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Les Diablerets lectures on resurgence in mathematics and physics

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Abstract: Notes for the Diablerets Winter School

Contents

1 Introduction and general aspects

Many problems in mathematics and physics are not solvable in closed form, and one has to resort to approximation schemes. Many of these approximations lead to formal power series in a small parameter which are generically divergent. It is crucial, both conceptually and technically, to make sense of these series. The most systematic way to do this is the theory of resurgence. So we have the first goal:

$$\boxed{\textbf{Make sense of divergent series}} \tag{1.1}$$

Let us consider some examples in mathematics and physics.

1) *ODEs*. ODEs can be often solved by a power series expansion. However, these are not always analytic. For example, solutions of linear ODEs around an irregular singular point will be factorially divergent. A typical example is the Airy ODE,

$$y''(x) = xy(x), \tag{1.2}$$

which leads to the Airy functions. This equation has two independent formal power series solutions around $x = \infty$, of the form

$$\begin{aligned} \Phi_{\rm Ai}(x) &= \frac{1}{2x^{1/4}\sqrt{\pi}} {\rm e}^{-2x^{3/2}/3} \varphi_1(x^{-3/2}), \\ \Phi_{\rm Bi}(x) &= \frac{1}{2x^{1/4}\sqrt{\pi}} {\rm e}^{2x^{3/2}/3} \varphi_2(x^{-3/2}), \end{aligned} \tag{1.3}$$

where

$$\begin{aligned} \varphi_{1,2}(z) &= \sum_{n=0}^{\infty} \frac{1}{2\pi} \left(\mp\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!} z^n \\ &= 1 \pm \frac{5}{48} z + \frac{385}{4608} z^2 \pm \frac{85085}{663552} z^3 + \cdots \end{aligned} \tag{1.4}$$

These series provide asymptotic expansions for the Airy functions $\mathrm{Ai}(x)$, $\mathrm{Bi}(x)$. For example, when $x > 0$, we have

$$\mathrm{Ai}(x) \sim \Phi_{\mathrm{Ai}}(x). \tag{1.5}$$

This means that, for a given value of x, we can obtain an approximate value of $\mathrm{Ai}(x)$ by *truncating* appropriately (optimally) the formal power series appearing in the r.h.s. The number of terms that we can keep in such an *optimal truncation* depends on the value of x (and increases with x), but there is always an unavoidable error that is made in approximating the value of the function with the truncated asymptotics.

2) *Perturbation theory in quantum mechanics.* Let us consider the *quartic oscillator.* It is described by the Hamiltonian,

$$\mathsf{H} = \frac{\mathsf{p}^2}{2} + \frac{\mathsf{q}^2}{2} + \frac{g\mathsf{q}^4}{4}, \qquad g > 0, \tag{1.6}$$

where p, q are Heisenberg operators on $L^2(\mathbb{R})$ with canonical commutation relations $[\mathsf{q}, \mathsf{p}] = \mathrm{i}\hbar$. There are various rigorous results on the spectrum of this Hamiltonian, regarded as an operator on $L^2(\mathbb{R})$. Since it involves a confining potential, with

$$V(q) = \frac{q^2}{2} + \frac{gq^4}{4} \to \infty, \qquad |q| \to \infty, \tag{1.7}$$

one can show (see for example [1]) that H^{-1} is compact and positive, so that H has a discrete, positive, non-degenerate spectrum $E_n(g)$, $n = 0, 1, 2, \cdots$, with

$$0 < E_0(g) < E_1(g) < \cdots \tag{1.8}$$

The asymptotic expansion of $E_n(g)$, for small g, can be calculated by using stationary perturbation theory. For example, for the ground state energy $E_0(g)$ one finds

$$E_0(g) \sim \varphi(g), \tag{1.9}$$

where

$$\varphi(g) = \sum_{n \geq 0} a_n g^n = \frac{1}{2} + \frac{3}{4}\left(\frac{g}{4}\right) - \frac{21}{8}\left(\frac{g}{4}\right)^2 + \frac{333}{16}\left(\frac{g}{4}\right)^3 + \mathcal{O}(g^4). \tag{1.10}$$

Here, we set $\hbar = 1$. It is known that the coefficients in this series, a_n, grow factorially [2],

$$a_n \sim \left(\frac{3}{4}\right)^n (-1)^{n+1} n!, \qquad n \gg 1. \tag{1.11}$$

Therefore, perturbation theory gives a divergent series. One important question is *whether* (and *how*) one can reconstruct the *exact* $E_0(g)$ from this asymptotic series. The only thing that we know from classical asymptotics is that one can *approximate* $E_0(g)$ by using for example an optimal truncation of the asymptotic series. It may happen however that, in order to reconstruct $E_0(g)$, one needs more information than just what is contained in the perturbative series. This is in fact generally the case, and as we will see leads to the introduction of *trans-series*.

3) *Perturbative series in quantum field theory.* A rich source of asymptotic series is the saddle-point approximation to integrals. For example, solutions to the Airy equation can be constructed as integrals,

$$y(x) = \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{C}} \mathrm{e}^{xw - w^3/3} \mathrm{d}w, \tag{1.12}$$

where $\mathcal{C}$ is such that the integral converges. It is easy to show that there are two saddle-points for this integral. To see this, we write

$$x = r\mathrm{e}^{\mathrm{i}\kappa} \tag{1.13}$$

and rescale the integrand

$$w = ur^{\frac{1}{2}}. \tag{1.14}$$

We find in this way

$$y(x) = \frac{r^{\frac{1}{2}}}{2\pi\mathrm{i}} \int_{\mathcal{C}} \mathrm{e}^{r^{3/2}(\mathrm{e}^{\mathrm{i}\kappa}u - u^3/3)} \mathrm{d}u. \tag{1.15}$$

We then study the integral

$$F_\kappa(\lambda) = \int_{\mathcal{C}} \mathrm{e}^{\lambda S_\kappa(u)} \mathrm{d}u \tag{1.16}$$

where

$$S_\kappa(u) = \mathrm{e}^{\mathrm{i}\kappa}u - \frac{u^3}{3}. \tag{1.17}$$

There are *two* saddle points:

$$u_0^{\mathrm{R}} = \mathrm{e}^{\mathrm{i}\kappa/2}, \qquad u_0^{\mathrm{L}} = -\mathrm{e}^{\mathrm{i}\kappa/2} \tag{1.18}$$

with

$$\begin{aligned} S_\kappa(u_0^{\mathrm{R}}) &= \frac{2}{3}\cos\left(\frac{3}{2}\kappa\right) + \mathrm{i}\frac{2}{3}\sin\left(\frac{3}{2}\kappa\right) = \frac{2}{3}\mathrm{e}^{3\mathrm{i}\kappa/2}, \\ S_\kappa(u_0^{\mathrm{L}}) &= -\frac{2}{3}\cos\left(\frac{3}{2}\kappa\right) - \mathrm{i}\frac{2}{3}\sin\left(\frac{3}{2}\kappa\right) = -\frac{2}{3}\mathrm{e}^{3\mathrm{i}\kappa/2}. \end{aligned} \tag{1.19}$$

