Resurgence and non-perturbative physics: from quantum mechanics to superconductors

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The missing chapter in QFT books

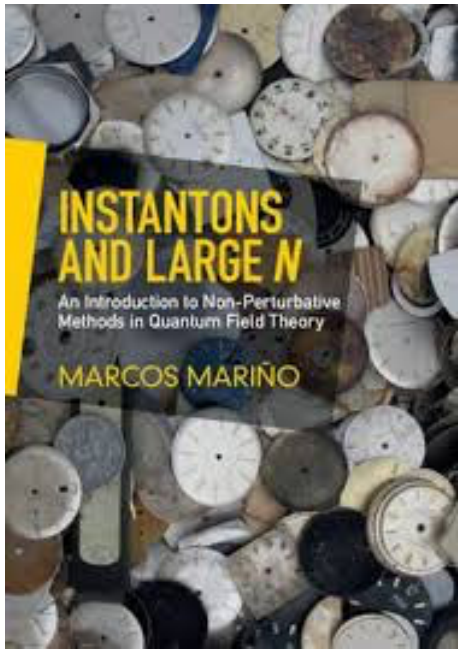
Perturbation theory is one of the few universal tools we have in quantum physics. One important fact of life is that, typically, the coefficients in these series grow **factorially**

$$E(g) = \sum_{n \geq 0} a_n g^n \qquad a_n \sim n!$$

Series with factorial growth have zero radius of convergence. Sometimes one can use the very first terms to obtain approximations to their sum (the so-called optimal truncation), but further knowledge of the coefficients does **not** improve things

This violates one of the basic moral principles of our civilization
(aka “travailler plus pour gagner plus”)

So on top of regularization and renormalization, in quantum theory we also need **resummation**, which is not usually covered in textbooks



