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Stringy geometry and emergent space

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Abstract: String theory provides a consistent theory of quantum gravity which involves a generalization of classical, Riemannian geometry. Therefore, it modifies and/or qualifies the notions of spacetime inherited from General Relativity. In this article, I discuss some aspects of spacetime in string theory. In particular, I emphasize how string theory can be regarded as a two-parameter deformation of classical geometry, and how spacetime appears as an emergent phenomenon in string/gauge theory dualities.

Contents

1 Introduction

String theory appeared forty years ago as a tentative, unified theory of all particles and interactions. It is still not clear if the description of nature provided by string theory will be vindicated by experiments in the near future, although in view of the absence of supersymmetry signals at the LHC, one should not be too optimistic. However, string theory seems to have the rare virtue of being *a* consistent theory of quantum gravity. It might not be the right theory of actually existing gravity, but it contains propagating gravitons and black hole solutions which can be studied in the quantum regime. The Bekenstein–Hawking entropy of many of these black holes can be correctly accounted for in terms of microstates. It has also been conjectured that in some cases, string theories are dual to gauge theories, therefore providing the first concrete realization of holographic dualities. Last, but not least, string theory has had a continuing impact on mathematics, which goes well beyond the traditional patterns of interaction between physics and mathematics. Although one might be skeptical of string theory models of particle physics, string theory has found in the meantime interesting applications in condensed matter theory and in hadronic physics.

As a theory of quantum gravity, string theory proposes a new picture of the underlying structure of spacetime, and some of the most successful applications of string theory in mathematics have involved "stringy" versions of geometry, with deep relations to enumerative geometry. The modern theory of enumerative invariants has grown out of the challenges (and the answers) proposed by string theory. More recently, holography has suggested that, in string theory, spacetime should be regarded as an "emergent" phenomenon. In this short article I will review some of these developments, and I will try to give some concrete examples in which string theory has modified or challenged our notions of space (insightful articles on this subject include [14, 19, 21].)

After a summary of string theory (hopefully for non-specialists), I discuss how geometry can be deformed to take into account the extended nature of the string. This leads to the notion of "stringy" geometry put forward in, for example, topological string theory. As I will emphasize, string geometry should also take into account string interactions, and this paves the way to a discussion of non-perturbative aspects of the theory. Our best guide for such a non-perturbative formulation is the string/gauge theory correspondence. In this framework, the space in which strings propagate seems to be an emergent notion, akin to a macroscopic description in thermodynamics. “Classical” geometry appears as an effective description at large distances. It becomes “stringy” and “brany” as we go to shorter distances and as we increase the strength of string interactions, and might finally “dissolve” into gauge-theoretic degrees of freedom as we reach the Planck scale. I illustrate these considerations in one particular example of a string/gauge duality, in which precise calculations seem to vindicate this picture of spacetime.

I should haste to say that my view of this subject is very partial, and this article does not pretend to be exhaustive or make justice to the enormous amount of work done on this problem. It should be read as an essay on the subject, rather than as an expository article or review.

2 Stringy geometry

2.1 From particles to strings

In order to understand some of the implications of string theory for physical conceptions of space, it is necessary to present some of its ingredients, and in particular to explain in which sense string theory is a generalization (or more precisely, a deformation) of theories of point particles. Our presentation will be necessarily sketchy and incomplete. Modern textbooks on superstring theory include [15, 20], and the summary presented below is based on [17], which discusses the interaction of string theory with modern mathematics.

In a classical theory of point particles, the fundamental ingredient is the *trajectory* of the particle in a given spacetime, which is typically represented by a differentiable, Riemannian manifold X. This trajectory can be represented by an application

$$\begin{aligned} x :& \mathcal{I} \to X, \\ &\tau \mapsto x(\tau), \end{aligned} \tag{2.1}$$

where $\mathcal{I} \subset \mathbb{R}$ is an interval, and $\tau \in \mathcal{I}$ is the time parametrizing the trajectory. This specification of the trajectory provides only the kinematical data. Determining the dynamics requires as well an *action functional* $S(x(\tau))$. For example, for a (non-relativistic) free particle one can take,

$$S(x(\tau)) = \int \mathrm{d}\tau \, G_{\mu\nu}(x(\tau))\dot{x}^\mu(\tau)\dot{x}^\nu(\tau), \tag{2.2}$$

where $G_{\mu\nu}$ is the metric of X. The classical equations of motion can be derived from the variational principle

$$\frac{\delta S}{\delta x} = 0, \tag{2.3}$$

and in the case of the free particle this means that the motion occurs along *geodesics* of the Riemannian manifold X.

The quantization of the theory is appropriately done by using Feynman’s path integral formalism. In this formalism, the quantum mechanical propagator (in Euclidean signature) is obtained by integrating the weight $\exp(-S(x))$ over all possible trajectories $x(\tau)$ with fixed boundary conditions,

$$x(\tau_0) = x_i, \qquad x(\tau_1) = x_f, \tag{2.4}$$

where τ_0, τ_1 are the endpoints of the interval. This is written, formally, as

$$K(x_i, x_f) = \int \mathcal{D}x(\tau)\, \mathrm{e}^{-\frac{1}{\hbar}S(x(\tau))}. \tag{2.5}$$

The weight $\exp(-S(x))$ might be regarded as a probability distribution for paths, and the propagator is closely related to the probability that a quantum particle which starts at x_i at $\tau = \tau_0$ is detected at x_f at $\tau = \tau_1$. One can also consider *periodic* trajectories, i.e. maps of the form

$$x : \mathbb{S}^1 \to X \tag{2.6}$$

where the circle has length β. This means that the map x is defined on a *closed* one-manifold. As is well-known, the path integral with these boundary conditions,

$$Z(\beta) = \int \mathcal{D}x(\tau)\, \mathrm{e}^{-\frac{1}{\hbar}S(x(\tau))} \tag{2.7}$$

gives the partition function at temperature $kT = \beta^{-1}$ (where k is the Boltzmann constant), which describes the properties of a particle in a thermal bath at this temperature.