We then find two different expansions of the form:

$$\Phi_{\mathrm{Ai,Bi}}(x) \sim \mathrm{e}^{\mp 2x^{3/2}/3}. \tag{1.20}$$

In quantum field theory and quantum mechanics, many quantities can be represented as path integrals, of the form

$$Z = \int \mathcal{D}\,\phi(x)\,\mathrm{e}^{-S(\phi(x))/\hbar} \tag{1.21}$$

The construction of formal saddle-point expansions of this path integral is well-understood in many theories. Saddle-points correspond to *classical* solutions, i.e. solutions of the classical EOM

$$\frac{\delta S(\phi)}{\delta \phi} = 0. \tag{1.22}$$

Non-trivial solutions to this equation are called *instantons*. Expansions around these solutions lead to perturbative expansions (in general, in the background of an instanton). There exist many techniques to construct these expansions in QFT, and one obtains in this way formal power series in $\hbar$ which are in general factorially divergent.

2 Elementary resurgent technology

2.1 Borel transform

The *Borel transform* acts on formal power series as follows

$$\begin{aligned} \mathcal{B} : \mathbb{C}[[z]] &\to \mathbb{C}[[\zeta]], \\ z^n &\mapsto \zeta^n/n! \end{aligned} \tag{2.1}$$

Therefore, if we write the starting series as

$$\varphi(z) = \sum_{n\geq 0} a_n z^n \tag{2.2}$$

its Borel transfom will be given by

$$\widehat{\varphi}(\zeta) = \sum_{n\geq 0} a_n \frac{\zeta^n}{n!}. \tag{2.3}$$

Definition 2.1. We say that a formal power series $\varphi(z)$ is *Gevrey-1* if there exist two constants $M, \rho > 0$ such that

$$|a_n| < M n! \rho^n. \tag{2.4}$$

The following result is elementary:

Lemma 2.2. *If $\varphi(z)$ is a Gevrey-1 series, its Borel transform is analytic in a neigbourhood of $\zeta = 0$.*

Example 2.3. Consider the series

$$\varphi(z) = \sum_{k=0}^{\infty} \frac{\Gamma(k+b)}{\Gamma(b)} A^{-k} z^k. \tag{2.5}$$

This series is Gevrey-1. The Borel transform is given by

$$\widehat{\varphi}(\zeta) = \sum_{k=0}^{\infty} \frac{\Gamma(k+b)}{k!\Gamma(b)} A^{-k} \zeta^k = (1-\zeta/A)^{-b}, \tag{2.6}$$

which has a singularity at $\zeta = A$. If $b = 1$, this singularity is a pole. If $0 < b < 1$, it is a branch point. The case $b = 0$ corresponds to a logarithmic singularity. More precisely, if

$$\varphi(z) = \sum_{k=0}^{\infty} \Gamma(k) A^{-k} z^k, \tag{2.7}$$

we have

$$\widehat{\varphi}(\zeta) = \sum_{k\geq 0} \frac{1}{k} A^{-k} \zeta^k = -\log\left(1 - \frac{\zeta}{A}\right). \tag{2.8}$$

□

One of the key ideas of resurgence is that *the singularities of the Borel transform contain information about additional sectors of the theory.* To find these additional sectors, we will introduce a basic category of formal power series:

Definition 2.4. *Resurgent function.* A resurgent function is a Gevrey-1 series $\varphi(z)$ whose Borel transform has the following property: on any line issuing from the origin, there is a finite set of points (the singularities of the Borel transform) such that $\widehat{\varphi}(\zeta)$ may be continued analytically along any path that follows the line, while circumventing (from above or from below) those singular points.

We will assume that our functions are resurgent. In fact, to simplify our life, we will most of the time assume that the singularities of the Borel transform are poles or logarithmic branch cuts. In this case, the resurgent function is called *simple*. The local expansion of $\widehat{\varphi}(\zeta)$ is of the form

$$\widehat{\varphi}(\zeta_\omega + \xi) = -\frac{a}{2\pi\xi} - \frac{1}{2\pi}\log(\xi)\sum_{n\geq 0}\widehat{c}_n\xi^n + \text{regular}, \tag{2.9}$$

where the series

$$\widehat{\varphi}_\omega(\xi) = \sum_{n\geq 0}\widehat{c}_n\xi^n \tag{2.10}$$

has a finite radius of convergence. We will regard $\widehat{\varphi}_\omega(\xi)$ as the Borel transform of

$$\varphi_\omega(z) = \sum_{n\geq 0} c_n z^n, \qquad c_n = n!\widehat{c}_n. \tag{2.11}$$

A more general case involves functions with branch cuts of the form

$$(\zeta_\omega - \zeta)^{-b}, \qquad 0 < b < 1. \tag{2.12}$$

In this case, we have the local expansion

$$\widehat{\varphi}(\zeta_\omega + \xi) = (-\xi)^{-b}\sum_{n\geq 0}\widehat{c}_n\xi^n + \text{regular}, \tag{2.13}$$

and we will regard $\widehat{\varphi}_\omega(\xi)$ as the Borel transform of

$$\varphi_\omega(z) = \sum_{n\geq 0} c_n z^n, \qquad c_n = \Gamma(n+1-b)\widehat{c}_n. \tag{2.14}$$

The reason for transforming these convergent series into formal power series will be understood shortly.

The key result so far is that, given a formal power series $\varphi(z)$, the expansion of its Borel transform around its singularities generates additional formal power series:

$$\varphi(z) \to \{\varphi_\omega(z)\}_{\omega\in\mathcal{S}}, \tag{2.15}$$

where $\mathcal{S}$ denotes the set of singular points.

Example 2.5. Let us consider the formal power series $\varphi_1(z)$ appearing in (1.4). In this case, the Borel transform can be computed explicitly,

$$\widehat{\varphi}(\zeta) = {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; -\frac{3\zeta}{4}\right). \tag{2.16}$$

This has a singularity at

$$\zeta_\omega = -\frac{4}{3}. \tag{2.17}$$

It is easy to see from the general theory of hypergeometric functions that this is a logarithmic singularity, and one finds

$$\widehat{\varphi}_1(\zeta) = -\frac{1}{2\pi}\log\left(\zeta + \frac{4}{3}\right){}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; \frac{3\zeta}{4}\right) + \text{regular} \tag{2.18}$$

therefore

$$\varphi_{-4/3}(z) = \varphi_2(z) = \sum_{n\geq 0} \frac{1}{2\pi} \left(\frac{3}{4}\right)^n \frac{\Gamma(n+5/6)\Gamma(n+1/6)}{n!} z^n. \tag{2.19}$$

This is the other formal power series that appears when solving the Airy equation.