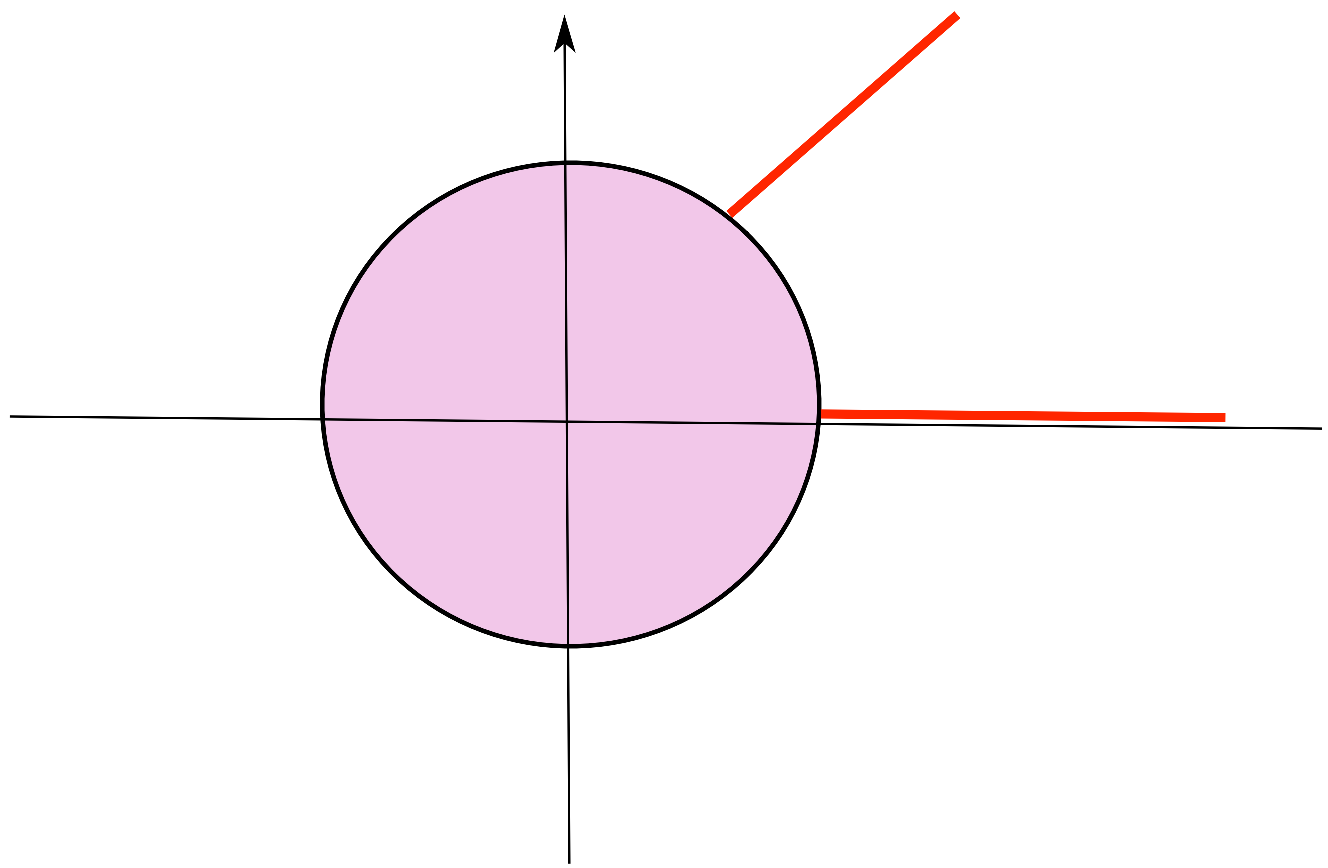
The Borel triangle

The Borel method is a systematic way of making sense of factorially divergent formal power series

$$
\begin{tikzcd}
\varphi(z) = \sum_{n\geq 0} a_n z^n \arrow[rr, "\text{Borel transform}"] & & \widehat{\varphi}(\zeta) = \sum_{n\geq 0} \frac{a_n}{n!}\zeta^n \arrow[dl, "\text{Laplace transform}"] \\
& s(\varphi)(z) = z^{-1}\int_0^{\infty} \mathrm{e}^{-\zeta/z}\widehat{\varphi}(\zeta)\mathrm{d}\zeta \arrow[ul, dotted] &
\end{tikzcd}
$$

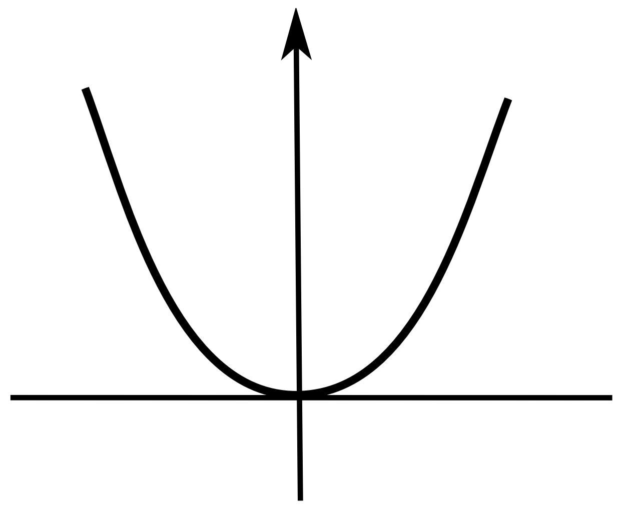
Borel resummation

The Borel transform $\widehat{\varphi}(\zeta)$ is **analytic** at the origin. Very often it can be analytically continued to the complex plane, displaying **singularities** (poles, branch cuts).



Note that the Borel transform takes us from the world of divergent formal power series to the world of analytic functions

In some cases Borel resummation works remarkably well, as in the energies of the quartic oscillator in quantum mechanics



$$\mathsf{H} = \frac{\mathsf{p}^2}{2} + \frac{\mathsf{x}^2}{2} + g\mathsf{x}^4$$

$$E_0(g) \sim \frac{1}{2} + \frac{3}{4}g - \frac{21}{8}g^2 + \cdots$$

Here, Borel resummation of the perturbative series in the coupling constant reproduces the **exact** spectrum (level by level)

