Prepared for submission to JHEP

Quantum theory and non-perturbative effects

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Abstract: Notes for the 7th Itzykson seminar on "Resurgence and quantization."

Contents

1 Introduction and general aspects

Our working knowledge of quantum theory, from Quantum Mechanics (QM) to Quantum Field Theory (QFT), relies heavily on approximation schemes like the WKB approximation and perturbation theory. Both of them lead to formal power series in a small parameter which are generically divergent. It is crucial, both conceptually and technically, to make sense of these series. This situation arises even in very basic examples in Quantum Mechanics. A well-known example is the *quartic oscillator*, with Hamiltonian,

$$\mathsf{H} = \frac{\mathsf{p}^2}{2} + \frac{\mathsf{q}^2}{2} + \frac{g\mathsf{q}^4}{4}, \qquad g > 0, \tag{1.1}$$

where p, q are Heisenberg operators on $L^2(\mathbb{R})$ with canonical commutation relations $[\mathsf{q}, \mathsf{p}] = \mathrm{i}\hbar$. There are various rigorous results on the spectrum of this Hamiltonian, regarded as an operator on $L^2(\mathbb{R})$. Since it involves a confining potential, with

$$V(q) = \frac{q^2}{2} + \frac{gq^4}{4} \to \infty, \qquad |q| \to \infty, \tag{1.2}$$

one can show (see for example [2]) that H^{-1} is compact, so that H has a discrete spectrum $E_n(g)$, $n = 0, 1, 2, \cdots$, with

$$0 < E_0(g) < E_1(g) < \cdots \tag{1.3}$$

The asymptotic expansion of $E_n(g)$, for small g, can be calculated by using stationary perturbation theory. For example, for the ground state energy $E_0(g)$ one finds

$$E_0(g) \sim \varphi(g), \tag{1.4}$$

where

$$\varphi(g) = \sum_{n\geq 0} a_n g^n = \frac{1}{2} + \frac{3}{4}\left(\frac{g}{4}\right) - \frac{21}{8}\left(\frac{g}{4}\right)^2 + \frac{333}{16}\left(\frac{g}{4}\right)^3 + \mathcal{O}(g^4). \tag{1.5}$$

Here, we set $\hbar = 1$. It is known that the coefficients in this series, a_n, grow factorially [1],

$$a_n \sim \left(\frac{3}{4}\right)^n (-1)^{n+1} n!, \qquad n \gg 1. \tag{1.6}$$

Therefore, perturbation theory gives a divergent series. One important question is *whether* (and *how*) one can reconstruct the *exact* $E_0(g)$ from this asymptotic series. The only thing that we know from classical asymptotics is that one can *approximate* $E_0(g)$ by using for example an optimal truncation of the asymptotic series. It may happen however that, in order to reconstruct $E_0(g)$, one needs more information than just what is contained in the perturbative series. This is in fact generally the case, and leads to the introduction of *trans-series*.

For a detailed introduction to trans-series and resurgence in the spirit of these notes, see [12].

2 Trans-series

2.1 Introducing trans-series: the case of ODEs

The need for trans-series can be seen already in the classical asymptotic theory of functions defined by ODEs. A typical example is the Airy function $\mathrm{Ai}(x)$, which solves the ODE,

$$\varphi'' = x\varphi. \tag{2.1}$$

This equation has a formal power series solution of the form

$$Z_{\mathrm{Ai}}(x) = \frac{1}{2x^{1/4}\sqrt{\pi}} \mathrm{e}^{-2x^{3/2}/3} \sum_{n=0}^{\infty} a_n x^{-3n/2}, \qquad a_n = \frac{1}{2\pi}\left(-\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!}. \tag{2.2}$$

When x is real, positive and large, this formal power series describes well the asymptotic behavior of $\mathrm{Ai}(x)$. However, it is well known that for x negative and large, the correct asymptotics is given by

$$\mathrm{Ai}(x) \sim \frac{|x|^{-1/4}}{\sqrt{\pi}} \cos\left(\frac{2}{3}|x|^{3/2} - \frac{\pi}{4}\right), \qquad x < 0,\ |x| \to \infty. \tag{2.3}$$