□

The above procedure can be repeated with the new formal power series $\varphi_\omega(z)$, to generate yet more power series. This might lead to a finite set of functions (this is the case of the Airy functions). Generically, however, one finds an infinite set of formal power series. We will label such series by the index ω, and we will write then as $\varphi_\omega(z)$. Let us assume that they are all simple resurgent functions. Then, we have the general relation

$$\widehat{\varphi}_\omega(\zeta_{\omega'} + \xi) = -\mathsf{S}_{\omega\omega'} \frac{\log(\xi)}{2\pi} \widehat{\varphi}_{\omega'}(\xi) + \text{regular}. \tag{2.20}$$

The constants $\mathsf{S}_{\omega\omega'}$ are called *Stokes constants*.

Example 2.6. In the Airy example, the Stokes constants are simply

$$\mathsf{S}_{12} = \mathsf{S}_{21} = 1. \tag{2.21}$$

The above structure can be extended to a more general type of singularity, with e.g. a branch cut structure.

We can now understand the origin of the name "resurgence". We have seen that the singularities of the Borel transform lead to new power series. It turns out that these new series "resurge" in the original series through the behavior of the coefficients a_k when k is large. In terms of the Borel transform (which is analytic at the origin), this is essentially an old theorem of Darboux, which relates the large order behavior of the coefficients of an analytic function at the origin, to the behavior near the closest singularity (see e.g. [3]).

Let us first state the result. Let $\varphi(z)$ a simple resurgent function as in (2.2). Let A be the singularity of the Borel transform which is closest to the origin in the complex plane (we will assume for simplicity that there is only one, although the generalization is straightforward). Let the behavior near this singularity be as in (2.9), with $\zeta_\omega = A$. Then, the coefficients a_k have the following asymptotic behavior,

$$a_k \sim \frac{1}{2\pi} \sum_{n\geq 0} A^{-k+n} c_n \Gamma(k-n), \qquad k \gg 1. \tag{2.22}$$

To understand this formula better, it is convenient to write explicitly the very first terms:

$$a_k \sim \frac{1}{2\pi} A^{-k}\Gamma(k) \left\{ c_0 + \frac{c_1 A}{k-1} + \frac{c_2 A^2}{(k-1)(k-2)} + \cdots \right\}, \qquad k \gg 1. \tag{2.23}$$

The first factor in the r.h.s. gives the leading factorial asymptotics, while the second factor gives a series of corrections in $1/k$ to the leading factorial behavior. These corrections involve the coefficients c_n of the power series obtained in (2.11). One can use this asymptotic formula in two ways: as a procedure to extract the numbers A, c_n from the knowledge of the series a_k, of conversely, as a way to obtain the large order asymptotics of these coefficients once A, c_n are known.

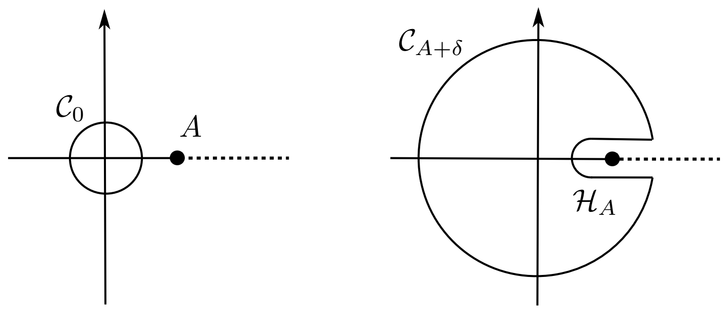

Figure 1. Contour deformation in (2.25).

Let us sketch a proof of this result. The coefficients of the Borel transform are given by the Cauchy formula

$$\frac{a_k}{k!} = \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{C}_0} \frac{\widehat{\varphi}(\zeta)}{\zeta^{k+1}} \mathrm{d}\zeta, \tag{2.24}$$

where $\mathcal{C}_0$ is a small circle around $\zeta = 0$. Let us choose a $\delta > 0$. We now enlarge the contour $\mathcal{C}_0$ to a contour $\mathcal{C}_{A+\delta} \cup \mathcal{H}_A$, where $\mathcal{C}_{A+\delta}$ is a circle of radius $A+\delta$, minus an arc, and $\mathcal{H}_A$ is a Hankel contour centered around A, see Fig. 1. By deforming the contour we find

$$\frac{a_k}{k!} = \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{C}_{A+\delta}} \frac{\widehat{\varphi}(\zeta)}{\zeta^{k+1}} \mathrm{d}\zeta + \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{H}_A} \frac{\widehat{\varphi}(\zeta)}{\zeta^{k+1}} \mathrm{d}\zeta. \tag{2.25}$$

The first integral can be estimated to be of order $\mathcal{O}\left((A+\delta)^{-k}\right)$. Since, as we will now show, the leading large k asymptotics goes like A^{-k}, and $A+\delta > A$, this is a subleading, exponentially small correction as k grows large, and it does not contribute to the leading $1/k$ asymptotics. The integral around the contour $\mathcal{H}_A$ can be evaluated by using (2.9). and we obtain

$$\frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{H}_A} \frac{\widehat{\varphi}(\zeta)}{\zeta^{k+1}} \mathrm{d}\zeta = \frac{1}{2\pi} \sum_{n\geq 0} \hat{c}_n \int_0^{\delta} \frac{\xi^n}{(A+\xi)^{k+1}} \mathrm{d}\xi, \tag{2.26}$$

where we have set $\zeta = A + \xi$. An easy estimate shows that, at fixed n,

$$\begin{aligned} \int_0^{\delta} \frac{\xi^n}{(A+\xi)^{k+1}} \mathrm{d}\xi &= \int_0^{\infty} \frac{\xi^n}{(A+\xi)^{k+1}} \mathrm{d}\xi + \mathcal{O}\left((A+\delta)^{-k}\right) \\ &= A^{n-k} \frac{\Gamma(k-n)\Gamma(n+1)}{\Gamma(k+1)} + \mathcal{O}\left((A+\delta)^{-k}\right). \end{aligned} \tag{2.27}$$

We conclude that the asymptotics of a_k is given by (2.22).

Remark 2.7. When the Borel transform has a branch cut as in (2.13), a similar argument leads to

$$a_k \sim \frac{1}{2\pi} \sum_{n\geq 0} A^{-k-b+n} c_n \Gamma(k+b-n). \tag{2.28}$$

The asymptotics (2.22) implies that the coefficients c_n determining the series $\varphi_A(z)$ are encoded in the large order behavior of a_k. This has multiple applications. In particular, when there is no easy method to determine $\varphi_A(z)$, one can extract it from the knowledge of the large order behavior. This has become one of the most powerful heuristic methods in resurgence.

