}$$

It has the following large κ expansion,

$$e(\lambda;\kappa) = \sum_{n=0}^{\infty} e_n(\lambda)\kappa^{-n}. \tag{4.49}$$

Note that, when $\kappa = 2$, λ and $e(\lambda;\kappa)$ become the coupling constant γ and the energy density $e(\gamma)$, respectively. The leading order function $e_0(\lambda)$ to (4.49) is given by the free gas result plus the Hartree term,

$$e_0(\lambda) = \frac{\pi^2}{12} - \lambda. \tag{4.50}$$

The subleading function $e_1(\lambda)$ is the sum of the contributions of a subset of many-body diagrams called ring diagrams, made out by circles of fermions joined by interaction lines (the second and third diagrams in Fig. 8 are of this type). These are the diagrams that appear in the so-called "random phase approximation". One finds

$$\begin{aligned} e_1(\lambda) \sim \lambda - \frac{\lambda^2}{3} - \frac{4\zeta(3)}{\pi^4}\lambda^3 - \frac{12\zeta(3)}{\pi^6}\lambda^4 - \frac{48\zeta(3)}{\pi^8}\lambda^5 - \frac{40(5\zeta(3)+\zeta(5))}{\pi^{10}}\lambda^6 \\ - \frac{120(7\zeta(3)+5\zeta(5))}{\pi^{12}}\lambda^7 - \frac{168(21\zeta(3)+35\zeta(5)+4\zeta(7))}{\pi^{14}}\lambda^8 + \mathcal{O}\left(\lambda^9\right). \end{aligned} \tag{4.51}$$

If the factorial divergence of $e_p(\gamma)$ was due to instanton effects, we would expect this series to be convergent. On the contrary, if this series diverges factorially, then it means that ring diagrams *by themselves* lead to a factorial growth, which is the typical sign of a renormalon effect. An explicit analysis of ring diagrams [28, 32] shows that in fact the coefficients in this series grow like $\pi^{-2\ell}\ell!$.

What is the physics associated to the lack of Borel summability in the attractive case? We argued in [28] that this was a signal of the Cooper instability, and that indeed the non-perturbative scale set by the first Borel singularity is the square root of the energy gap of the interacting system. Indeed, already a BCS-type calculation in this model shows that

$$\text{gap} \sim \exp\left(-\frac{2\pi^2}{\gamma}\right). \tag{4.52}$$

Therefore, in this example renormalons are directly connected to the physics of the problem. The non-Borel summability of the ground state energy is an indication that we are expanding around the “wrong” vacuum, since the system is in a superconducting phase and not in a normal phase.

Note that the Gaudin–Yang model is not renormalizable in the conventional sense of quantum field theory, since the diagrams are finite (or at least, the sum of all diagrams contributing to a given order is finite). However, one can introduce an effective RG by integrating out degrees of freedom and calculate the beta function. In the case of attractive fermions, the model behaves as an asymptotically free theory and the Parisi–'t Hooft conjecture turns out to be true for all κ, see [32] for details.

4.4 Renormalons in integrable field theories

I will now summarize some more recent results in integrable field theories which are important for two reasons:

1. These models have renormalon singularities which are *not* of the form predicted by Parisi and 't Hooft.

2. The trans-series can be sometimes calculated in detail, and one can verify the resurgence conjecture according to which exact results agree with (lateral) resummations of the trans-series.

These results can be obtained by exploiting the integrability the model, in particular by using a particular observable which can be studied in much detail: the free energy of the model once we introduced a chemical potential coupled to a conserved charge. The idea to use this type of observables to explore renormalon physics first appeared in [33, 34] (in a large N calculation) and more recently in [25, 26]. The results mentioned above were mostly obtained in [29]

We will consider asymptotically free, integrable models in two dimensions. These include e.g. the $O(N)$ sigma model and the $O(N)$ Gross–Neveu model. We will write the beta function for the coupling constant g as

$$\beta_g(g) = \mu \frac{\mathrm{d}g}{\mathrm{d}\mu} = -\beta_0 g^3 - \beta_1 g^5 - \cdots, \tag{4.53}$$

With this convention, asymptotically free theories have $\beta_0 > 0$. All of our perturbative calculations will be done in the $\overline{\text{MS}}$ scheme.

Here we are interested in a particular observable first considered by Polyakov and Wiegmann [35]. Let H be the Hamiltonian of the model, and let Q be a conserved charge, associated to a global conserved current. Let h be an external field coupled to Q. h can be regarded as a chemical potential, and as usual in statistical mechanics we can consider the ensemble defined by the operator

$$\mathsf{H} - h\mathsf{Q}. \tag{4.54}$$

The corresponding free energy per unit volume is then defined by

$$F(h) = -\lim_{V,\beta\to\infty} \frac{1}{V\beta} \log \mathrm{Tr} e^{-\beta(\mathsf{H}-h\mathsf{Q})}, \tag{4.55}$$

where V is the volume of space and β is the total length of Euclidean time. This is the ground state energy of the model in the presence of the additional coupling. In an asymptotically free theory, we expect to be able to calculate

$$\mathcal{F}(h) = F(h) - F(0) \tag{4.56}$$

in conventional perturbation theory when $h \gg \Lambda$, where Λ is the dynamically generated scale in the $\overline{\mathrm{MS}}$ scheme:

$$\Lambda = \mu \left(2\beta_0 g^2\right)^{-\beta_1/(2\beta_0^2)} \mathrm{e}^{-1/(2\beta_0 g^2)} \exp\left(-\int_0^g \left\{\frac{1}{\beta_g(x)} + \frac{1}{\beta_0 x^3} - \frac{\beta_1}{\beta_0^2 x}\right\} \mathrm{d}x\right). \tag{4.57}$$