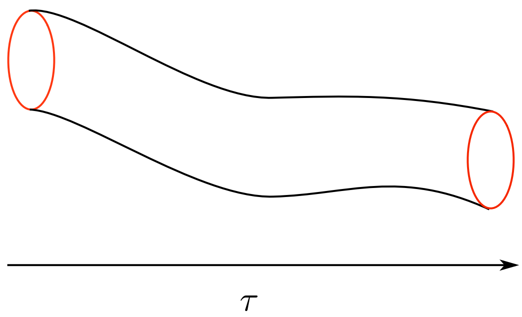

Figure 1. A closed string propagating in time.

In string theory, point-particles are replaced by one-dimensional objects. Classically, the embedding of such an object in a spacetime manifold is described by a map

$$\begin{aligned} x : \mathcal{I} \times \mathcal{S} &\to X, \\ (\tau, \sigma) &\mapsto x(\tau, \sigma) \end{aligned} \tag{2.8}$$

where $\sigma \in \mathcal{S} \subset \mathbb{R}$ parametrizes now the string. The dynamics is specified again by a classical action $S(x)$, which in the simplest case takes the form

$$S(x, h) = \frac{1}{\ell_s^2} \int \mathrm{d}\tau \mathrm{d}\sigma\, \sqrt{h} G_{\mu\nu} h^{ab} \partial_a x^\mu \partial_b x^\nu. \tag{2.9}$$

Here, h is a metric on the two-dimensional manifold $\Sigma = \mathcal{I} \times \mathcal{S}$, and it plays the role of an auxiliary field. The quantity ℓ_s, which has the dimensions of a length, sets the scale of this one-dimensional extended object and for this reason it is called the *length of the string*. It is convenient to parametrize $\mathcal{S} = [0, \pi]$. A *closed string* is a loop with no free ends, and in this case the appropriate boundary condition is

$$x(\tau, 0) = x(\tau, \pi), \qquad \tau \in \mathcal{I}. \tag{2.10}$$

A freely propagating closed string, as it evolves in time, spans the surface with the topology of a cylinder, see Fig. 1.

There are two types of string interaction: in a *splitting* process one single string splits into two, and in a *joining* process two stings merge into one. As in quantum physics, the strength of such an interaction is measured by a constant $g_{\rm st}$ called the *string coupling constant*. For closed strings, the basic process of joining or splitting is described by a "pair of pants" diagram as in the left hand side of Fig. 2, and it has a single power of $g_{\rm st}$ associated to it. When this process takes place n times, it has the factor $g_{\rm st}^n$.

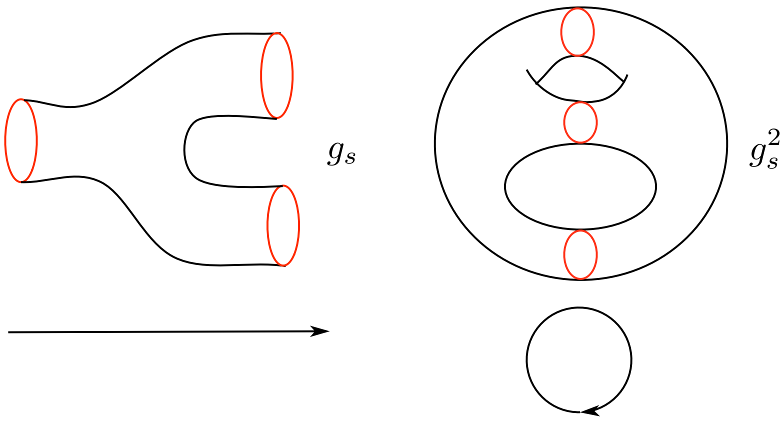

Figure 2. A closed string interaction takes place when one single string splits into two, as shown in the left hand side of the figure. Such a process is weighted by a power of g_s. In the right hand side we show a periodic configuration in which a closed string splits and then joins again before coming back to itself, spanning a Riemann surface of genus 2. Since there were two string interactions involved, this process has a weight g_s^2.

As in the theory of point particles, periodic configurations generalizing (2.6) play a very important role. For a closed string, a periodic configuration is described by a single string which, after various processes of splitting and joining, comes back to itself. This process produces a *closed, orientable Riemann surface* Σ_g, and it is easy to see that it has the weight $g_{\rm st}^{2g-2}$, where g is the genus of the resulting Riemann surface. In the right hand side of Fig. 2 we show a periodic configuration in which a closed string evolves, splits into two strings which merge back to a single closed string, and this string goes back to the starting point. The time evolution produces a Riemann surface of genus 2, and since there were two string interactions (one splitting and one joining) the whole process has a factor $g_{\rm st}^2$ associated to it.

We conclude that, in string theory, a periodic map is just a map from a closed Riemann surface Σ_g to the spacetime X:

$$x : \Sigma_g \to X. \tag{2.11}$$

The Riemann surface Σ_g is called the *worldsheet* of the string, while the manifold X is called its *target space*.