Example 2.8. In the case of the series $\varphi_1(z)$ in (1.4), the above result leads to the asymptotics

$$\begin{aligned} a_k &\sim \frac{1}{2\pi}\left(-\frac{4}{3}\right)^{-k}(k-1)!\sum_{n\geq 0} c_n\left(-\frac{4}{3}\right)^n\frac{\Gamma(k-n)}{\Gamma(k)} \\ &\sim \frac{1}{2\pi}\left(-\frac{4}{3}\right)^{-k}(k-1)!\left(1-\frac{4}{3}\cdot\frac{5}{48}\frac{1}{k-1}+\frac{16}{9}\cdot\frac{385}{4608}\frac{1}{(k-1)(k-2)}+\cdots\right). \end{aligned} \tag{2.29}$$

Since we have a closed form for a_k, we can calculate the asymptotics directly:

$$a_k = \frac{1}{2\pi}\left(-\frac{3}{4}\right)^k\frac{\Gamma(k+\frac{5}{6})\Gamma(k+\frac{1}{6})}{k!}, \tag{2.30}$$

and one indeed verifies that

$$\frac{\Gamma(k+\frac{5}{6})\Gamma(k+\frac{1}{6})}{k!(k-1)!} \sim 1-\frac{4}{3}\cdot\frac{5}{48}\frac{1}{k-1}+\frac{16}{9}\cdot\frac{385}{4608}\frac{1}{(k-1)(k-2)}+\cdots, \qquad k\gg 1. \tag{2.31}$$

□

2.2 Borel resummation and Stokes automorphism

Definition 2.9. Let ζ_ω be a singularity of $\widehat{\varphi}(\zeta)$. A ray in the Borel plane which starts at the origin and passes through ζ_ω is called a *Stokes ray.* It is of the form $\mathrm{e}^{\mathrm{i}\theta}\mathbb{R}_+$, where $\theta = \arg(\zeta_\omega)$.

Note that a Stokes ray might pass through many singularities.

Definition 2.10. Let $\varphi(z)$ a Gevrey-1 formal power series series, $z \in \mathbb{C}$, and $\theta = \arg z$. If $\widehat{\varphi}(\zeta)$ analytically continues to an L^1-analytic function along the ray $\mathcal{C}^\theta := \mathrm{e}^{\mathrm{i}\theta}\mathbb{R}_+$ we define its Laplace transform by

$$s(\varphi)(z) = \int_0^\infty \widehat{\varphi}(z\zeta)\mathrm{e}^{-\zeta}\mathrm{d}\zeta = \frac{1}{z}\int_{\mathcal{C}^\theta}\widehat{\varphi}(\zeta)\mathrm{e}^{-\zeta/z}\mathrm{d}\zeta. \tag{2.32}$$

The function $s(\varphi)(z)$ is often called the *Borel resummation* of the formal power series φ.

Let us first note that, if $s(\varphi)(z)$ exists, its asymptotic behavior for small z can be obtained by expanding the integrand and integrating term by term:

$$s(\varphi)(z) \sim \sum_{n\geq 0} a_n z^n. \tag{2.33}$$

This is the formal power series that we started with. Therefore, if we are lucky, Borel resummation produces an actual function which reproduces the original series. It is then a way to "make sense" of our original formal power series.

If we vary $\theta = \arg z$ and we do not encounter singularities of $\widehat{\varphi}$, the function $s_\theta(\varphi)(z)$ is locally analytic. However, when $\theta = \arg(z)$ crosses a Stokes ray, the Borel resummation has a discontinuity. To define this discontinuity more precisely, we introduce *lateral Borel resummations.*

Definition 2.11. Let $\mathcal{C}^\theta_\pm$ be contours starting at the origin and going slightly above (respectively, below) the Stokes ray. Then,

$$s_\pm(\varphi)(z) = \frac{1}{z}\int_{\mathcal{C}^\theta_\pm}\widehat{\varphi}(\zeta)\mathrm{e}^{-\zeta/z}\mathrm{d}\zeta. \tag{2.34}$$

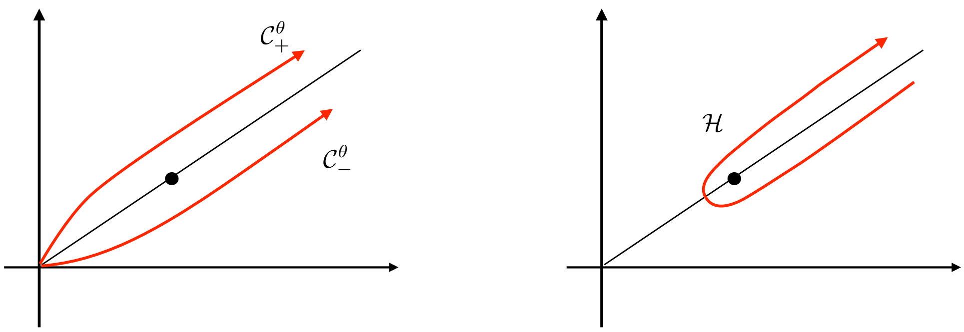

Figure 2. Contour deformation in the calculation of the discontinuity.

The discontinuity is then simply given by

$$s_+(\varphi)(z) - s_-(\varphi)(z). \tag{2.35}$$

Note that, since $s_\pm(\varphi)(z)$ have the same asymptotics for small z, given in (2.33), the discontinuity must be invisible as an asymptotic expansion. As we will now see, this difference is *exponentially small* and closely related to the local structure of the Borel transform. Indeed, let us assume that $\varphi(z)$ is a simple resurgent function, and that there is an isolated logarithmic singularity ζ_ω in the Stokes ray. The difference between the two contours $\mathcal{C}_+^\theta - \mathcal{C}_-^\theta$ can be deformed into a Hankel contour $\mathcal{H}$ around the logarithmic branch cut. We then have,

$$\begin{aligned} s_+(\varphi)(z) - s_-(\varphi)(z) &= \frac{1}{z}\oint_{\mathcal{H}} \widehat{\varphi}(\zeta)\mathrm{e}^{-\zeta/z}\mathrm{d}\zeta = \mathrm{i}\frac{\mathrm{e}^{-\zeta_\omega/z}}{z}\int_{\mathcal{C}_-^\theta} \widehat{\varphi}_\omega(\xi)\mathrm{e}^{-\xi/z}\mathrm{d}\xi \\ &= \mathrm{i}\,\mathrm{e}^{-\zeta_\omega/z} s_-(\varphi_\omega)(z). \end{aligned} \tag{2.36}$$

In the first line we have written $\zeta = \zeta_\omega + \xi$. More generally, if there are many singularities, we find

$$s_+(\varphi_\omega)(z) - s_-(\varphi_\omega)(z) = \mathrm{i}\sum_{\omega'} \mathsf{S}_{\omega\omega'}\mathrm{e}^{-\zeta_{\omega'}/z} s_-(\varphi_{\omega'})(z), \tag{2.37}$$

where the sum over ω' runs over the singularities in the Stokes ray.