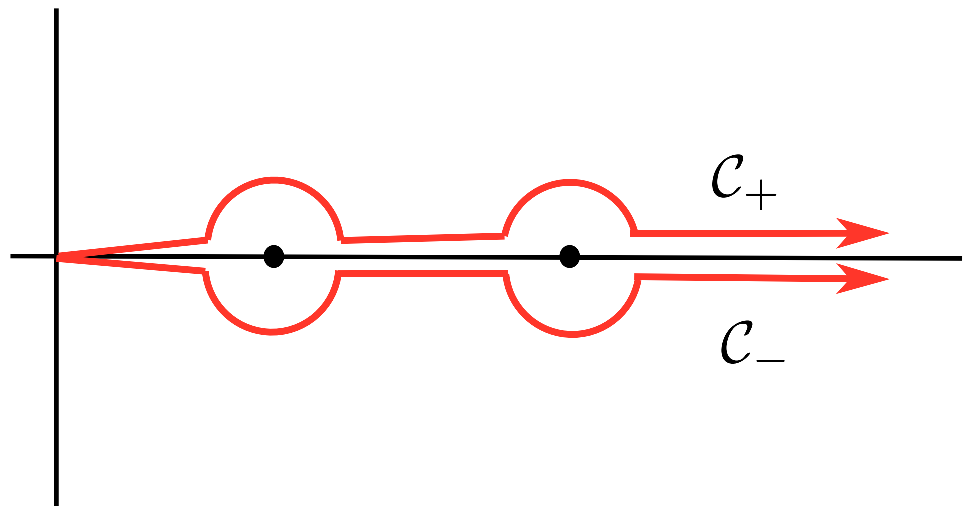
However, there are cases where the procedure is not conclusive

If the Borel transform has singularities along the positive real axis, the integral giving the resummation is ill-defined, and the series is not Borel summable.

One can however perform so-called **lateral Borel resummations** by deforming the integration contour

$$s_{\pm}(\varphi)(z) = z^{-1} \int_{\mathcal{C}_{\pm}} \mathrm{e}^{-\zeta/z} \widehat{\varphi}(\zeta) \mathrm{d}\zeta$$

Borel plane



If e.g. the Borel transform has a simple pole at $\zeta = A$, the two lateral resummations differ by an imaginary, **exponentially small quantity**

$$s_+(\varphi)(z) - s_-(\varphi)(z) = \frac{2\pi \mathrm{i}}{z} \mathrm{e}^{-A/z}$$

This is an example of the **non-perturbative ambiguity** in non-Borel summable theories

To solve this ambiguity, we have to enlarge the original data and consider **trans-series**

Trans-series

We have to incorporate the exponentially small terms from the very beginning:

$$\Phi(z) = \varphi(z) + \sum_{\ell=1}^{\infty} C^{\ell} \mathrm{e}^{-\ell A/z} \varphi_{\ell}(z)$$

trans-series parameter

$$\varphi(z) = \sum_{n \geq 0} a_n z^n$$

$$\varphi_{\ell}(z) = z^{-b_{\ell}} \sum_{n=0}^{\infty} a_{\ell,n} z^n$$

factorially divergent formal power series

Physically, this corresponds to adding non-perturbative sectors

An important result of resurgence is that all these formal power series are closely related. In particular, the large order behavior of the **perturbative** series is determined by the first trans-series

$$a_n \sim \frac{a_{1,0}}{2\pi} A^{-n-b_1} \Gamma(n+b_1) \qquad\qquad n \gg 1$$

factorial growth

Note that the value of the trans-series parameter *C* depends on the choice of lateral resummation, in such a way that the final answer is unambiguous. There is no well-defined weight of a non-perturbative correction! [David]

The research program of resurgence is based on the idea that

all observables in a quantum theory can be obtained by (lateral) Borel resummations of trans-series

Pros and cons

Potentially, this gives a universal definition of non-perturbative physics built upon perturbative tools

"Semiclassical decoding": in contrast to e.g. numerical methods, the building blocks of trans-series can be understood analytically as contributions from the perturbative sector, plus a series of non-perturbative corrections

"It is nice to know that the computer understood the answer, but I would like to understand it too"

Beautiful mathematics, analyticity, and a bootstrap program

Very good at detecting the presence of non-perturbative effects (e.g. Shenker's argument for D-branes in string theory)