The quantization of a theory of strings is rather delicate. Formally, one considers a path integral over all possible configurations of the fields, as in (2.5), with the appropriate boundary conditions. For simplicity we will consider periodic boundary conditions, i.e. the analogue of (2.7). Since we have to consider all possible configurations of the string, we have to take into account all possible splitting/joining processes, and this means that we should sum over all possible genera for the Riemann surfaces spanned by the string. In the computation of Z one considers *disconnected* Riemann surfaces, but in the so-called "free energy," defined by $F = \log Z$, one should sum only over connected Riemann surfaces, labelled by the genus g. This means that the free energy is given by a formal infinite series over the different genera

$$F = \sum_{g=0}^{\infty} g_{\rm st}^{2g-2} F_g, \tag{2.12}$$

which is sometimes called the genus expansion of the string free energy. To calculate F_g for a fixed genus g, we should integrate over all metrics on Σ_g and all configurations of maps x. The space of all metrics on Σ_g, after taking into account the relevant symmetries, turns out to be equivalent to the moduli space of Riemann surfaces $\overline{M}_g$ constructed in algebraic geometry, and one has

$$F_g = \int_{\overline{M}_g} \mathcal{D}h_{ab}\, \mathcal{D}x\, {\rm e}^{-S(x,h)}. \tag{2.13}$$

Note that, in (2.13), the integration over the metric has been in fact reduced to an integration over $3g-3$ complex moduli parametrizing $\overline{M}_g$. It is important to note that the "total string amplitude" F in (2.12) depends on two parameters: the string length ℓ_s (which appears in the string action) and the string coupling constant $g_{\rm st}$. In particular, if we regard string theory as a two-dimensional quantum field theory, we see from (2.9) that the squared string length ℓ_s^2 plays the same role as $\hbar$. When $\ell_s \to 0$, the length of the string vanishes and we recover a theory of point particles. Of course, the coupling constant $g_{\rm st}$ can be also regarded as a quantum parameter, and in the point-particle limit of string theory it becomes a standard coupling constant governing quantum interactions of particles. The existence of these two quantum parameters makes it clear that string theory is a consistent deformation of a quantum theory of point particles.

The structure we have described is roughly speaking the so-called *bosonic string* with target space X, and it can be easily generalized after we identify its key ingredients. On one hand, the map x can be regarded as a two-dimensional quantum field described by the action (2.9). This field theory has the property of being invariant under the full conformal group in two dimensions, i.e. it is an example of a two-dimensional *conformal field theory* (CFT). On the other hand, we also introduced a metric in the two-dimensional surface where this field lives. The combination of these two ingredients is a particular example of a *two-dimensional conformal field theory coupled to two-dimensional gravity*. In more abstract terms, a string theory is just such a system, and depending on the conformal field theory involved –the so-called "matter content" of the string theory– we will have different string theories. For example, in *supersymmetric* string theories the conformal field theory is supersymmetric, i.e. there are extra fermionic fields as well as a symmetry exchanging the bosonic and the fermionic fields. In some cases these theories lead to supersymmetry in space-time, and the resulting theories are called *superstring theories*.

One surprising aspect of string theories is that they only make sense as quantum theories if one imposes constraints on the field content. For example, the bosonic string is consistent only if the target space has 26 dimensions, i.e. if the field x has 26 components. Superstring theories require X to have 10 dimensions. If string theory is regarded as a model of the real world, the target space X should be identified as the physical space-time, and the operators of the CFT give rise to quantum fields propagating on X. We conclude that string theory models requires extra physical dimensions. It is widely believed that consistent theories of strings have to be superstring theories, since the bosonic string is unstable (it has a tachyonic state in its spectrum.) There are five different consistent superstring theories, which are called type I, type

IIA, type IIB and the two different heterotic strings. In this article we will not need the details of how these theories are constructed (see for example [20]). One crucial property of all these string theories is that they contain, in its perturbative spectrum, interacting gravitons, which at large distances are described by General Relativity or modifications thereof. Therefore, these theories provide in principle a framework to study the quantum regime of interacting gravitons. In addition, they lead naturally to gauge interactions, so they have all the necessary ingredients to describe particles and the forces among them, including gravity. One can go further and assume that X is the product of a tiny, compact manifold six-dimensional K, and a four-dimensional Minkowski spacetime. In this way one finds unified theories in four dimensions, reasonably close to supersymmetric versions of the Standard Model.

There are strong indications that the five, apparently different string theories, are in reality different descriptions (in different regimes) of the same underlying theory, which is sometimes called M-theory. Unfortunately, we do not have a basic description of this fundamental theory, although we know some of its ingredients. For example, this theory contains, on top of strings or extended one-dimensional objects, other extended objects like *membranes* (two-dimensional objects) and also five-branes.

In addition to the "physical" string theories that we have described, there are other, simplified models of string theory which play an important rôle in mathematical physics. One of them is "non-critical" (super)string theory, in which the string propagates on a space of dimension less or equal than one. We should note that, in string theory, the dimension of spacetime is given by the so-called central charge of the corresponding CFT, and a non-critical string is simply a CFT with $c \leq 1$ coupled to two-dimensional gravity, in the way we have sketched above.

Another important simplified model is *topological string theory*, first introduced by Witten. In this case, one considers a two-dimensional *topological* field theory, i.e. a two-dimensional QFT whose correlation functions are independent of the metric, and couples the resulting theory (which is in particular conformal) to two-dimensional gravity. Topological string theory is the physical counterpart of the modern theory of enumerative invariants (i.e. Gromov–Witten theory and their close cousins.) It offers a precise arena in which one can explore "stringy" versions of space.

2.2 Strings and geometry

As we have just seen, string theory can be understood a *deformation* of a theory of point particles, in which the deformation parameter is the string length ℓ_s. When this parameter goes to zero, string theories become theories of point particles. For example, superstring theories become, in the point-particle limit, supergravity theories.

The natural geometric arena for a theory of point particles are differentiable manifolds. Since string theory is a deformation of a theory of point particles, it is natural to suppose that it will lead to a natural deformation of the usual structures appearing in the theory of manifolds. A precise and general formulation of these deformed structures has not been constructed yet, but a simplified version thereof has been studied in detail in the context of topological string theory.

The basic idea of "stringy" geometry is that, in string theory, a manifold X comes equipped with all possible maps of Riemann surfaces to X (satisfying some additional conditions, like for example holomorphy.) In this framework, a *point* in X should be regarded as a limiting case of such maps, when the length of the string ℓ_s goes to zero size (see Fig. 3).