This can not be reproduced by the formal power series (2.2) only. We need an additional formal power series given by

$$Z_{\mathrm{Bi}}(x) = \frac{1}{2x^{1/4}\sqrt{\pi}} \mathrm{e}^{2x^{3/2}/3} \sum_{n=0}^{\infty} (-1)^n a_n x^{-3n/2}. \tag{2.4}$$

A crucial fact here is that $Z_{\mathrm{Ai}}(x)$ and $Z_{\mathrm{Bi}}(x)$ have different leading exponential behavior. In this particular case, the general *trans-series solution* to the Airy equation is the formal combination

$$C_1 Z_{\mathrm{Ai}}(x) + C_2 Z_{\mathrm{Bi}}(x). \tag{2.5}$$

In general, formal solutions of ODEs involving trans-series can be constructed near an irregular singular point (see for example [4]). The simplest example is Euler's equation,

$$\frac{\mathrm{d}\varphi}{\mathrm{d}z} + A\varphi(z) = \frac{A}{z} \tag{2.6}$$

which has an irregular singular point at $z = \infty$. There is a formal power-series solution to this equation of the form

$$\varphi_0(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}}, \qquad a_n = A^{-n} n!. \tag{2.7}$$

This solution is an asymptotic series, with zero radius of convergence, since the coefficients grow factorially with n. It is easy to see that one can construct a family of formal solutions to the Euler's ODE based on $\varphi_0(z)$,

$$\varphi(z) = \varphi_0(z) + C\mathrm{e}^{-Az} \tag{2.8}$$

where C is an arbitrary constant parametrizing the family of solutions.

A more interesting, non-linear example is the Painlevé II equation,

$$u''(\kappa) - 2u^3(\kappa) + 2\kappa u(\kappa) = 0 \tag{2.9}$$

There is a formal solution to PII which goes like $u(\kappa) \sim \sqrt{\kappa}$ at the irregular singularity at $\kappa \to \infty$:

$$u^{(0)}(\kappa) = \sqrt{\kappa} - \frac{1}{16\,\kappa^{\frac{5}{2}}} - \frac{73}{512\,\kappa^{\frac{11}{2}}} - \frac{10657}{8192\,\kappa^{\frac{17}{2}}} - \frac{13912277}{542888\,\kappa^{\frac{23}{2}}} + \cdots, \qquad \kappa \to \infty. \tag{2.10}$$

One can consider a more general, formal solution to the equation, based on the "perturbative" solution (2.10), and with the structure

$$u(\kappa; C) = \sum_{\ell=0}^{\infty} C^{\ell} u^{(\ell)}(\kappa) = \sqrt{\kappa} \sum_{\ell=0}^{\infty} C^{\ell} \kappa^{-\frac{3\ell}{4}} \mathrm{e}^{-\ell A \kappa^{3/2}} \epsilon^{(\ell)}(\kappa), \quad \kappa \to \infty, \tag{2.11}$$

where C is a constant,

$$A = \frac{4}{3} \tag{2.12}$$

and

$$\epsilon^{(\ell)}(\kappa) = \sum_{n=0}^{\infty} u_{\ell,n} \kappa^{-3n/2}. \tag{2.13}$$

(we normalize the solution with $u_{1,0} = 1$). The "perturbative" part $u^{(0)}(\kappa)$ is given by (2.10). It is easy to see that the higher $u^{(\ell)}(\kappa)$ satisfy *linear* equations.

We could give a formal definition of trans-series in the context of ODEs, but trans-series appear in many different contexts where a formal approach is less useful. However, from the above examples, we can already note some important properties:

1. Trans-series involve at least two "small parameters": the first small parameter is the one appearing in the original, "perturbative" series (in the case of the Euler equation, it is simply $1/z$). There is also an *exponentially small parameter* (for example, e^{-Az} in the case of the Euler equation). We will call the series involving an exponentially small parameter "non-perturbative corrections," or "instanton corrections". The quantity A involved in the exponentially small parameter will be sometimes called *instanton action*.