Too universal a language! It does not give concrete tools to e.g. obtain trans-series. These have to be found elsewhere

Works better when we have a lot of control of perturbation theory, which is not often the case

Not yet clear if resurgence is a discovery tool or an organizational tool

An example: the WKB method

The WKB method leads to a reformulation of one-dimensional QM in terms of trans-series

$$H(x,p) = p^2 + V(x) = E$$ WKB curve

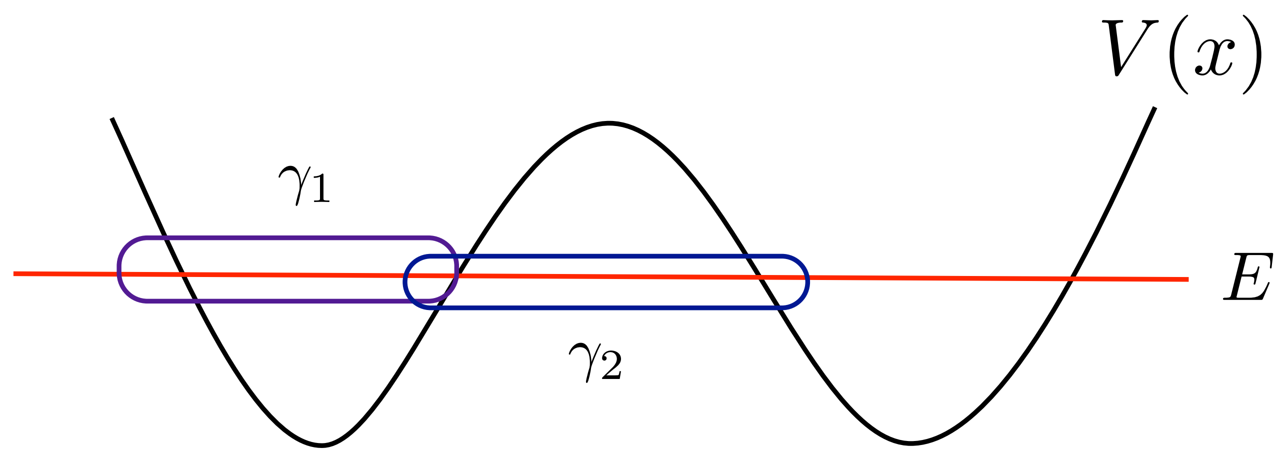
The all-orders WKB method leads to an asymptotic form for the wavefunction

$$\psi(x,\hbar) \sim \frac{1}{\sqrt{p(x,\hbar)}} \exp\left(\frac{\mathrm{i}}{\hbar}\int^x p(x',\hbar)\mathrm{d}x'\right)$$

"quantum" one-form

$$p(x,\hbar) \sim p(x) + \sum_{n\geq 1} p_n(x)\hbar^{2n}$$

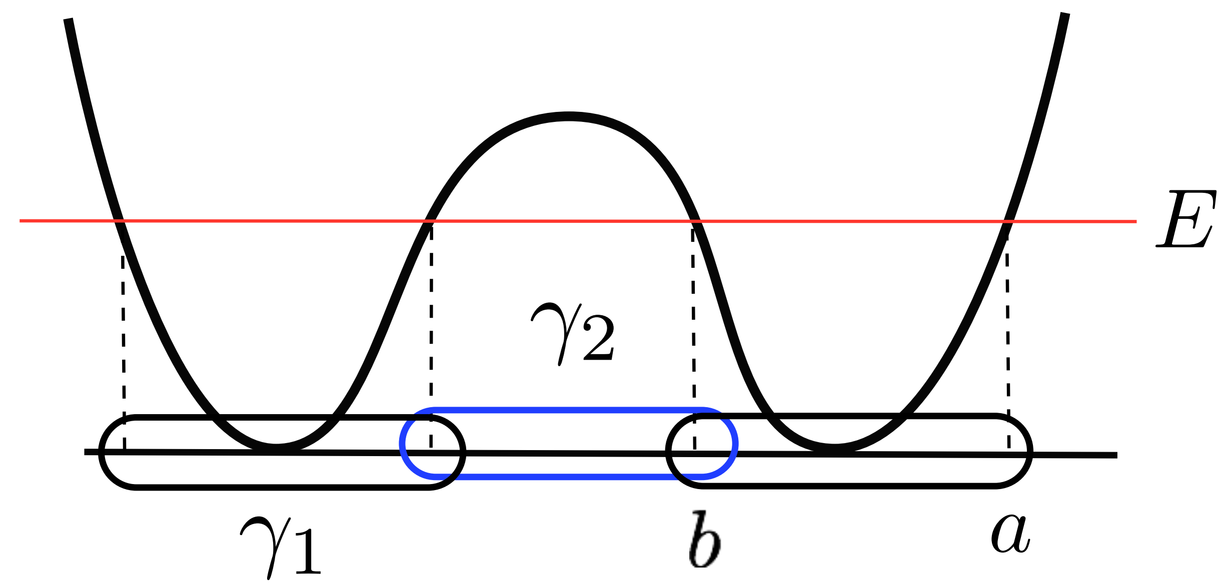
We can integrate the quantum one-form against the one-cycles of the curve to obtain **quantum periods** (aka Voros symbols)