In order to elaborate on this, let us consider one of the most important examples of "stringy" geometry (see [4, 13] for comprehensive references on these topics). As we mentioned above, there is a simplified version of string theory, called topological string theory, which can be studied on

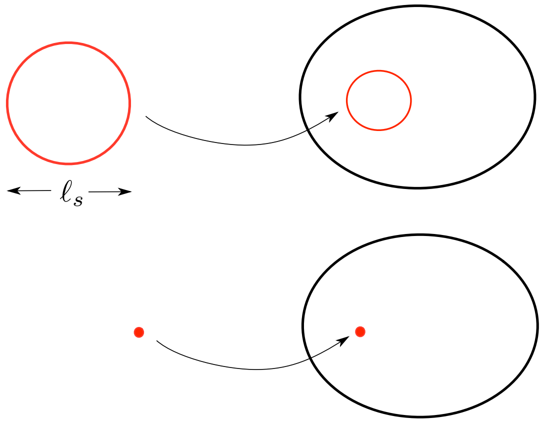

Figure 3. In string theory, a point in a manifold should be regarded as an embedded string in the limit in which the string length goes to zero size, $\ell_s \to 0$.

a special type of spaces called Calabi–Yau manifolds. Technically, a Calabi–Yau manifold is a complex, Kähler manifold which is Ricci-flat. In string theory, Calabi–Yau manifolds of real dimension equal to six (also called Calabi–Yau threefolds) play an important rôle. In some models, they provide the six additional dimensions that are needed in order to go from our spacetime of four dimensions to the ten-dimensional spacetime of the superstring. However, one can also study the "stringy" geometry of these manifolds from a more mathematical point of view. Let X be a compact, Calabi–Yau threfold. If $\omega \in H^{1,1}(X)$ is the Kähler form of X, it is customary to introduce a *complexified* Kähler form as

$$J = \omega + \mathrm{i}B, \qquad B \in H^{1,1}(X), \tag{2.14}$$

where B is sometimes called the B-field. Let $\{S_a\}$ be a basis of the two-homology of $H_2(X, \mathbb{Z})$. The "complexified" sizes or Kähler parameters of the S_a are defined as

$$t_a = \int_{S_a} J, \quad a = 1, \cdots, h^{1,1}(X), \tag{2.15}$$

and they have dimension of an area. Positivity of the metric requires that $\mathrm{Re}\, t_a > 0$.

In topological string theory, the quantum theory of embedded strings on a Calabi–Yau X reduces to a study of holomorphic maps from the worldsheet of the string Σ_g to X, and a one-loop calculation of the small fluctuations around it. This is due to the fact that the underlying sigma model describing topological string theory is a topological field theory of the cohomological type, and for these theories the semiclassical approximation to the path integral gives the exact result. When one considers Riemann surfaces of genus zero, the free energy reduces to a sum over the different holomorphic embeddings, and it has the structure

$$F_0(t_a) = \frac{1}{6\ell_s^6} \sum_{a,b,c} C_{abc} t_a t_b t_c + \sum_{Q \in H_2(X,\mathbb{Z})} N_{0,Q} \mathrm{e}^{-\int_Q J/\ell_s^2}. \tag{2.16}$$

In this expression, C_{abc} is the intersection number of the three homology classes

$$C_{abc} = S_a \cap S_b \cap S_c, \tag{2.17}$$

and the sum over Q is over two-homology classes. The numbers $N_{0,Q}$ appearing in this expression are the *Gromov–Witten invariants* of the Calabi–Yau X for genus zero and for the class Q. These invariants have an enumerative interpretation as an appropriate counting of holomorphic curves of genus zero in the homology class Q. Physically, the Gromov–Witten invariants are due to "periodic" strings, holomorphically embedded in X. An important property of (2.16) is that, when $\ell_s \to 0$, the contribution from the Gromov–Witten invariants is exponentially suppressed. In this limit, the genus zero free energy is dominated by the first term in (2.16), which involves a purely classical property of the manifold X, namely, its triple intersection numbers C_{abc}. When ℓ_s is not zero, this point-particle or "classical" quantity gets a very rich, infinite series of "stringy corrections" which contain information about the enumerative geometry of X.

The above example is particularly important since it has led to a new branch of mathematics, Gromov–Witten theory and its refinements and generalizations, which has been directly motivated by string theory. However, the physical principle behind this example is at work in many other situations: quantities computed in string theory are typically given by a point-particle quantity (involving the "classical" or Riemannian geometry of the spacetime manifold), together with various corrections coming from the extended nature of the string. These corrections involve a generalized notion of geometry in which the manifold has to be enriched with an additional, "stringy" structure.

The "stringy" structure of the geometry that we have just described has led to many interesting qualifications to the classical notions of space and distance. Note that, if L is the characteristic scale of the manifold where the strings propagate (i.e. the "size" of the manifold), the dimensionless parameter measuring the importance of string corrections is

$$\frac{L}{\ell_s}. \tag{2.18}$$

When the size of the manifold is much larger that the size of the string, stringy corrections are negligible. However, as the size of the manifold decreases, these corrections become more and more important and classical geometric intuition is no longer reliable. For example, in a compact Calabi–Yau threefold X, one can use the corrections involving embedded strings to define the "stringy volume" of cycles of even dimension. One of these cycles is a six-cycle, i.e. the manifold X itself. In a classical geometry with a single Kähler parameter, if we decrease the volume of X to zero, all lower-dimensional cycles inside X will squeeze to zero volume, too. However, if we take into account stringy corrections, one can find examples in which this is not the case, and the ambient manifold can have zero "stringy" volume while lower-dimensional cycles still have a non-zero "stringy" volume [9, 10]. This phenomenon occurs in the regime in which the characteristic size of the manifold is comparable to the length of the string, so classical geometry receives important corrections.