2. All series in the first small parameter are factorially divergent.

3. The different series appearing in the trans-series are not independent. For example, the large order behavior of the terms in the perturbative series are controlled by the first (sometimes second) trans-series. For example, in the case of the Euler equation, we have $a_n = n! A^{-n}$, where A is the quantity characterizing the strength of the small exponential. In the Painlevé II equation, the coefficients of the perturbative series grow as
$$u_{0,2n} \sim (2n)! A^{-2n}, \tag{2.14}$$
where A is now given in (2.12). This is what motivates the name of "resurgence".

2.2 Other examples of trans-series

Trans-series are in fact ubiquitous in mathematics and physics. A simple example, closely related to the example of ODEs, is the case of one-dimensional integrals of the form
$$\mathcal{I}_{\mathcal{C}} = \int_{\mathcal{C}} g(z) \mathrm{e}^{-f(z)/\hbar} \mathrm{d}z, \tag{2.15}$$
where $\mathcal{C}$ is an appropriate contour. Let us consider the saddle-points z_n of $f(z)$, i.e. the different solutions of
$$f'(z_n) = 0. \tag{2.16}$$
We will assume that $f''(z_n) \neq 0$ at a saddle point. Then, there is a path of steepest $\mathcal{C}_n(\hbar)$ which depends on the saddle point and the value of $\hbar$ (more precisely, the argument of $\hbar$). Then, the integral $\mathcal{I}_{\mathcal{C}_n(\hbar)}$ along this path has an asymptotic expansion in powers of $\hbar$, which leads to a formal series $\mathcal{S}_n(\hbar)$. The corresponding trans-series is a linear combination of these,
$$\sum_n C_n \mathcal{S}_n(\hbar). \tag{2.17}$$

More complicated examples arise in Quantum Mechanics. The QM path integral can be regarded as an infinite-dimensional generalization of integrals of the form (2.15), and trans-series appear when one considers contributions from different saddles to the path integral. There is typically a "trivial" or "perturbative" saddle associated to trivial or constant configurations, and then non-trivial saddles. In the context of the Euclidean path integral, these saddles are called *intstantons*. A typical example is the double-well potential in QM, with Hamiltonian
$$H = \frac{p^2}{2} + W(x), \qquad W(q) = \frac{g}{2}\left(q^2 - \frac{1}{4g}\right)^2, \qquad g > 0. \tag{2.18}$$
In perturbation theory one finds two degenerate ground states, located around the minima
$$q_{\pm} = \pm \frac{1}{2\sqrt{g}}. \tag{2.19}$$
The ground state energy obtained in stationary perturbation theory is a formal power series of the form
$$\varphi_0(g) = \frac{1}{2} - g - \frac{9}{2} g^2 - \frac{89}{2} g^3 - \cdots \tag{2.20}$$
This series is obtained by doing a path integral around the constant trajectory $q = q_{\pm}$. However, one can consider a saddle-point of the Euclidean path integral, given by a path going from q_- to q_+ (or viceversa),
$$q_{\pm}^{t_0}(t) = \pm \frac{1}{2\sqrt{g}} \tanh\left(\frac{t - t_0}{2}\right). \tag{2.21}$$

This gives the first trans-series correction to the ground state energy, which has the form

$$\varphi_1(g) = -\left(\frac{2}{g}\right)^{1/2} \frac{\mathrm{e}^{-1/6g}}{\sqrt{2\pi}} \left(1 + \mathcal{O}(g)\right). \tag{2.22}$$

Our final example of trans-series occurs in a peculiar QFT, namely, CS theory on a three-manifold M. This a gauge theory with action

$$S = -\frac{k}{4\pi} \int_M \mathrm{Tr}\Big(\mathcal{A} \wedge \mathrm{d}\mathcal{A} + \frac{2\mathrm{i}}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\Big). \tag{2.23}$$