$$\Pi_a(\hbar) = \oint_{\gamma_a} p(x,\hbar)\mathrm{d}x \sim \sum_{n\geq 0} \Pi_a^{(n)} \hbar^{2n}$$

These are the building blocks of the trans-series

The double-well potential is a typical non-Borel summable problem



Physically, we know that there are non-perturbative effects due to tunneling, i.e. to instantons in the path integral. Mathematically, this means that the "perturbative" quantum period Π_1 is not Borel summable:

$$s_+(\Pi_1) - s_-(\Pi_1) = -\mathrm{i}\hbar \log\left(1 + \mathrm{e}^{-s(\Pi_2)/\hbar}\right)$$

The singularities are located at multiples of the instanton action $\Pi_2^{(0)}$

Exact quantization conditions for the spectrum of the double-well can be obtained as **vanishing conditions for Borel-resummed tran-series** [Voros, Zinn-Justin]

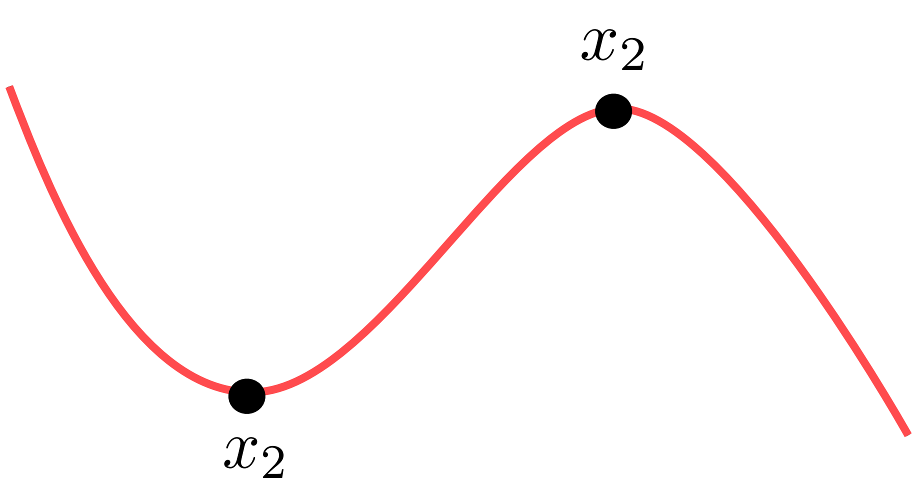
$$\frac{1}{\hbar}\left(s_+(\Pi_1)(\hbar) + s_-(\Pi_1)(\hbar)\right) \pm \tan^{-1}\left(\mathrm{e}^{-\frac{1}{2\hbar}s(\Pi_2)(\hbar)}\right) = 2\pi\left(k+\frac{1}{2}\right)$$

Moreover, discontinuity formulae in the resurgent WKB method lead to a connection with GMN wall-crossing formulae in SUSY gauge theories, and to a powerful reformulation of quantum periods in terms of TBA-like integral equations [Ito-M.M.-Shu, Grassi-Gu-M.M.]