2.3 Quantum strings

Most of the studies of changing notions of space and distances in string theory have focused on "stringy" corrections in *classical* string theory, in which the string coupling constant vanishes and spacetime string interactions are negligible. In the classical limit, only the first term in (2.12) is kept, corresponding to Riemann surfaces of genus zero, i.e. with the topology of a sphere. For example, in (2.16) we have only considered genus zero Gromov–Witten invariants. This is clearly not the whole story, and one should consider interacting strings. In considering the string free energy (2.12), one should sum over all possible genera of the Riemann surfaces. We should also expect modifications in the geometry of spacetime due to a non-zero string coupling constant, and not only to a non-zero string length. We will refer to the resulting geometry as a "quantum" geometry, to distinguish it from the "stringy" geometry analyzed above.

The effects of interacting strings on the geometry have been much less studied, due to various limitations. Perhaps the most important one is the fact that the genus expansion appearing in string theory (like in the formal infinite sum in (2.12)) is asymptotic, and leads to a divergent series. This is a typical feature of quantum perturbation theories, and is usually solved by considering an appropriate non-perturbative definition of the theory. The problem is that, in string theory, there is no such a definition in most of the situations. This makes it difficult to use the string perturbation series as a guide to quantum geometry.

Although we do not have general, non-perturbative definitions of superstring theory, we have some indications about the ingredients that enter into such a definition. One solid hint that has emerged in recent years is that, at the non-perturbative level, string theory requires extended objects of higher dimension, like membranes and more generally p-branes (i.e. extended objects with p dimensions). This suggests that a quantum geometry taking into account interactions between strings will involve Riemann surfaces of higher genus, as it follows from string perturbation theory, but also embedded, extended objects of higher dimensions in the spacetime manifold. For example, periodic configurations of membranes will lead to embedded three-dimensional configurations in the target manifold X. M-theory, which underlies the different superstring theories, is known to contain membranes and five-branes. These objects lead to new corrections in the computation of physical amplitudes. In many cases, these corrections have a non-analytic dependence on the string coupling, i.e. they involve small exponentials of the form

$$\mathrm{e}^{-1/g_{\mathrm{st}}}, \tag{2.19}$$

which are invisible in the perturbative framework. We will revisit these issues in the light of the string/gauge theory correspondence, in the next section.

3 Strings from gauge theories and emergent space

3.1 The string/gauge theory correspondence

As we have mentioned, string theory contains interacting gravitons in its perturbative spectrum, and it provides in principle a framework to study the quantum regime of these interacting gravitons. However, string theory is not defined non-perturbatively, and this has limited the applications of string theory to quantum gravity. In order to fully assess the implications of string theory, one needs a non-perturbative definition of the theory.

The search for a non-perturbative definition of string theory is almost as old as string theory itself. In the late eighties, it was found that, in some situations, non-critical strings can be described non-perturbatively in terms of quantum gauge theories in zero dimension (also known as matrix integrals) and in one dimension (also known as matrix quantum mechanics), see [6, 8] for a review of those developments. These gauge theories in low dimension have a $U(N)$ symmetry, and it was discovered by 't Hooft [22] that the observables of any $U(N)$ gauge theory can be organized in a $1/N$ expansion which looks very much like a genus expansion of a string theory. In the non-perturbative descriptions of non-critical strings, the perturbative genus expansion of string theory is in fact identified to the $1/N$ expansion of the corresponding gauge theories. This means, in particular, that the string free energy (2.12), which as we saw is a formal (and in fact divergent) power series expansion, can be obtained as the asymptotic expansion of the free energy for the "dual" gauge theory, in the $1/N$ expansion.

These discoveries, remarkable as they were, involved string theories in no more than one dimension, and didn't provide a clear picture of how gravitational physics could emerge from the "dual" degrees of freedom. However, in 1998, Juan Maldacena postulated an exact relationship between various superstring theories in ten dimensions and supersymmetric gauge theories in lower dimensions [16]. The most celebrated example is the correspondence between $\mathcal{N} = 4$ supersymmetric Yang–Mills theory in four dimensions, and the the type IIB superstring theory on the manifold $\mathrm{AdS}_5 \times \mathbb{S}^5$, where AdS_n denotes the n-dimensional Anti de Sitter space (a maximally symmetric space with negative cosmological constant). Maldacena's correspondence is based on a study of extended objects in superstring theory called D-branes. These objects support gauge theory degrees of freedom, and at the same time they can be effectively described by a curved background involving AdS spaces. In a suitable scaling limit, both descriptions are equivalent, and this leads to a duality between a superstring AdS background, and the gauge theory living in the D-branes. A similar duality can be obtained by considering membranes in M-theory.

The conjecture of Maldacena has triggered an enormous literature, and has many interesting implications. This conjecture, if true, provides a non-perturbative definition of string theory, therefore a non-perturbative definition of a quantum theory of gravity. In this definition the fundamental degrees of freedom of string theory are not gravitational, but rather gauge-theoretic. This correspondence has been compared to the relationship between a thermodynamic system, like a gas or a fluid, and its microscopic description in terms of elementary constituents, like atoms or molecules. It suggests that strings, and their spacetime physics, are not fundamental phenomena, but rather "effective" or "emergent" descriptions of very different degrees of freedom. In particular, the ten-dimensional space-time of superstring theory "emerges" from the quantum dynamics of a gauge theory in lower dimensions (see [5, 14, 21] for discussions of emergence in string theory and in the gauge/string correspondence). It is believed that the original proposal in [16] can be extended to a more general correspondence between a gravity theory in an AdS space of $d + 1$ dimensions, and a CFT in d dimensions living on the boundary of AdS. Finding and studying examples of such a correspondence remains a very active research area.