Here, $\mathcal{A}$ is a G-connection on the three-manifold M, where G is a gauge group. We will mostly consider $G = (S)U(N)$, and in this case our conventions are such that $\mathcal{A}$ is a Hermitian $N \times N$ matrix-valued one-form. Gauge invariance of the action requires

$$k \in \mathbb{Z}. \tag{2.24}$$

This is a very special type of QFT: as Witten showed in his groundbreaking work [16], CS theory is a topological, and essentially solvable, QFT. The partition function of the theory on M is defined by the path integral

$$Z(M) = \int \mathcal{D}\mathcal{A}\, \mathrm{e}^{\mathrm{i}S(\mathcal{A})}, \tag{2.25}$$

and it can be computed in closed form on many simple three-manifolds. At the same time, one can compute the path integral by using standard pertubative techniques. In a series of ingenious papers, L. Rozansky and his collaborators managed to write the exact answer for the partition function as a sum of contributions which can be interpreted as coming from different saddle points of the path integral. In CS theory, saddle points correspond to flat G-connections on M. For example, when $G = SU(2)$ and M is a Seifert homology sphere, characterized by the coprime integers P_s, $s = 1, \cdots, r$, the contribution of the trivial connection can be written as [11]

$$Z_{\text{pert}} \propto \sum_{n \geq 0} a_n \left(\frac{2\pi \mathrm{i} P}{k}\right)^n, \tag{2.26}$$

where

$$P = \prod_{s=1}^r P_s, \qquad a_n = \frac{f^{(2n)}(0)}{n!}, \qquad f(z) = \Big(2\sinh\frac{z}{2}\Big)^{2-r} \prod_{s=1}^r \left(2\sinh\frac{z}{2P_s}\right). \tag{2.27}$$

(We assumed for simplicity that $|H_1(M, \mathbb{Z})| = 1$). This is a divergent series which can be promoted to a trans-series by taking into account the contributions of non-trivial connections [9, 11]. We will give more details below (We note that in the formula above, the level k is the renormalized level, which differs from the bare level by a shift, but this is not important to our considerations.)

3 Borel resummation

When trans-series are made of formal series diverging factorially, a useful procedure to obtain a "true" function is the *Borel transform*. Given a formal series $\varphi(z)$, the Borel transform $\widehat{\varphi}(\zeta)$ is defined as the series

$$\widehat{\varphi}(\zeta) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \zeta^n. \tag{3.1}$$

If the coefficients grow as $a_n \sim n! A^{-n}$, the series $\widehat{\varphi}(\zeta)$ has a finite radius of convergence $\rho = |A|$ and it defines an analytic function in the circle $|\zeta| < |A|$. In particular, an important consequence of the "resurgent" character of trans-series is that the singularities of the Borel transform of the perturbative series give information about the rest of the trans-series, i.e., in physical parlance, about the structure of the other instanton sectors.

Example 3.1. A particularly beautiful example of this is the case of CS theory on Seifert spaces. The perturbative series for the partition function has been written down in (2.26). In order to understand the structure of this series, it turns out to be more convenient to use a slightly modified version of the Borel transform, given by

$$BZ_{\rm pert}(\xi) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n+\frac{1}{2})} \xi^{n-\frac{1}{2}} \,. \tag{3.2}$$

From the explicit expression for a_n, it is easy to see that

$$BZ_{\rm pert}(\xi) = \frac{2}{\sqrt{\xi}} f\left(2Pz\right), \tag{3.3}$$

where

$$z = \sqrt{\frac{2\pi {\rm i}\xi}{P}}. \tag{3.4}$$

The poles of $f(2Pz)$ are closely related to non-trivial flat connections on M, as discussed in detail in [9] in various examples. $\square$

