NON-PERTURBATIVE TOPOLOGICAL STRINGS

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Introduction and motivation

String theory is only defined perturbatively, and it has been known for some time that its perturbative expansion diverges factorially

String Perturbation Theory Diverges

David J. Gross and Vipul Periwal

Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544

(Received 11 April 1988)

This is usually a signal of exponentially small non-perturbative effects in the string coupling constant, coming from sectors of the theory which are invisible in a perturbative framework

This is not a peculiarity of string theory. Factorially divergent series are the norm in quantum mechanics and quantum field theory, as first noted by Dyson in 1957.

Bender and Wu made a quantitative connection between the factorial growth of perturbation theory and non-perturbative effects

Large-Order Behavior of Perturbation Theory

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(Received 28 June 1971)

This idea was developed in quantum field theory to give some of our best insights on non-perturbative physics (instantons, renormalons, non-perturbative condensates…)

At the same time, it has been given appropriate mathematical form in the **theory of resurgence** of Jean Ecalle.

In this talk I will summarize recent work which applies this framework to understand the non-perturbative aspects of topological string theory on Calabi-Yau (CY) threefolds

To do this, I need to introduce some mathematical tools

From wild series to analytic functions

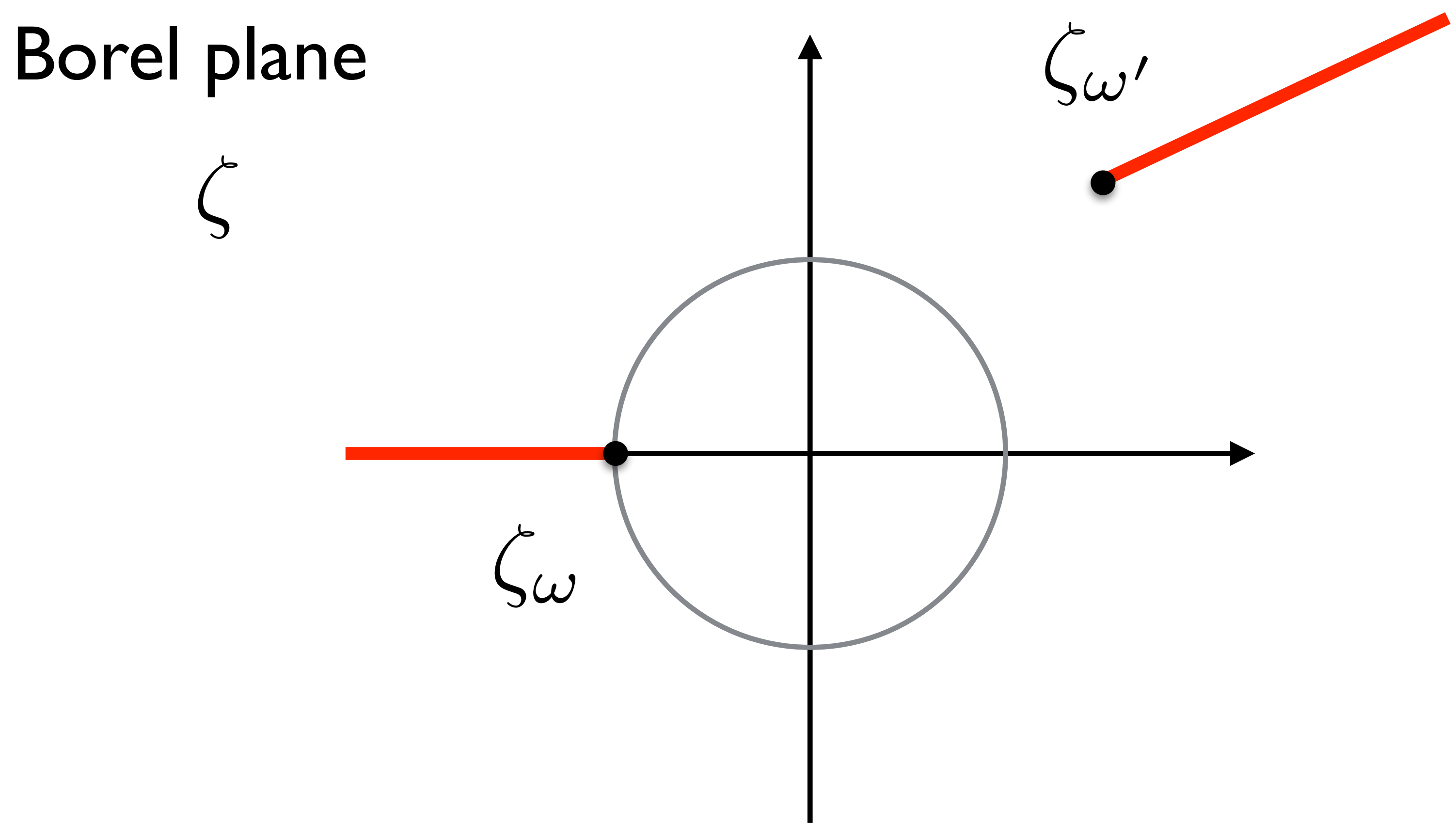
Let us consider a formal power series with factorially growing coefficients