2.3 Trans-series

One of the most important implications of resurgence is that, in order to reconstruct actual functions through Borel resummation, we need, in addition to the "obvious" perturbative series, all additional series $\varphi_\omega(z)$ appearing in the resurgent structure. This suggest the following definition

Definition 2.12. Let $\varphi_\omega(z)$ resurgent functions. A *trans-series* is a (possibly infinite) formal linear combination of formal power series

$$\Phi(z;\boldsymbol{C}) = \sum_\omega C_\omega \mathrm{e}^{-\zeta_\omega/z}\varphi_\omega(z), \tag{2.38}$$

where $\boldsymbol{C} = (C_{\omega_1}, \cdots)$ is a vector of complex numbers.

Example 2.13. The need for trans-series can be seen already in the classical asymptotic theory of functions defined by ODEs. Let us suppose that we want to reconstruct the Airy function $\mathrm{Ai}(x)$ out of Borel resummations of the formal power series $\varphi_{1,2}(z)$. It turns out that the answer depends on the argument of x.

1. If $|\arg(x)| < 2\pi/3$, the Airy function is obtained as

$$\mathrm{Ai}(x) = \frac{1}{2x^{1/4}\sqrt{\pi}} \mathrm{e}^{-2x^{3/2}/3} s(\varphi_1)(z), \qquad z = x^{-3/2}. \tag{2.39}$$

2. If $|\arg(x) - \pi| < \frac{\pi}{3}$ we have instead

$$\mathrm{Ai}(x) = \frac{1}{2x^{1/4}\sqrt{\pi}} \left\{ \mathrm{e}^{-2x^{3/2}/3} s(\varphi_1)(z) + \mathrm{i}\mathrm{e}^{2x^{3/2}/3} s(\varphi_2)(z) \right\}, \qquad z = x^{-3/2}. \tag{2.40}$$

   For example, when $x < 0$, we have

$$\mathrm{Ai}(-x) = \frac{1}{2x^{1/4}\sqrt{\pi}} \left\{ \mathrm{e}^{-\pi\mathrm{i}/4 + 2\mathrm{i}x^{3/2}/3} s(\varphi_1)(\mathrm{i}x^{-3/2}) + \mathrm{e}^{\pi\mathrm{i}/4 - 2\mathrm{i}x^{3/2}/3} s(\varphi_2)(\mathrm{i}x^{-3/2}) \right\}. \tag{2.41}$$

   This leads to the well-known oscillatory behavior of the Airy function along the negative real axis,

$$\mathrm{Ai}(-x) \sim \frac{x^{-1/4}}{\sqrt{\pi}} \cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right), \qquad x \to \infty. \tag{2.42}$$

□

Example 2.14. *Non-linear ODEs.* In the Airy case, there are only two formal power series in the full resurgent structure. However, in the case of *non-linear* ODEs one needs an infinite number of formal power series to reconstruct solutions to the ODE. An example of this situation is the Painlevé II equation,

$$u''(\kappa) - 2u^3(\kappa) + 2\kappa u(\kappa) = 0, \tag{2.43}$$

which plays an important rôle in many areas of mathematical physics. There is a formal solution to PII which goes like $u(\kappa) \sim \sqrt{\kappa}$ as $\kappa \to \infty$:

$$u^{(0)}(\kappa) = \sqrt{\kappa} - \frac{1}{16\,\kappa^{\frac{5}{2}}} - \frac{73}{512\,\kappa^{\frac{11}{2}}} - \frac{10657}{8192\,\kappa^{\frac{17}{2}}} - \frac{13912277}{542888\,\kappa^{\frac{23}{2}}} + \cdots, \qquad \kappa \to \infty. \tag{2.44}$$

In the case of non-linear ODEs, the trans-series can be systematically constructed as follows. Let us consider a formal solution with the structure,

$$u(\kappa; C) = \sum_{\ell=0}^{\infty} C^\ell u^{(\ell)}(\kappa), \tag{2.45}$$

where C is a constant, and $u^{(0)}$ is the formal solution obtained above. By plugging this *ansatz* in the original equation, we find that each $u^{(\ell)}$ satisfies a linear ODE. For example, one finds immediately that

$$(u^{(1)})'' + 2\kappa u^{(1)} - 6(u^{(0)})^2 u^{(1)} = 0. \tag{2.46}$$

Work of Jean Écalle, Ovidiu Costin and others (see [4] for a wonderful introduction, and [5] for a in-depth treatment) shows that the $u^{(\ell)}(\kappa)$ obtained in this way are trans-series, in the sense that they can be obtained through the Borel transform of the formal series $u^{(0)}(\kappa)$. They lead to exponentially small corrections to the leading asymptotic behavior given by $u^{(0)}$, and they have the form

$$u^{(\ell)}(\kappa) = \kappa^{-\frac{3\ell}{4}} \mathrm{e}^{-\ell A \kappa^{3/2}} \epsilon^{(\ell)}(\kappa), \quad \kappa \to \infty, \tag{2.47}$$

where

$$A = \frac{4}{3}, \qquad \epsilon^{(\ell)}(\kappa) = \sum_{n=0}^{\infty} u_{\ell,n} \kappa^{-3n/2} \tag{2.48}$$

It can be shown that general solutions to Painlevé II with the asymptotics (2.44) can be written as an infinite sum of lateral Borel resummations

$$u(\kappa) = \sum_{\ell=0}^{\infty} C^{\ell} s_{+}(u^{(\ell)})(\kappa) \tag{2.49}$$

for an appropriate value of C. □

Let us make some conceptual comments on trans-series:

1. Trans-series involve at least two "small parameters": the first small parameter is the one appearing in the original, "perturbative" series. There is also an *exponentially small parameter* $\mathrm{e}^{-\zeta_\omega/z}$. In this sense, trans-series go beyond classical asymptotics by including exponentially small corrections

2. All series in the first small parameter are factorially divergent.

3. The different series appearing in the trans-series are not independent. For example, the large order behavior of the terms in the leading perturbative series are controlled by the first series in trans-series (the one corresponding to the smallest singularity, in absolute value).

We will now state a principle of *semiclassical decoding*.

Definition 2.15. (Semiclassical decoding). Let $f(z)$ be a function with the asymptotic expansion

$$f(z) \sim \varphi(z) = \sum_{n \geq 0} a_n z^n. \tag{2.50}$$

We say that $f(z)$ admits a *semiclassical decoding* if $\varphi(z)$ can be promoted to a trans-series $\Phi(z; \boldsymbol{C})$, which is lateral Borel summable, and such that

$$f(z) = s_{\pm}(\Phi)(z; \boldsymbol{C}_{\pm}) \tag{2.51}$$

for some vectors of complex constants $\boldsymbol{C}_{\pm}$.