Let us suppose that $\widehat{\varphi}(\zeta)$ has an analytic continuation to a neighbourhood of the positive real axis, in such a way that the Laplace transform

$$s(\varphi)(z) = \int_0^\infty {\rm e}^{-\zeta} \widehat{\varphi}(z\zeta)\, {\rm d}\zeta = z^{-1} \int_0^\infty {\rm e}^{-\zeta/z} \widehat{\varphi}(\zeta)\, {\rm d}\zeta, \tag{3.5}$$

exists in some region of the complex z-plane. In this case, we say that the series $\varphi(z)$ is *Borel summable* and $s(\varphi)(z)$ is called the *Borel sum* of $\varphi(z)$. Notice that, by construction, $s(\varphi)(z)$ has an asymptotic expansion around $z=0$ which coincides with the original series $\widehat{\varphi}(\zeta)$, since

$$s(\varphi)(z) = z^{-1} \sum_{n\geq 0} \frac{a_n}{n!} \int_0^\infty {\rm d}\zeta\, {\rm e}^{-\zeta/z} \zeta^n = \sum_{n\geq 0} a_n z^n. \tag{3.6}$$

This procedure makes it possible in principle to reconstruct a well-defined function $s(\varphi)(z)$ from the asymptotic series $\varphi(z)$ (at least for some values of z). As we pointed out above, in some cases the formal series $\varphi(z)$ is the asymptotic expansion of a well-defined function $f(z)$ (like in the example of the quartic oscillator).

In some cases, the Borel transform has poles on the positive real axis, and the Borel transform defined above does not exist, strictly speaking. One can then deform the contour of integration and consider contours $\mathcal{C}_\pm$ that avoid the singularities and branch cuts by following paths slightly above or below the positive real axis. This leads to the *lateral Borel resummations*,

$$s_\pm(\varphi)(z) = z^{-1} \int_{\mathcal{C}_\pm} {\rm d}\zeta\, {\rm e}^{-\zeta/z} \widehat{\varphi}(\zeta). \tag{3.7}$$

which are in general complex.

4 Semiclassical decoding

Let us suppose that we have a trans-series around $z = 0$, of the form

$$\Phi(z; C) = \varphi_0(z) + \sum_{\ell=1}^{\infty} C^\ell \mathrm{e}^{-\ell A/z} \varphi_\ell(z), \tag{4.1}$$

in which lateral Borel resummations exist for all $\varphi_\ell(z)$, $\ell = 0, 1, 2, \cdots$, at least for some range of values of z. Then, we can consider the series,

$$s_\pm(\Phi)(z; C) = s_\pm(\varphi)(z) + \sum_{\ell=1}^{\infty} C^\ell \mathrm{e}^{-\ell A/z} s_\pm(\varphi_\ell)(z). \tag{4.2}$$

If this series converges for some range of z and C, we say that the trans-series is lateral Borel summable. We will now state a principle of *semiclassical decoding.*

Definition 4.1. (Semiclassical decoding). Let $f(z)$ be a function with the asymptotic expansion

$$f(z) \sim \varphi_0(z) = \sum_{n \geq 0} a_n z^n. \tag{4.3}$$

We say that $f(z)$ admits a *semiclassical decoding* if $\varphi_0(z)$ can be promoted to a trans-series $\Phi(z)$, which is Borel summable in the sense above, and such that

$$f(z) = s_\pm(\Phi)(z; C_\pm) \tag{4.4}$$

for some values $C_\pm$.

Note that, if we perform lateral Borel resummations, the value of C depends on the choice of $\pm$ direction. This fact is the resurgent version of the Stokes phenomenon, and the difference $C_+ - C_-$ is the Stokes parameter.

The principle of semiclassical decoding solves the problem of dealing with asymptotic, divergent series in quantum theory. When semiclassical decoding holds, one recovers the exact, non-perturbative information by just considering the Borel-resummed trans-series. The simplest situation corresponds to the case in which $C = 0$, there are no singularities along the positive real axis, and the Borel resummation of the perturbative series reproduces the exact result. This is famously the case for the perturbative series (1.5) of the quartic oscillator.