$$\varphi(z) = \sum_{n\geq 0} a_n z^n \qquad a_n \sim n!$$

These are sometimes called Gevrey-1 series. The **Borel transform** is a deceptively simple way of transforming these series into "nice" functions

$$\varphi(z) = \sum_{n\geq 0} a_n z^n \xrightarrow{\text{Borel transform}} \widehat{\varphi}(\zeta) = \sum_{n\geq 0} \frac{a_n}{n!}\zeta^n$$

The Borel transform $\widehat{\varphi}(\zeta)$ is analytic at the origin. We now demand that it can be "endlessly analytically continued" to the complex plane, displaying a set of **singularities** (poles, branch cuts)



The closest singularity to the origin controls the factorial growth of perturbation theory (Darboux' theorem)

A crucial insight is that **the singularities of the Borel transform contain information about the non-perturbative sectors** of the theory

Concretely, the local expansion of the Borel transform at the singularities leads to **new formal power series**. A typical example is a logarithmic singularity at $\zeta = \zeta_\omega$

$$\widehat{\varphi}(\zeta) = -\mathsf{S}_\omega \, \widehat{\varphi}_\omega(\zeta - \zeta_\omega) \frac{\log(\zeta - \zeta_\omega)}{2\pi \mathrm{i}} + \text{regular}$$

singularity ζ_ω → new power series $\varphi_\omega(z)$

inverse Borel transform of the local expansion

singularity ζ_ω → Stokes constant S_ω

depends on the normalization of the power series

With this, we build up a **non-perturbative amplitude**

$$\Phi_\omega(z) = \mathsf{S}_\omega \, \mathrm{e}^{-\zeta_\omega/z} \varphi_\omega(z)$$

This is an example of a **trans-series**, involving exponentially small terms

Resurgent structures

Given a factorially divergent power series, we can collect the set of all formal power series and Stokes constants associated to the singularities of its Borel transform. I will call this collection its **resurgent structure**. It encodes in a mathematically precise way all the non-perturbative information which can be obtained from perturbation theory.

Understanding the resurgent structure of quantum theories has been an ongoing pursuit for many years [Voros, Zinn-Justin, Brezin, Parisi, 't Hooft, ...]. We expect that topological fields and strings will provide workable and interesting examples of these structures

Topological string theory

Let *M* be a Calabi-Yau (CY) threefold. At each genus *g* one can consider the topological string free energy $F_g(X)$, which depends on the flat homogeneous coordinates *X* (I will often consider one-modulus CYs for simplicity)

At large *X* this has an expansion encoding Gromov-Witten invariants of *M*, which "count" holomorphic curves of genus *g* and degree *d*:

$$F_g(X) = \sum_d N_{g,d}\, \mathrm{e}^{-dX}$$

I recall that in the mirror manifold $M^\star$ one can calculate **periods** by integrating the holomorphic 3-form over a symplectic basis of 3-cycles. This determines both X and the genus zero free energy [Candelas-de la Ossa-Green-Parkes]

$$X^I = \int_{\alpha^I} \Omega \qquad \mathcal{F}_I = \int_{\beta_I} \Omega = \frac{\partial F_0}{\partial X^I}$$