3.2 Emergent space: an example

In order to understand a little bit more quantitatively the emergence of space in the string/gauge theory correspondence, it is useful to look at a concrete example. I will focus on the duality between the gauge theory known as ABJM theory, and type IIA superstring on an AdS_4 background. ABJM theory is built upon Chern–Simons gauge theory, which is defined by the action

$$S(A) = -\frac{\mathrm{i}k}{4\pi} \int \mathrm{d}^3 x \, \mathrm{Tr} \left(A \wedge \mathrm{d}A + \frac{2\mathrm{i}}{3} A^3 \right). \tag{3.1}$$

Here, A is a $U(N)$ gauge connection, and k is the so-called Chern–Simons level. It has to be an integer number, in order to prevent the appearance of gauge anomalies. In ABJM theory one considers two different $U(N)$ connections with Chern–Simons actions, so that the gauge group is $U(N) \times U(N)$, but with opposite levels k and $-k$. In addition, one considers a supersymmetric extension of the theory, with additional matter fields in the bifundamental representation of $U(N) \times U(N)$, and coupling the two gauge groups. The resulting theory is a superconformal field theory depending on two integer parameters: the rank N appearing in the gauge group, and the level k. Note that k plays the rôle of the inverse gauge coupling, in the sense that ABJM theory is weakly coupled when k is large. As in many other gauge theories, ABJM theory admits a 't Hooft or $1/N$ expansion, in which N and k are taken to be large, while the quotient,

$$\lambda = \frac{N}{k}, \tag{3.2}$$

also called 't Hooft parameter, is kept fixed. In particular, it can be shown, order by order in perturbation theory, that the free energy F of ABJM theory on a three-manifold, defined as the logarithm of the partition function Z, has the following asymptotic $1/N$ expansion:

$$F = \log Z \sim \sum_{g=0}^{\infty} F_g(\lambda) N^{2-2g}. \tag{3.3}$$

If we compare this expansion to (2.12), we find that it has precisely the structure of a genus expansion in (super)string theory. Similar $1/N$ expansions hold for other observables. At large N but λ fixed, the main contribution comes from the first term, and one has

$$F \approx F_0(\lambda) N^2, \qquad N \to \infty. \tag{3.4}$$

This is called the *planar approximation* to the gauge theory, in which only a subset of diagrams are kept [22].

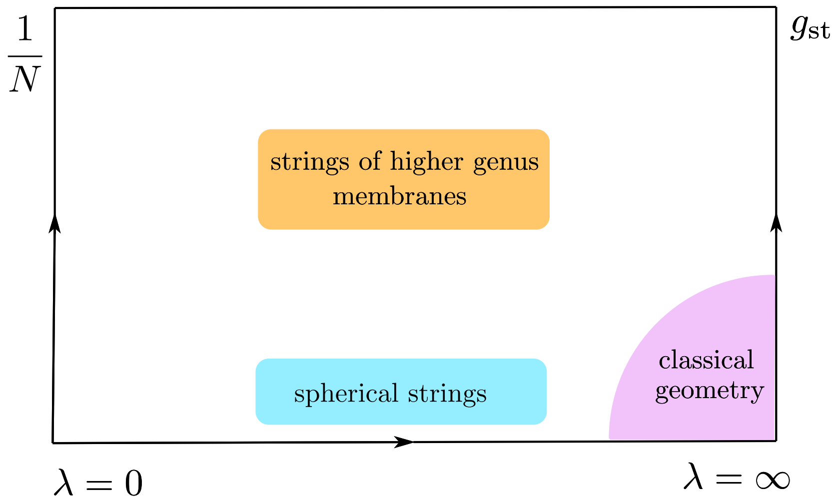

Figure 4. The two-dimensional parameter space of superstring theory on the manifold (3.5), and its relation to the gauge theory parameter space. The description of the theory in terms of classical, Riemannian geometry corresponds to the point-particle limit of non-interacting strings. This is the regime of the gauge theory in which λ and N are very large. Away from this regime, the classical picture is modified by "stringy", "quantum" and "brany" corrections.

According to [1], ABJM theory is dual to type IIA superstring theory on the manifold

$$\mathrm{AdS}_4 \times \mathbb{CP}^3, \tag{3.5}$$

where $\mathbb{CP}^3$ is the projective space in three complex dimensions. The manifold (3.5) has one single scale, the common "radius" L of AdS_4 and $\mathbb{CP}^3$. Therefore, the type IIA superstring on (3.5) is characterized by two dimensionless parameters. One of them is the quotient L/ℓ_s, while the other is the string coupling constant $g_{\rm st}$. According to the string/gauge theory duality of [1], there is a dictionary between the two parameters of the superstring theory and the two parameters of the gauge theory. The dictionary reads as follows:

$$g_{\rm st}^2 = \frac{\sqrt{32\pi^2\lambda}}{k^2}, \qquad \lambda = \frac{1}{32\pi^2}\left(\frac{L}{\ell_s}\right)^4. \tag{3.6}$$

According to this dictionary, the regime in which the scale of the spacetime L is large as compared to the string length, i.e. the point-particle approximation to string theory, corresponds to the regime in which λ, the 't Hooft parameter, is large. In addition, the regime in which the string is weakly coupled, i.e. the string coupling constant $g_{\rm st}$ is small, corresponds to the regime in which k is large. Since λ is fixed, this corresponds to the planar limit of the gauge theory, in which N is large. The two-dimensional parameter space of the theories is shown in Fig. 4. The gauge/string correspondence also postulates that the $1/N$ expansion of the gauge theory corresponds to the genus expansion of string theory. This means in particular that the free energy in (2.12) corresponds, order by order, to the expansion of the free energy (3.3). The three-manifold on which (3.3) is computed is precisely the *boundary* of the AdS_4 geometry appearing in (3.5).