Standard perturbative theory gives a formal power series of the form

$$\mathcal{F}_{\mathrm{p}}(\bar{g}) = -\frac{h^2}{2\pi} \sum_{n\geq 0} a_n \bar{g}^{n-b} \tag{4.58}$$

where $\bar{g}$ is the running coupling constant evaluated at the scale h, and $b = 0, 1$ for fermionic (bosonic) models, respectively.

It turns out that one can calculate the non-perturbative corrections to $\mathcal{F}(h)$ *exactly* (in practice, it is difficult to calculate the full trans-series, but there is a systematic way of computing the trans-series in $\bar{g}$). The reason is that $\mathcal{F}(h)$ is encoded in a Bethe integral equation which can be studied in some detail. The equation describes the Fermi density of Bethe roots,

$$\epsilon(\theta) - \int_{-B}^{B} \mathrm{d}\theta' \, K(\theta - \theta')\epsilon(\theta') = h - m\cosh(\theta), \quad \theta \in [-B, B]. \tag{4.59}$$

In this equation, m is the mass of the charged particles, and with a clever choice of Q, it is directly related to the mass gap of the theory. One has of course, due to asymptotic freedom,

$$m = c\Lambda, \tag{4.60}$$

where c is a non-trivial constant which can be found explicitly in integrable models [36–38]. The kernel of the integral equation is given by

$$K(\theta) = \frac{1}{2\pi \mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}\theta} \log S(\theta), \tag{4.61}$$

where $S(\theta)$ is the S-matrix appropriate for the scattering of the charged particles. The endpoints $\pm B$ are fixed by the condition

$$\epsilon(\pm B) = 0. \tag{4.62}$$

The free energy is then given by

$$\mathcal{F}(h) = -\frac{m}{2\pi}\int_{-B}^{B} \epsilon(\theta)\cosh(\theta)\mathrm{d}\theta. \tag{4.63}$$

It turns out that the non-perturbative corrections to $\mathcal{F}(h)$ can be extracted from the singularities of $K(\theta)$ along the positive imaginary axis. The procedure is somewhat involved and is explained in [29]. The key point is that these singularities involve both a pole and a branch cut, and there are two different prescriptions to obtain their contribution, depending on the choice of branch cut. The fact that the non-perturbative corrections are ambiguous is typical of trans-series, as we have argued in these lectures, and we expect this ambiguity to be correlated to the choice of Borel resummation for the perturbative part.

Let us present the example of the $O(N)$ Gross–Neveu model [39], with $N > 5$. We choose a charge corresponding to an $O(N)$ rotation in, say, the 12 internal plane, as in [38]. In this model one already finds non-conventional power corrections. The non-perturbative part of the free energy reads [29]

$$\mathcal{F}_{\mathrm{np}}(h) = \mp \frac{\mathrm{i}m^2}{8} - \frac{h^2}{2\pi}\mathcal{C}^{\pm}\left(\mathrm{e}^{-\frac{1}{|\beta_0|\bar{g}^2}}\right)^{\frac{N-4}{N-2}} \bar{g}^{\frac{4}{N-4}}\left(1 + \mathcal{O}(\bar{g})\right). \tag{4.64}$$

Here, the $\pm$ signs correspond to the choices of branch cut mentioned above, and

$$\mathcal{C}^{\pm} = \mathcal{C}\exp\left(\mp\frac{\pi \mathrm{i}}{N-4}\right). \tag{4.65}$$

where $\mathcal{C}$ is a known, non-trivial function of N. The first term in (4.64) is proportional to Λ^2, and it is a typical Parisi–'t Hooft renormalon (4.30) with $\ell = 2$. However, the second correction is *not* of the form (4.30), since it would lead to a fractional value of $\ell = 2(N-2)/(N-4)$. The *imaginary part* of this correction gives a Borel singularity in an unconventional location of the Borel plane, but it can be checked very explicitly that it is exactly what is needed to control the large order behavior of $\mathcal{F}_{\mathrm{p}}(h)$.

Let us observe that, at large N, the two exponential corrections become identical and they control the behavior of the free energy at large N (presumably encoded by ring diagrams, although these have not been calculated explicitly). This suggests that the Parisi–'t Hooft conjecture is a large N approximation, but it does not hold at large N.

Let us also note that the *real part* of the second correction in (4.64) gives a contribution to the free energy which agrees with the result of the Bethe ansatz. This suggests that the trans-series (4.64) provides, after lateral resummation, the exact free energy. In [40] this was verified in much more detail for the same observable, in the $O(4)$ non-linear sigma model.

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