Example 4.2. By results of J. Écalle, O. Costin and others, solutions to ODEs with an asymptotic expansion near an irregular singular point admit a semiclassical decoding. A nice example is provided by the Hastings–McLeod solution to Painlevé II, with the asymptotic behavior,

$$\begin{aligned} u_{\mathrm{HM}}(\kappa) &\sim \kappa^{1/2}, \qquad \kappa \to \infty, \\ u_{\mathrm{HM}}(\kappa) &\sim \mathrm{e}^{-2\sqrt{2}(-\kappa)^{3/2}/3}, \qquad \kappa \to -\infty, \end{aligned} \tag{4.5}$$

The asymptotics near $\kappa \to \infty$ has an all-order generalization of the form

$$u_{\mathrm{HM}}(\kappa) \sim u^{(0)}(\kappa), \tag{4.6}$$

where $u^{(0)}(\kappa)$ is given in (2.10). Then, it turns out that $u_{\mathrm{HM}}(\kappa)$ admits a semiclassical decoding in terms of the trans-series (2.11), and one has

$$u_{\mathrm{HM}}(\kappa) = s_+(u)(\kappa; C), \qquad C = \frac{\mathrm{i}}{2\sqrt{2\pi}}. \tag{4.7}$$

An important question in quantum theory is whether well-defined functions in QM and QFT admit a semiclassical decoding. For example, it is known that the energy levels of many QM systems in one dimension admit a semiclassical decoding (a particularly important example is the double-well oscillator). In physical parlance this means that one can reconstruct the observable with information coming from semiclassical considerations, i.e. from perturbation theory (around the trivial vacuum) and from perturbative series around non-trivial instantons.

The program of semiclassical decoding was very active in QFT after the discovery of instantons, but then it suffered an important drawback in the late 70's when it was shown that Yang–Mills theory in *infinite volume* does not admit a simple semiclassical decoding (and the related problem of renormalons). The revival of this program in the last few years has been based on looking at examples where semiclassical decoding is likely to occur, like matrix models in the $1/N$ expansion, topological strings in the genus expansion, topological QFTs, and QFTs with new types of IR cutoffs (see the work of Argyres, Dunne, Unsal and collaborators on QFTs compactified on $\mathbb{R}^d \times \mathbb{S}^1$, where one imposes twisted boundary conditions instead of periodic boundary conditions).

In the case of CS theory, semiclassical decoding was an important issue from the very beginning. CS theory is ideally suited for this type of analysis, since one knows the exact partition function in many examples (many of the papers by Rozansky and his collaborators in the 1990's address this issue explicitly). In the case of Seifert rational homology spheres with $SU(2)$ gauge group, a result of Lawrence and Rozansky [11] gives the *exact* partition function as

$$Z(M) \propto \frac{1}{2\pi \mathrm{i}} \int_C f(y) \mathrm{e}^{-\frac{y^2}{4\hat{g}_s}} \mathrm{d}y - \sum_{m=1}^{2P-1} \mathrm{Res}_{y=2\pi \mathrm{i} m} \left(\frac{f(y) \mathrm{e}^{-\frac{y^2}{4\hat{g}_s}}}{1-\mathrm{e}^{-ky}} \right). \tag{4.8}$$

Here,

$$C = \mathrm{e}^{\mathrm{i}\pi/4}\mathbb{R}, \tag{4.9}$$

and

$$\hat{g}_s = \frac{2\pi \mathrm{i} P}{k}. \tag{4.10}$$

As shown in [9], the first integral can be written as a Borel resummation of the perturbative series, and the remaining residues are polynomials in $1/k$, therefore "truncated" trans-series associated to non-trivial flat connections on M. In this case, the partition function can be fully decoded semiclassically, in an explicit analytic way.

5 Two examples of semiclassical decoding

An important ingredient in the program of semiclassical decoding is the access to the exact (or "non-perturbative" value) of the quantum observables. We will see two examples of this in QM and in string theory, where the access to the exact value is highly non-trivial.