What are the implications of this duality for the structure of space-time? According to the dictionary (3.6), the Riemannian geometry of the space (3.5) corresponds to a very particular regime of the dual gauge theory, namely, the regime in which λ is large (so we are in the point-particle limit) and k is large (so that strings do not interact). In particular, in this regime, N, the rank of the gauge group, is very large. This suggests that the standard notions of Riemannian geometry appear as emergent phenomena in the limit of a large number of constituents in the gauge theory.

Let us now suppose that we decrease slightly the value of λ, but we keep N very large. This is equivalent to going away from the point particle limit, and to making manifest the "stringy" nature of the geometry. Explicit calculations indicate that, indeed, away from this limit, the gauge theory excitations organize themselves into embedded, spherical curves in the spacetime (3.5), similar to what happens for topological strings in a Calabi–Yau manifold. Consequently, the free energy is given by the result obtained from General Relativity on (3.5), together with an infinite number of corrections coming from these embedded curves [7].

If we now increase the value of the string coupling constant, curves of higher genera start contributing. From the point of view of the gauge theory, they come from the sub-leading corrections in the $1/N$ expansion (3.3). As we mentioned above, the resulting genus expansion in string theory is badly divergent, but now it makes sense as the asymptotic expansion of a well-defined quantity, namely, the free energy of the gauge theory.

It turns out that, as $g_{\rm st}$ is increased, the corrections to Riemannian geometry are not only due to strings of all genera, but also to extended objects of two spatial dimensions, i.e. to *membranes*. These configurations are invisible in string perturbation theory, since they have the non-analytic dependence on the string coupling constant in (2.19). Nevertheless, we expect such corrections in the non-perturbative picture of type IIA superstring theory given by M-theory [2]. In fact, it can be shown that, if we consider only the corrections to classical geometry due to embedded strings, the resulting answer is inconsistent, and both membranes and strings have to be included [11]. The detailed analysis of this example of gauge/string correspondence shows that the emergent space of string theory is not just the standard arena of Riemannian geometry, but a "stringy" and "brany" space.

As $g_{\rm st}$ becomes large, the contribution of membranes becomes more and more important, and the description based on fundamental strings is no longer appropriate. Rather, we should find a formulation of the theory in which membranes and fundamental strings are on equal footing. Such a formulation is provided, in principle, by M-theory. At large distances, M-theory is described by *eleven*-dimensional supergravity [23]. Therefore, a remarkable aspect of the transition from type IIA string to M-theory is that the space itself *changes*: as the string interactions becomes stronger, and membranes become more and more important, an eleventh dimension appears. In this particular example, the spacetime background for M-theory in the long distance regime is

$$\mathrm{AdS}_4 \times \left(\mathbb{S}^7/\mathbb{Z}_k\right). \tag{3.7}$$

A seven-dimensional sphere (modded by a discrete group $\mathbb{Z}_k$), of radius L, has grown out of the six-dimensional projective space $\mathbb{CP}^3$. The parameter k, which as we noted in (3.6), is related to the value of the string coupling constant, is now purely geometric, since it defines the modding of the seven-sphere by a discrete group. In M-theory, the fundamental length is no longer the string length, but the Planck length ℓ_p, and the dimensionless parameter L/ℓ_p is now related to N and k by

$$\left(\frac{L}{\ell_p}\right)^6 = 32\pi^2 kN. \tag{3.8}$$

This relation holds at large N. As we will see later on, it receives corrections. The long-distance regime of M-theory corresponds to $L/\ell_p \gg 1$. As L/ℓ_p becomes smaller, the description in terms of eleven-dimensional supergravity is less and less reliable, and should be replaced by a more fundamental description of M-theory (in the same way that supergravity is replaced by string theory when L/ℓ_s becomes smaller.) Although we know some properties of this description (for example, it contains membranes and five-branes) we do not know its precise formulation.

Figure 5. The parameter space of M-theory on the manifold (3.7). In the long distance regime $L/\ell_p \gg 1$, the theory is well described by classical Riemannian geometry (i.e. classical gravity). This corresponds to the large N limit of the gauge theory at fixed gauge coupling (the "thermodynamic limit" of the gauge theory.) When L is of the order of the Planck length, quantum gravity effects become important, as well as the contributions of extended objects, like strings and membranes.

The transition from (3.5) to (3.7) supports the view that, in string/M-theory, space is a not an ultimate entity, but rather an emergent or "effective" description. From the point of view of the gauge/string correspondence, both type IIA superstring on (3.5) and eleven-dimensional supergravity on (3.7) have to be considered as "effective" descriptions of two different regimes (or "phases") of the gauge theory. The type IIA string description is convenient if we want to describe the gauge theory in the 't Hooft expansion, in which N and k are large and we keep the ratio N/k fixed. Eleven-dimensional supergravity is useful if we want to describe the gauge theory in the regime in which k is fixed (but not necessarily large) and N is large. This is a sort of "thermodynamical" regime of the gauge theory, different from the 't Hooft regime. Note that, when N is large and k is fixed but not large, the geometry of the type IIA string is ten-dimensional, but it is very "quantum" (since higher genus strings are not suppressed). However, this "quantum" ten-dimensional geometry can be effectively described (as long as N is still large) by the classical geometry (3.7) in eleven-dimensional supergravity.