When semiclassical decoding holds, one recovers the exact information by just considering Borel-resummed trans-series.

The simplest situation corresponds to the case in which $C = 0$, there are no singularities along the positive real axis, and the Borel resummation of the perturbative series reproduces the exact result. This is famously the case for the perturbative series (1.10) of the quartic oscillator.

An important question in quantum theory is whether well-defined functions in QM and QFT admit a semiclassical decoding. What we could call the *strong program of resurgence* is the conjectural statement that *every observable in QM/QFT can be written as the lateral Borel resummation of an appropriate trans-series.*

The program of semiclassical decoding was very active in QFT after the discovery of instantons, but then it suffered an important drawback in the late 70's when it was shown that asymptotically free theories in *infinite volume* do not admit a simple semiclassical decoding, and that trans-series made out of instantons are not applicable in that case. However, there are recent results indicating that more general trans-series, not based on classical solutions of the equations of motion, can be used to reconstruct the exact answer.

3 Resurgence and the WKB method

An important application of resurgence is to the WKB method in one-dimensional quantum mechanics, which is then upgraded to the so-called "exact WKB method". One consequence of this application is the result that energy levels and resonances in one-dimensional quantum mechanics can be "semiclassically decoded" in terms of the perturbative WKB series and exponentially small corrections to it.

3.1 The WKB method

The WKB method is a systematic approximation scheme to solve the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x). \tag{3.1}$$

The result of this approximation is an expression for the wavefunction as a formal power series in $\hbar$. In particular, as we will see, in the analysis of bound state problems, the WKB method gives estimates of the energy spectrum when $\hbar$ is small as compared to the classical action of the problem. This happens when the quantum number labelling the bound state is very large. The resulting approximation often provides an accurate description of the energy spectrum, and it has the advantage that it can be formulated in terms of classical quantities.

In the WKB method, we treat $\hbar$ as a small parameter and we perform a systematic expansion of all quantities in a power series in $\hbar$, around $\hbar = 0$. However, if we write (3.1) as

$$\hbar^2\psi''(x) + p^2(x)\psi(x) = 0, \qquad p(x) = \sqrt{2m(E - V(x))}, \tag{3.2}$$

it is clear that we can not send $\hbar \to 0$ directly, since in this limit the equation becomes algebraic. Let us however write the wavefunction as

$$\psi(x) = \exp\left[\frac{\mathrm{i}}{\hbar}\int^x Y(x')\mathrm{d}x'\right]. \tag{3.3}$$

When we do this, we transform the Schrödinger equation into a Riccati equation for $Y(x)$,

$$Y^2(x) - \mathrm{i}\hbar\frac{\mathrm{d}Y(x)}{\mathrm{d}x} = p^2(x), \tag{3.4}$$

which can be solved in power series in $\hbar$:

$$Y(x) = \sum_{k=0}^{\infty} Y_k(x)\hbar^k. \tag{3.5}$$

We will regard this as a *formal* power series, i.e. we will not address issues of convergence for the moment being. The functions $Y_k(x)$ can be computed recursively as

$$\begin{aligned} Y_0(x) &= \pm p(x), \\ Y_1(x) &= \frac{\mathrm{i}}{2}\frac{Y_0'(x)}{Y_0(x)}, \\ Y_{n+1}(x) &= \frac{1}{2Y_0(x)}\left(\mathrm{i}\frac{\mathrm{d}Y_n(x)}{\mathrm{d}x} - \sum_{k=1}^{n} Y_k(x)Y_{n+1-k}(x)\right), \qquad n \geq 1. \end{aligned} \tag{3.6}$$

The two choices of sign for $Y_0(x)$ will give the two independent solutions of the Schrödinger equation. Let us split the formal series $Y(x)$ into two series of even and odd powers of $\hbar$,

$$P(x) = \sum_{k=0}^{\infty} Y_{2k}(x)\hbar^{2k}, \qquad Y_{\mathrm{odd}}(x) = \sum_{k=0}^{\infty} Y_{2k+1}(x)\hbar^{2k+1}, \tag{3.7}$$

so that

$$Y(x) = P(x) + Y_{\mathrm{odd}}(x). \tag{3.8}$$

The Riccati equation (3.4) splits into two different equations:

$$\begin{aligned} &\text{even:} \quad Y_{\mathrm{odd}}^2(x) + P^2(x) - \mathrm{i}\hbar Y_{\mathrm{odd}}'(x) = p^2(x), \\ &\text{odd:} \quad 2Y_{\mathrm{odd}}(x)P(x) - \mathrm{i}\hbar P'(x) = 0. \end{aligned} \tag{3.9}$$

The second equation can be solved in closed form:

$$Y_{\mathrm{odd}}(x) = \frac{\mathrm{i}\hbar}{2}\frac{P'(x)}{P(x)} = \frac{\mathrm{i}\hbar}{2}\frac{\mathrm{d}}{\mathrm{d}x}\log P(x). \tag{3.10}$$

Therefore,

$$\frac{\mathrm{i}}{\hbar}\int^x Y(x')\mathrm{d}x' = -\frac{1}{2}\log P(x) + \frac{\mathrm{i}}{\hbar}\int^x P(x')\mathrm{d}x', \tag{3.11}$$

and the wavefunction reads

$$\psi(x) = \frac{1}{\sqrt{P(x)}}\exp\left(\frac{\mathrm{i}}{\hbar}\int^x P(x')\mathrm{d}x'\right). \tag{3.12}$$

We can think of $P(x)\mathrm{d}x$ as a "quantum differential" which promotes the Liouville differential form $p(x)\mathrm{d}x$ to a formal power series in $\hbar$.