5.1 The WKB expansion

Let us consider a standard Schrödinger Hamiltonian in one dimension

$$\mathsf{H} = \mathsf{p}^2 + V(\mathsf{q}), \tag{5.1}$$

with a discrete set of eigenvalues $E_1, E_2, \cdots$ (we have set for convenience $m = 1/2$). The all-orders WKB method provides, first of all, a *quantum differential*, which is a formal power series in $\hbar$,

$$\lambda(q;\hbar) = \sum_{n\geq 0} \lambda_n(q)\hbar^n, \qquad \lambda_0(q) = p(q)\mathrm{d}q \tag{5.2}$$

By integrating this differential around the "perturbative" B-cycle (corresponding to the classical motion), we obtain an asymptotic series in $\hbar^2$ as a function of the energy,

$$\mathrm{vol}(E;\hbar) = \oint_B \lambda(q;\hbar) = \sum_{n=0}^{\infty} \mathrm{vol}_n(E)\hbar^{2n}, \tag{5.3}$$

where

$$\mathrm{vol}_0(E) = \oint_B p\mathrm{d}q, \tag{5.4}$$

is the classical volume in phase space (the odd powers of $\hbar$ in the quantum differential are total derivatives and do not contribute to the period integrals). This series, in contrast to the standard perturbative series in QM, diverges *doubly-factorially* for a fixed value of E,

$$\mathrm{vol}_n(E) \sim (2n)!(\mathcal{A}(E))^{-2n}, \qquad n \gg 1. \tag{5.5}$$

The all-orders quantization condition

$$\mathrm{vol}(E;\hbar) = 2\pi\hbar\left(n + \frac{1}{2}\right), \tag{5.6}$$

can be solved order by order in $\hbar$ to obtain E as a function of ν, where

$$\nu = \hbar\left(n + \frac{1}{2}\right). \tag{5.7}$$

This leads to a formal power series in $\hbar^2$ which we will write as

$$E^{(0)}(\nu,\hbar) = \sum_{n\geq 0} E_n(\nu)\hbar^{2n}. \tag{5.8}$$

Note that this seris is different from the one obtained in standard perturbation theory, where one performs an expansion in a coupling g, as in (1.5), and for a fixed level n. In fact, the WKB problem is a two-parameter problem: we have a series in $\hbar$ where each term is a (non-trivial!) function of the energy E (or, in the inversion, of the parameter ν).

There exist exact WKB methods, pioneered by André Voros, which lead to "exact" quantization conditions for many QM problems. These "exact" conditions lead to trans-series expansions for the functions $\mathrm{vol}(E;\hbar)$ or $E(\nu,\hbar)$. For example, in the case of the double well, one has (see [3])

$$E(\nu,\hbar) = E^{(0)}(\nu,\hbar) + \frac{\hbar}{2\pi}\frac{\partial E^{(0)}(\nu)}{\partial\nu}\exp\left(\frac{\mathrm{i}}{\hbar}\oint_A \lambda(q;\hbar)\right) + \cdots \tag{5.9}$$

Here, A is the non-perturbative or "tunneling" cycle.

Note however that $\mathrm{vol}(E;\hbar)$ is defined by the trans-series, and as a *function* there is no general, exact prescription to compute it. It is worth quoting one of the true classics of resurgence on this issue. Referring to the series (5.3) for the quantum volume, Balian, Parisi and Voros asked:

"The following question now arises about the divergent series (5.3): is there a natural way to sum it to a smooth function $\mathrm{vol}_{\mathrm{ex}}(E, \hbar)$ such that the equation

$$\mathrm{vol}_{\mathrm{ex}}(E, \hbar) = 2\pi\hbar \left(n + \frac{1}{2} \right) \tag{5.10}$$

yields the *exact* eigenvalues E_n for each n? Such a $\mathrm{vol}_{\mathrm{ex}}(E, \hbar)$ would appear as an exact version of the Thomas–Fermi distribution. Actually this is a very tricky question...". They also note that such a function would lead to an analytic relation between energy and quantum number, while in the quantum theory, strictly speaking, only *integer* values of n make sense.