In both the type IIA string and the M-theory picture, classical or Riemannian geometry appears as a limiting regime, and the corrections based on strings and membranes become important as we move slightly away from it. Let us focus on the M-theory picture, which is in many ways simpler, since its parameter space is one-dimensional (see Fig. 5). The corrections to Riemannian geometry on the space (3.7) become important when L, the size of the target space, is of the order of the Planck length ℓ_p. These include quantum gravity corrections, as well as corrections due to extended objects like strings and membranes (on equal footing). The structure of such corrections can be made very precise, since in the case of ABJM theory, the full expansion of the free energy at large N and fixed k can be obtained from a gauge theory calculation. It has the form (see [12, 18] for reviews)

$$\begin{aligned} F(N,k) = & -\frac{1}{384\pi^2 k}\zeta^{3/2} + \frac{1}{6}\log\left[\frac{\pi^3 k^3}{\zeta^{3/2}}\right] + A(k) + \sum_{n=1}^{\infty} c_{n+1} k^n \zeta^{-3n/2} \\ & + \mathcal{O}\left(\mathrm{e}^{-\sqrt{N/k}}, \mathrm{e}^{-\sqrt{kN}}\right). \end{aligned} \tag{3.9}$$

In this equation,

$$\zeta = 32\pi^2 k \left(N - \frac{k}{24} - \frac{1}{3k}\right), \tag{3.10}$$

$A(k)$ is a known function of k, and c_{n+1} are calculable coefficients. It is natural to identify

$$\zeta = \left(\frac{L}{\ell_p}\right)^6, \tag{3.11}$$

which at large N agrees with the dictionary (3.8), but it incorporates some subleading corrections in $1/N$. If this identification is made, then the first term in (3.9) is precisely the contribution to the free energy obtained from classical (super)gravity, i.e. from classical Riemannian geometry. The remaining terms in the first line are quantum corrections in supergravity: the logarithmic term is a one-loop correction, while the n-th term in the series,

$$\zeta^{-3n/2} = \left(\frac{\ell_p}{L}\right)^{9n}, \tag{3.12}$$

has precisely the form of an $(n+1)$-loop correction in a quantum theory of gravity in eleven dimensions. Finally, the terms in the second line (which can be computed explicitly) are non-perturbative corrections at large N. The corrections which go like

$$\mathrm{e}^{-\sqrt{N/k}} \sim \mathrm{e}^{-(L/\ell_s)^2} \tag{3.13}$$

are due to "stringy" effects. They can be obtained from perturbative string theory, and they are similar to the corrections due to holomorphic curves appearing in (2.16). The corrections which go like

$$\mathrm{e}^{-\sqrt{kN}} \sim \mathrm{e}^{-(L/\ell_p)^3} \sim \mathrm{e}^{-1/g_{\mathrm{st}}} \tag{3.14}$$

are due to membrane instantons. They are non-perturbative in the string coupling constant, and they are invisible in string perturbation theory. Clearly, when N is very large, the free energy is dominated by the contribution of classical Riemannian geometry. As N becomes small, i.e., as we enter the Planckian regime, both quantum corrections to classical gravity (in the first line of (3.9)) as well as corrections due to strings and branes (in the second line of (3.9)) become more and more important. The "classical" Riemannian geometry of the eleven-dimensional space (3.7) provides an accurate description only when N is large.

It is interesting to note that, since N is the rank of a gauge group, it can only take positive integer values. According to the dictionary (3.8), this seems to imply that the scale L should be quantized in units of the Planck length. Therefore, as N becomes small, we might encounter an intrinsic discreteness in space, inherited from the gauge theory realization. Unfortunately, not much is known about the ultra-Planckian regime of the correspondence between gauge theory and string/M-theory. Perhaps, in this regime, the spacetime of string/M-theory, after being substantially corrected by string and membrane excitations, finally "dissolves" into gauge-theoretic degrees of freedom. However, recent calculations show that, in the case of ABJM theory and its string/M-theory dual, it is possible to resum the expansion in (3.9) in such a way that the free energy of the gauge theory makes sense for arbitrary complex values of N. Therefore, at least in this concre example, one might be able to avoid the discretization of space at Planckian lengths [3].

4 Conclusions and open problems

In this article I have summarized some of the ways in which string theory qualifies or modifies our notions of space. In going from point-like particles to extended strings, the framework of classical or Riemannian geometry gets modified in ways which in some cases can be made mathematically precise. In Gromov–Witten theory or quantum cohomology, classical topological and geometrical notions have to be enlarged in order to take into account the structure of embedded strings in spacetime. One finds in this way a "stringy" geometry which incorporates the corrections due to the finite length of the strings. However, a full string theory geometry should also be "quantum," i.e. it should incorporate the string interactions and the effects of a finite string coupling constant. This requires a non-perturbative formulation of string theory.

A useful framework to address the full quantum nature of the strings is the string/gauge theory correspondence. In these dualities, space can be regarded as an "effective" or emergent description of a completely different system, namely a gauge theory. The emergent space comes equipped with corrections to the classical geometry due to embedded strings, as expected from perturbative string theory, but also to membranes and other extended objects, as expected from M-theory and non-perturbative string theory. An emergent notion of space also appears when we consider the relation between type IIA superstring and eleven-dimensional supergravity, in which an eleventh dimension appears dynamically in order to describe the strongly coupled regime of the superstring. From the point of view of string/gauge theory dualities, both descriptions, with their different spacetimes, should be regarded as different "effective" pictures for two different regimes of the underlying gauge theory.

In most our discussion of the string/gauge theory duality, we have assumed that the gauge theory side has a conceptual and ontological primacy, since it provides the "microscopic" description of "macroscopic" gravity theories. This is a reasonable assumption, since the gauge theory is a relatively well-defined object, and it can be used to provide a non-perturbative description of the elusive string/M-theory side of the correspondence. However, there might still be a description of M-theory/string theory in which spacetime recovers a more fundamental status, and the string/gauge duality would then become an equivalence of two different theories, rather than a hierarchy along the lines of the microscopic/macroscopic divide (see [5] for a related discussion).

In order to answer this and related questions, we need a better understanding of the way in which the gauge theory encodes the spacetime of the dual string/M-theory, and of the behavior of the gauge/string correspondence at small distances. In spite of all the work done on holography, this remains a relatively unexplored subject. The lessons extracted from this line of work might provide important hints about a possible fundamental formulation of M-theory, and, hopefully, also about the nature of actually existing spacetime.

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