Where do trans-series come from?

As in WKB, a typical source of trans-series are expansions around different saddle points (i.e. **instanton sectors**) of the path integral.

$$Z(\hbar) = \int \mathrm{d}x \, \mathrm{e}^{-S(x)/\hbar}$$



$$A = S(x_2) - S(x_1)$$

$$\sqrt{\frac{2\pi}{\hbar}} \mathrm{e}^{-S(x_1)/\hbar} \left(1 + \mathcal{O}(\hbar)\right)$$

$$\mathrm{i}\sqrt{\frac{2\pi}{\hbar}} \mathrm{e}^{-S(x_2)/\hbar} \left(1 + \mathcal{O}(\hbar)\right)$$

Resurgence implies that instanton corrections should tell you about the behavior of perturbation theory at large order. It is believed that instantons encode the factorial growth of perturbation theory due to the growth in the **total number of diagrams** [Bender-Wu]

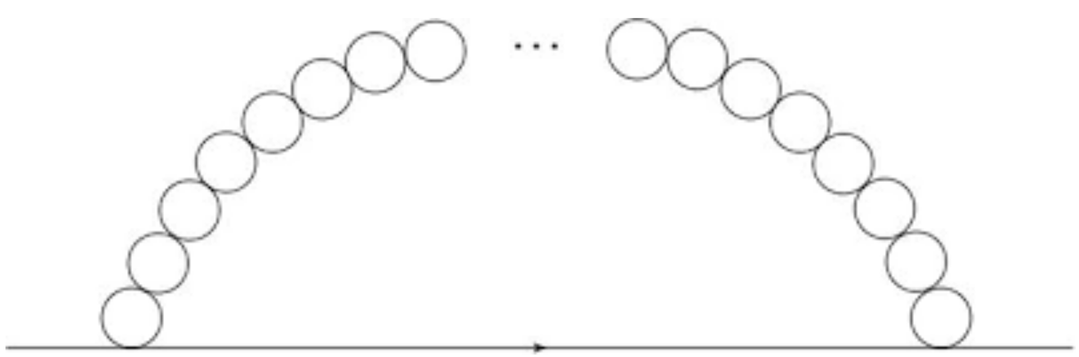
For a while, people had the hope that trans-series in QFTs would come just from perturbative and instanton sectors. Sometimes this is the case, like the $\lambda\Phi^4$ theory in two dimensions [Serone et al.]

This **"instanton dominance"** seems to hold as well in some string theories, like non-critical strings, where instanton sectors correspond to D-branes, and control the large order behavior of string perturbation theory

Some evidence for this picture was also given in the type IIA dual to ABJM theory [Drukker-M.M.-Putrov]

However, the dream of instanton dominance was shattered in the 1970s-1980s by the discovery of a new and mysterious source of singularities in the Borel plane: **renormalon singularities**

One manifestation of these singularities are bubble diagrams in renormalizable theories, which grow factorially due to the integration over momenta. This is a different source of growth than the overall proliferation of diagrams.



$$\int_0^1 (-\log(k))^n \, \mathrm{d}k = n!$$

Parisi argued that in asymptotically free (AF) theories there should be a renormalon singularity at

$$\zeta = \frac{1}{|\beta_0|}$$

This obstructs Borel summability and leads to non-perturbative ambiguities. It should also control the factorial growth of the perturbative series in these theories

This has been verified in recent years by calculating perturbative series at large order, either numerically [Bauer-Bali-Pineda] or analytically, in integrable 2d theories [Fateev et al., Volin, M.M.-Reis].

So in AF theories we expect to find trans-series with the structure:

$$\sum_{n\geq 0} a_n g^n + C g^{-b} \mathrm{e}^{-\frac{1}{|\beta_0| g}} \sum_{n\geq 0} a_{1,n} g^n + \cdots$$

The non-perturbative corrections do **not** come from saddles or instantons. If the underlying observable is a product of operators, one can use the OPE, but in general **one can not compute these corrections from first principles in the path integral**

This is in my view a crucial open problem in this subject (and in QFT in general). It has been proposed that special compactifications might solve this problem [Dunne, Unsal et al.]