1. Is it possible to provide a non-perturbative definition of the quantum volume function in arbitrary QM problems?

2. In the cases in which such a non-perturbative definition exists, is semiclassical decoding valid for this definition?

In [15], Voros has provided an indirect construction of the quantum volume function for polynomial potentials, based on a fixed-point mechanism. More recently, advances in supersymmetric gauge theory and topological string theory have led to remarkable exact results in QM and integrable systems. As a consequence of results in [13], for example, one can determine the exact quantum volume, as a *convergent* series, for the Hamiltonian

$$\mathsf{H} = \mathsf{p}^2 + 2\gamma \cosh(\mathsf{q}). \tag{5.11}$$

Moreover, in [7], an infinite family of operators ρ associated to toric CY manifolds were defined, for which an exact quantum volume can be calculated in closed form, as a convergent series (see also [10]). It is then an interesting question whether this explicit (analytic) and exact quantum volumes functions can be decoded semiclassically.

5.2 Topological string theory

In topological string theory on a CY manifold, the closed string free energy is given by an asymptotic series in the string coupling constant g_{st}, where each term is itself a function depending on a geometric modulus λ (we assume for simplicity that the CY has a single modulus λ, although the general case is straightforward),

$$F(\lambda, g_{\mathrm{st}}) = \sum_{g \geq 0} F_g(\lambda) g_{\mathrm{st}}^{2g-2}, \tag{5.12}$$

This genus expansion turns out to be divergent, namely, for a fixed λ (inside the radius of convergence of $F_g(\lambda)$), the $F_g(\lambda)$ grow like

$$F_g(\lambda) \sim (2g)!(A_{\mathrm{st}}(\lambda))^{-2g}, \tag{5.13}$$

where $A_{\mathrm{st}}(\lambda)$ is a spacetime instanton action. This is a generic property of the genus expansion in string theories [8, 14].

An important conceptual question in topological string theory is the following: is there a well-defined function which has the formal genus expansion as its asymptotic expansion? This is the question of the non-perturbative completion of the topological string. Recently, a concrete answer to this question was given in [7] for arbitrary toric CYs, in which the non-perturbative completion is provided by an ideal Fermi gas in one dimension. The quantum Hamiltonian of this gas is obtained by quantizing the mirror curve to the CY. Let us restrict ourselves to genus one mirror curves for simplicity. They are of the form

$$\mathcal{O}_X(\mathrm{e}^x, \mathrm{e}^y) + \kappa = 0, \tag{5.14}$$

where $\mathcal{O}_X(\mathrm{e}^x, \mathrm{e}^y)$ is a polynomial in exponentiated variables. Quantization is achieved by promoting x, y to canonically conjugate Heisenberg operators on $L^2(\mathbb{R})$,

$$[\mathsf{x}, \mathsf{y}] = \mathrm{i}\hbar, \tag{5.15}$$

and doing Weyl quantization of $\mathcal{O}(\mathrm{e}^x, \mathrm{e}^y)$. One obtains in this way a formally self-adjoint operator O_X. We then have a correspondence,

$$X \to \rho_X = \mathsf{O}_X^{-1} = \mathrm{e}^{-\mathsf{H}_X}, \tag{5.16}$$

and ρ_X turns out to be of trace class on $L^2(\mathbb{R})$. One then defines "fermionic" spectral traces $Z(N, \hbar)$ of ρ_X by using the power series expansion of the Fredholm determinant

$$\det(1 + \kappa\rho_X) = 1 + \sum_{N \geq 0} Z(N, \hbar)\kappa^N. \tag{5.17}$$

A conjecture in [7] states that $Z(N, \hbar)$ has (5.12) as its asymptotic expansion, in the 't Hooft-like regime

$$N \to \infty, \qquad \hbar \to \infty, \qquad \frac{N}{\hbar} = \lambda, \tag{5.18}$$

where

$$\hbar = \frac{4\pi^2}{g_{\mathrm{st}}}. \tag{5.19}$$

Therefore, $Z(N, \hbar)$ is a non-perturbative completion of the genus expansion (5.12). We can then ask whether one can perform a semiclassical decoding of this quantity in terms of a suitable trans-series. This turns out to be possible, as shown in [6], and based on the trans-series construction of [5].

Acknowledgments

This work is supported in part by the Swiss National Science Foundation, subsidies 200020-149226, 200021-156995, and by the NCCR 51NF40-141869 "The Mathematics of Physics" (SwissMAP).

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