3.2 Quantization conditions

In confining potentials, leading to a discrete spectrum, approximate quantization conditions can be obtained by considering period integrals of the quantum differential. Let us suppose that we are in a situation with two turning points $x_\pm$, defined by the condition $V(x) = E$. Let us consider a cycle $\mathcal{A}$ in the complex x-plane surrounding the interval of classical motion, as shown in Fig. 3. The Bohr–Sommerfeld quantization condition says that the approximate spectrum can be obtained by solving the equation

$$\mathrm{vol}_0(E) = 2\pi\hbar\left(n + \frac{1}{2}\right), \tag{3.13}$$

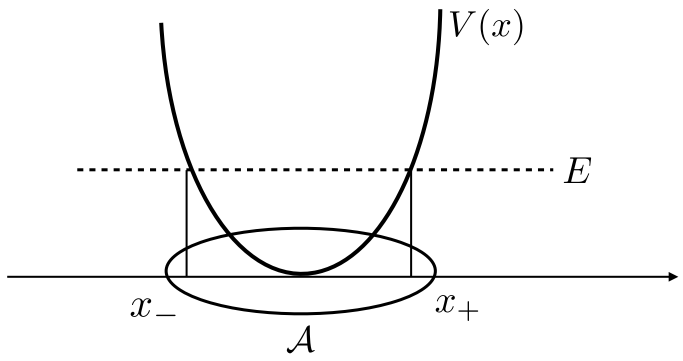

Figure 3. A contour $\mathcal{A}$ in the complex plane, encircling the turning points.

where

$$\mathrm{vol}_0(E) = \oint_{\mathcal{A}} p(x)\mathrm{d}x, \tag{3.14}$$

is the classical volume in phase space. This quantity can be promoted to a formal power series in $\hbar^2$:

$$\mathrm{vol}(E;\hbar) = \oint_{\mathcal{A}} P(x)\mathrm{d}x = \sum_{n=0}^{\infty} \mathrm{vol}_n(E)\hbar^{2n}, \tag{3.15}$$

This series diverges *doubly-factorially* for a fixed value of E,

$$\mathrm{vol}_n(E) \sim (2n)!, \qquad n \gg 1. \tag{3.16}$$

With this series one can obtain, at least formally, an all-orders quantization condition of the form

$$\mathrm{vol}(E;\hbar) = 2\pi\hbar\left(n + \frac{1}{2}\right) \tag{3.17}$$

which was written down by Dunham in 1929.

However, the series in the l.h.s. does not define a function, and therefore does not lead to a well defined quantization condition. Understanding this in detail is crucial to improve systematically the semi-classical, Bohr–Sommerfeld approximation. This procedure was pioneered by André Voros, and eventually led to "exact" quantization conditions for many QM problems.

An important example of this situation is the famous double-well potential. This is a symmetric, quartic potential with two degenerate minima and a local maximum at its middle point. We can take for example

$$V(x) = \frac{g^2}{2}\left(x^2 - \frac{1}{4g^2}\right)^2, \tag{3.18}$$

where g is a coupling constant. Let us consider energies below the top of the barrier in the middle:

$$E < \frac{1}{32g^2}, \tag{3.19}$$

so there are four turning points which will be labelled a, b, c, d from left to right, see Fig. 4. Due to the symmetry of the potential, we have $a = -d$, $b = -c$. Let us suppose that we want

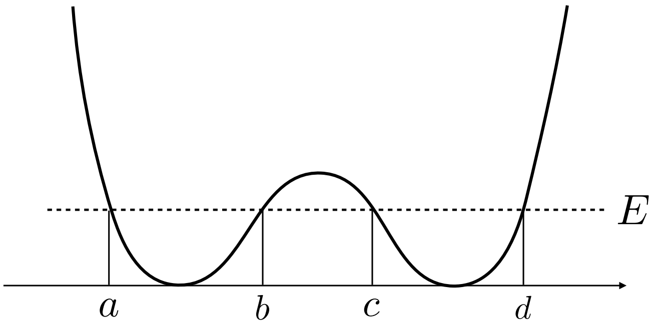

Figure 4. The double-well potential.

to calculate (even approximately) the energy of the ground state of this system. The Bohr–Sommerfeld quantization condition tells us that there are *two* degenerate ground states, whose approximate energy is given by (3.13), and the $\mathcal{A}$ cycle encircles the region of allowed motion $[a, b]$.

It turns out that this is even *qualitatively wrong*. The elementary spectral theory of the Schrödinger operator tells us that the ground state is non-degenerate.

One could think that incorporating the all-order corrections in (3.17) would solve the problem. However, it is possible to show that the resulting formal power series is *not* Borel summable along the real axis of $\hbar$. We can however consider the two lateral Borel resummations

$$s_{\pm}(\Pi_{\mathcal{A}})(\hbar), \tag{3.20}$$

where we denoted

$$\Pi_{\mathcal{A}} = \oint_{\mathcal{A}} P(x)\mathrm{d}x. \tag{3.21}$$

It turns out that a careful study of the all-orders WKB method makes it necessary to consider as well period integrals involving the "forbidden" classical region between the points b and c. Let us not that, in this region, the momentum is imaginary, and we make a choice of branch cut such that

$$p(x) = -\mathrm{i}p_1(x), \qquad p_1(x) = \sqrt{2m(V(x)-E)} > 0. \tag{3.22}$$

We now consider a cycle $\mathcal{B}$ around the forbidden integral $[b, c]$, and define the quantum period as

$$\Pi_{\mathcal{B}} = 2\int_b^c P_1(x)\mathrm{d}x \tag{3.23}$$

where $P_1(x)$ is obtained from $P(x)$ by the above choice of branch cut. It turns out that this formal power series in $\hbar$ is Borel summable for real $\hbar$. Then, by using the so-called Voros–Silverstone connection formulae, one finds the following constraint that has to be satisfied by the energy in order to obtain a square-integrable wavefunction,

$$1 + \exp\left(\pm\frac{\mathrm{i}}{\hbar}s_{\pm}(\Pi_{\mathcal{A}})(\hbar)\right) = \pm\mathrm{i}\epsilon\exp\left(-\frac{1}{2\hbar}s(\Pi_{\mathcal{B}})(\hbar)\right). \tag{3.24}$$

Here ϵ is the parity of the state. If we label the states in increasing order of energy, with a non-negative integer $n = 0, 1, \cdots$, then $\epsilon = (-1)^n$. This equation is sometimes called an "exact quantization condition", since it allows an exact calculation of the spectrum (in this form, it is valid for states whose energy is lower than the potential barrier between the two wells).

Let us make some comments on these equations.

1. One can deduce from the equation the Stokes discontinuity of the $\mathcal{A}$-quantum period. Indeed, we find

$$\mathrm{e}^{\mathrm{i}(s_+(\Pi_\mathcal{A})-s_-(\Pi_\mathcal{A}))/\hbar} = 1 + \mathrm{e}^{-s(\Pi_\mathcal{B})(\hbar)/\hbar}, \tag{3.25}$$

which can be written as

$$\mathcal{X}_\mathcal{A}^+ = \mathcal{X}_\mathcal{A}^- \left(1 + \mathcal{X}_\mathcal{B}\right), \tag{3.26}$$

where

$$\mathcal{X}_\mathcal{A}^\pm = \mathrm{e}^{\mathrm{i}s_\pm(\Pi_A)(\hbar)/\hbar}, \qquad \mathcal{X}_\mathcal{B} = \mathrm{e}^{-s(\Pi_\mathcal{B})(\hbar)/\hbar} \tag{3.27}$$

Equivalently, we have the discontinuity

$$s_+(\Pi_\mathcal{A})(\hbar) - s_-(\Pi_\mathcal{A})(\hbar) = -\mathrm{i}\hbar \log\left(1 + \mathrm{e}^{-s(\Pi_\mathcal{B})(\hbar)/\hbar}\right) \tag{3.28}$$

The discontinuity equation (3.26) is a particular case of the Delabaere–Pham formula, which was found in [6, 7] in the context of the exact WKB method. It has been rediscovered in recent developments in algebraic geometry and supersymmetric gauge theory (see e.g. [8, 9]).