It has been assumed that renormalons, as their name indicates, occur only in renormalizable field theories.

However, in recent work we have shown that renormalon singularities appear in many more theories. These include the $\lambda\Phi^4$ theory with *O(n)* symmetry in two dimensions, as well as non-relativistic QFTs describing interacting many-body systems [M.M.-Reis]

Resurgence for superconductors

Consider a spin 1/2 Fermi gas with a weak attractive interaction, described by

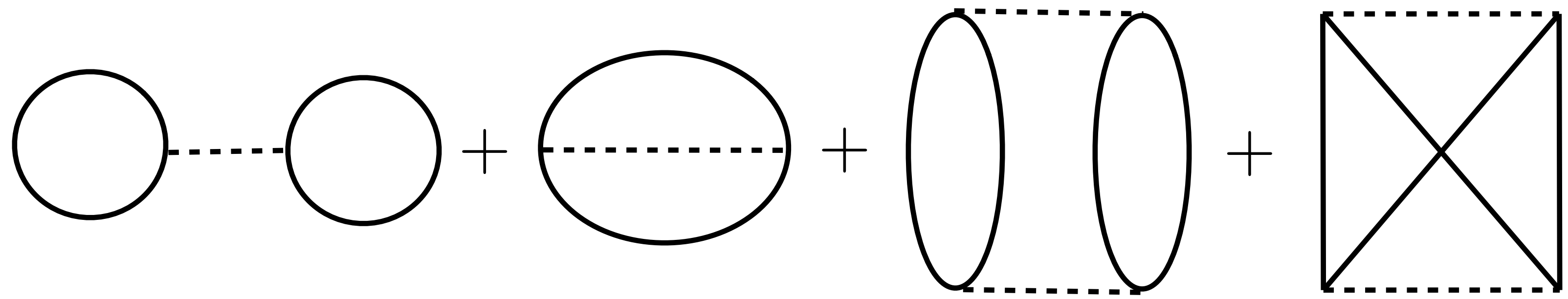
$$\mathcal{H} = -\sum_{\sigma=\uparrow,\downarrow} \bar{\psi}_\sigma \frac{\nabla^2}{2m} \psi_\sigma - g \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow$$

According to standard BCS theory, the ground state consists of Cooper pairs, whose binding energy is given by the **superconducting energy gap**

$$\Delta \sim \mathrm{e}^{-\delta/g}$$

This is non-perturbative in the coupling! Can we understand it in the framework of resurgence?

Let us consider the perturbative series for the ground state energy of this interacting Fermi system

$E_0 = E_{\text{free}} +$  $+ \cdots$

We conjectured that

1) this series is factorially divergent and non-Borel summable

2) the structure of singularities in the Borel plane is controlled by the (squared) superconducting gap

More precisely, the first singularity occurs at $\zeta = 2\delta$

A simple model

To test this conjecture, we considered the model in one dimension, aka the **Gaudin-Yang (GY)** model. It can be exactly solved with the Bethe ansatz

Dimensionless coupling in the GY model:

$$\gamma = \frac{g}{n}$$

n ← density of particles

$$e(\gamma) = \frac{E(g)}{n^3}$$

$$\Delta \sim \exp\left(-\frac{\pi^2}{2\gamma}\right)$$

superconducting gap

The series

Calculating the perturbative expansion is not completely obvious (only the first three terms had been computed in the last 50 years). However, by using techniques developed by D. Volin in similar problems, we found the first 50 terms of the expansion:

$$
\begin{aligned}
e(\gamma) = {} & \frac{\pi^2}{12} - \frac{\gamma}{2} + \frac{\gamma^2}{6} - \frac{\zeta(3)}{\pi^4}\gamma^3 - \frac{3\zeta(3)}{2\pi^6}\gamma^4 - \frac{3\zeta(3)}{\pi^8}\gamma^5 - \frac{5(5\zeta(3)+3\zeta(5))}{4\pi^{10}}\gamma^6 \\
& - \frac{3\left(12\zeta(3)^2+35\zeta(3)+75\zeta(5)\right)}{8\pi^{12}}\gamma^7 - \frac{63\left(12\zeta(3)^2+7\zeta(3)+35\zeta(5)+12\zeta(7)\right)}{16\pi^{14}}\gamma^8 \\
& - \frac{3\left(404\zeta(3)^2+240\zeta(5)\zeta(3)+77\zeta(3)+735\zeta(5)+882\zeta(7)\right)}{4\pi^{16}}\gamma^9 \\
& - \frac{27\left(160\zeta(3)^3+1800\zeta(3)^2+3720\zeta(5)\zeta(3)+143\zeta(3)+2310\zeta(5)+6363\zeta(7)+1700\zeta(9)\right)}{32\pi^{18}}\gamma^{10} \\
& + \mathcal{O}\left(\gamma^{11}\right)
\end{aligned}
$$

Large order behavior

Using these data we can determine (numerically) the large order behavior of the series as

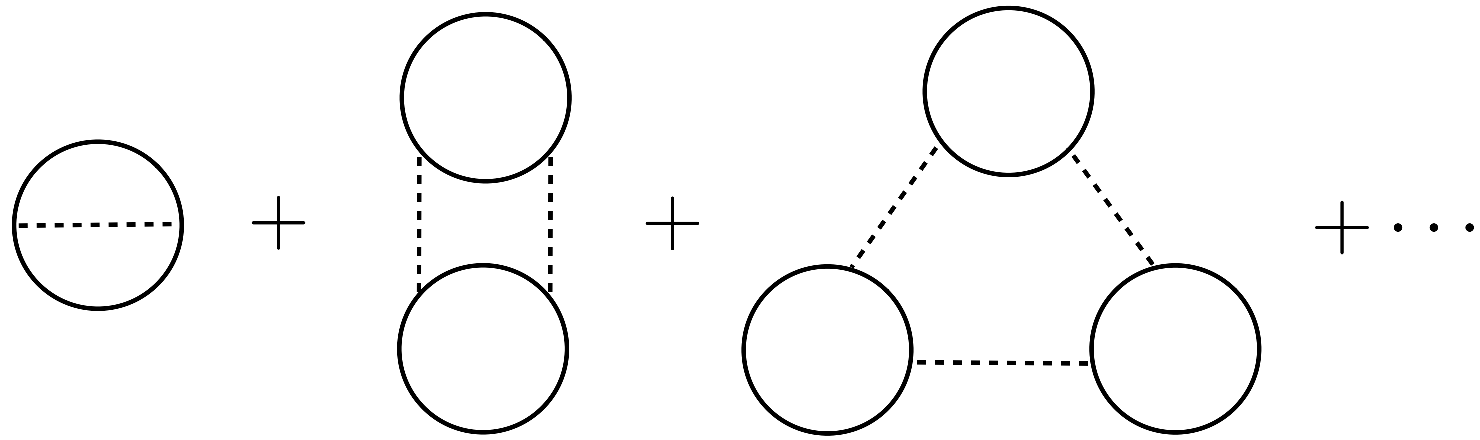
$$e(\gamma) \sim \sum_{k\geq 0} c_k \gamma^k \qquad\qquad c_k \sim -\frac{1}{\pi}(\pi^2)^{-k+1}\Gamma(k-1)$$

The series is **factorially divergent** and **non-Borel summable**. The corresponding trans-series is, at leading order,

$$\gamma\, \mathrm{e}^{-\pi^2/\gamma}$$

This has the exponential dependence of the squared energy gap, in agreement with our conjecture!

This seems to be a **renormalon** singularity, and we have identified multi-bubble diagrams leading to the correct factorial growth. In one dimension, these are the ring diagrams appearing in the RPA:



We have therefore grounds to believe that superconductivity is a renormalon effect, controlled by a trans-series

Conclusions

Resurgence is, first of all, the art of working with divergent series, by enlarging them to trans-series. It suggests a research program where non-perturbative physics can be "decoded" in terms of trans-series.

There are two known sources for trans-series in quantum theory: instantons, which are relatively well-understood, and renormalons, which we don't know how to incorporate in the path integral.

Renormalons are more universal than previously thought. They seem to be for example responsible for the emergence of the superconducting gap. These new scenarios for renormalon physics might clarify their origin.

Thank you for your attention