2. From the discontinuity formula, we can read that the singularities of the Borel transform of $\Pi_\mathcal{A}$ are located at

$$2k \int_b^c p_1(x)\mathrm{d}x, \qquad k \in \mathbb{Z}_{>0}. \tag{3.29}$$

3. If we neglect the r.h.s. of (3.24), which is exponentially small, we recover the all-orders quantization condition (3.17). Interestingly, the correction given by the r.h.s. removes the degeneracy of the states. Each solution to (3.17) gives rise to two different energy levels, with n even and odd, respectively. This is an example in which the conventional perturbative series in $\hbar$ misses an important qualitative effect, which is only recovered after including the exponentially small effect.

4. The correction in the r.h.s. of (3.24) can be interpreted in many ways. In conventional wave mechanics, it is due to the quantum "tunneling" of the particle between the two separated wells. In the path integral formulation of quantum mechanics, it is due to a non-trivial solution of the Euclidean equations of motion connecting the two minima.

4 Resurgence and quantum topology

In topological quantum field theories, partition functions and certain correlation functions calculate topological invariants of manifolds. In many cases, these invariants can be obtained by a precise algebraic construction, but their "classical" geometric meaning is often hidden in this construction. It turns out that the semiclassical limit of the quantum theory makes it possible to recover this geometric meaning. Therefore, in this family of examples, the semiclassical decoding of the exact invariant connects an algebraic, combinatorial picture, with a geometric picture.

A famous example of such a connection is *Kashaev's volume conjecture* [10]. The invariant is in this case the so-called *colored Jones polynomial* of a knot $\mathcal{K}$ in $\mathbb{S}^3$, and denoted by $J_{\mathcal{K}}(N,k)$. It is a function of a positive integer N and an integer k, also called the level. The Jones polynomial can be defined by a series of combinatorial rules. For example, if $N=1$, it can be obtained in a simple way from a planar projection of the knot and some simple *skein rules*. From the point of view of the underlying quantum theory, k plays the rôle of $\hbar^{-1}$. It turns out that, when $\mathcal{K}$ is a hyperbolic knot (i.e. its complement admits a hyperbolic metric) co, and $k=N$, one has the all-orders asymptotic behavior

$$J_{\mathcal{K}}(N,N) \sim N^{3/2} \exp\left(\frac{V(\mathcal{K})}{2\pi} N\right) \sum_{\ell \geq 0} S_\ell(\mathcal{K}) N^{-\ell}, \qquad N \gg 1. \tag{4.1}$$

Here, $V(\mathcal{K})$ is the complexified hyperbolic volume of the complement of the knot. Note that the above limit is in particular a limit of small $\hbar$, therefore it is a semiclassical limit. The series appearing in the r.h.s. is an asymptotic series, factorially divergent, and one can ask about its resurgent structure, with the long-term hope of providing a proof of the volume conjecture[1]. This was first pointed out by S. Garoufalidis in e.g. [12]

Another, in fact simpler example of such an asymptotics concerns the WRT invariant of a three-manifold M, $W_M(k)$, and it depends again on a level k. This invariant can be defined combinatorially from a surgery presentation of M. However, when k is large, one expects the asymptotics

$$W_M(k) \sim k^a \exp\left(\frac{\mathrm{i}k}{4\pi} \mathrm{CS}(M)\right) \sum_{\ell \geq 0} w_\ell(M) k^{-\ell}, \qquad k \gg 1, \tag{4.2}$$

where $\mathrm{CS}(M)$ is the Chern–Simons invariant of M and a depends on the topological properties of M. This asymptotic expansion was obtained by Witten in his construction of the WRT invariant in [13]. The numbers $w_\ell(M)$ are interesting numerical invariants of M (for example, $w_0(M)$ is related to Ray–Singer torsion of M). One can also ask what is the resurgent structure of this asymptotic expansion, and this was addressed in some examples in [14].

In order to understand the resurgent structure of the volume conjecture (4.1), it turns out to be more useful to consider a different invariant of hyperbolic knots, also called the state integral (or Andersen–Kashaev invariant).

References

[1] F. A. Berezin and M. Shubin, *The Schrödinger Equation*, vol. 66. Springer Science & Business Media, 2012.

[2] C. M. Bender and T. T. Wu, *Anharmonic oscillator. 2: A Study of perturbation theory in large order*, *Phys. Rev.* **D7** (1973) 1620–1636.

[3] P. Henrici, *Applied and computational complex analysis, 2.* Wiley, 1978.

[4] T. M. Seara and D. Sauzin, *Resumació de Borel i teoria de la ressurgencia*, *Butl. Soc. Catalana Mat.* **18** (2003) 131–153.

[5] O. Costin, *On Borel summation and Stokes phenomena for rank-one nonlinear systems of ordinary differential equations*, *Duke Math. J.* **93** (06, 1998) 289–344.

[6] E. Delabaere and F. Pham, *Resurgent methods in semi-classical asymptotics*, *Annales de l'IHP* **71** (1999) 1–94.

[1] The original volume conjecture focused on the leading exponential term, and the existence of an interesting all-orders asymptotics was pointed out in [11]

[7] E. Delabaere, H. Dillinger and F. Pham, *Exact semiclassical expansions for one-dimensional quantum oscillators*, *J. Math. Phys.* **38** (1997) 6126–6184.

[8] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, *arXiv:0811.2435* (2008) .

[9] D. Gaiotto, G. W. Moore and A. Neitzke, *Wall-crossing, Hitchin systems, and the WKB approximation*, *Adv. Math.* **234** (2013) 239–403.

[10] R. Kashaev, *The hyperbolic volume of knots from the quantum dilogarithm*, *Lett. Math. Phys.* **39** (1997) 269–275.

[11] S. Gukov, *Three-dimensional quantum gravity, Chern-Simons theory, and the A-polynomial*, *Comm. Math. Phys.* **255** (2005) 577–627.

[12] S. Garoufalidis, *Chern-Simons theory, analytic continuation and arithmetic*, *Acta Math. Vietnam.* **33** (2008) 335–362.

[13] E. Witten, *Quantum field theory and the Jones polynomial*, *Comm. Math. Phys.* **121** (1989) 351–399.

[14] S. Gukov, M. Mariño and P. Putrov, *Resurgence in complex Chern-Simons theory*, `1605.07615`.