String perturbation theory tells us that the **total free energy** is given by a genus expansion in a small parameter, a.k.a. the string coupling constant

$$F(X, g_s) = \sum_{g \geq 0} F_g(X) g_s^{2g-2}$$

General arguments [Gross-Periwal, Shenker] indicate that this series grows doubly-factorially, at fixed X

$$F_g(X) \sim (2g)!, \qquad g \gg 1$$

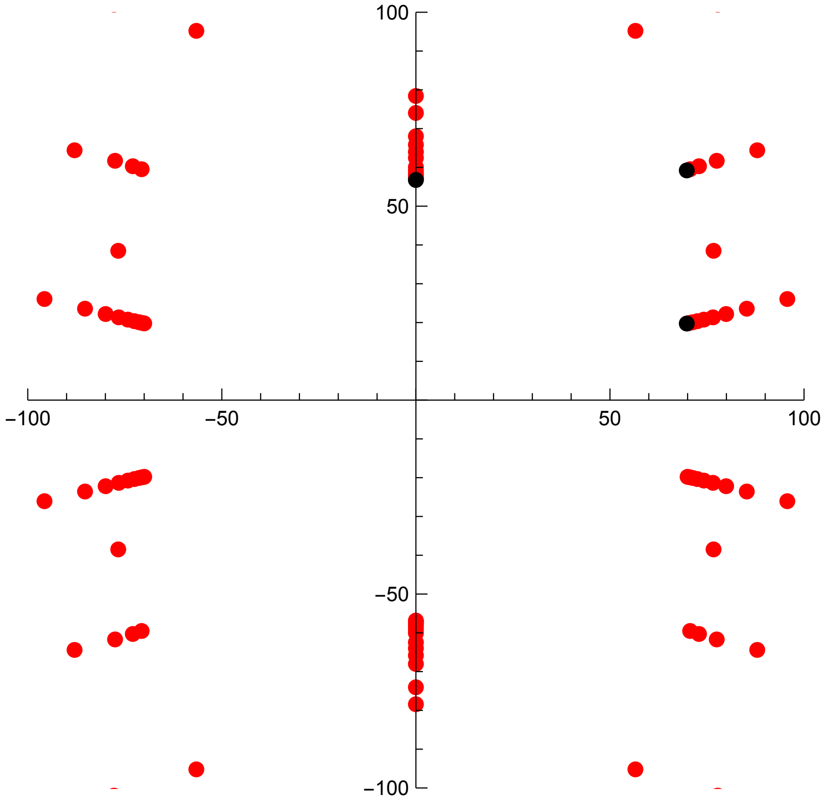
What is the resurgent structure associated to this series?

Borel plane and CY periods

Conjecture: the Borel singularities for the perturbative series of TS free energies are **integral periods** of the mirror CY

$$\mathcal{A} = c^I \mathcal{F}_I + d_I X^I$$

Determining which integral periods are actual singularities is more difficult and it sometimes relies on numerical calculations



Trans-series for topological strings

How do we determine the trans-series associated to the singularities?

The **holomorhic anomaly equations** of BCOV make it possible to obtain the perturbative series. One can then try to solve them with a "trans-series ansatz" [Couso-Edelstein-Schiappa-Vonk] as in Ecalle's theory of ODEs

$$F = \sum_{g \geq 0} F_g(X) g_s^{2g-2} + \mathrm{e}^{-\mathcal{A}/g_s} \sum_{n \geq 0} F_n^{(1)}(X) g_s^{n-1} + \cdots$$

perturbative series   instanton correction

In recent work [Gu-M.M, Gu-Kashani-Poor-Klemm-M.M.] we obtained an all-orders, **exact solution** for the trans-series, for **any** CY (compact or not)

$$\mathcal{A} = c^I \mathcal{F}_I + d_I X^I$$

$$\Phi_{\mathcal{A}} = \mathsf{S}\left(1 + g_s c^J \partial_J F(X^I - g_s c^I)\right) \mathrm{e}^{F(X^I - g_s c^I) - F(X^I)}$$

This is a universal formula for the "one-instanton amplitude" associated to $\mathcal{A}$. It can be written in terms of perturbative data only, and it suggests that the the CY periods X^I are **quantized** in units of the string coupling constant, as in large *N* dualities

Stokes constants

An important piece of the resurgent structure are the Stokes constants. In some interesting examples (e.g. WKB, complex Chern-Simons theory) they turn out to be **integers** related to **BPS invariants** [Gaiotto-Moore-Neitzke, Gukov-M.M.-Putrov, Garoufalidis-Gu-M.M.-Wheeler].

We expect a similar picture here. We can show e.g. that the genus zero Gopakumar-Vafa invariants $n_{0,d}$ arise as Stokes constants associated to towers of Borel singularities at

$$dX^1 + mX^0 \qquad\qquad m \in \mathbb{Z}$$

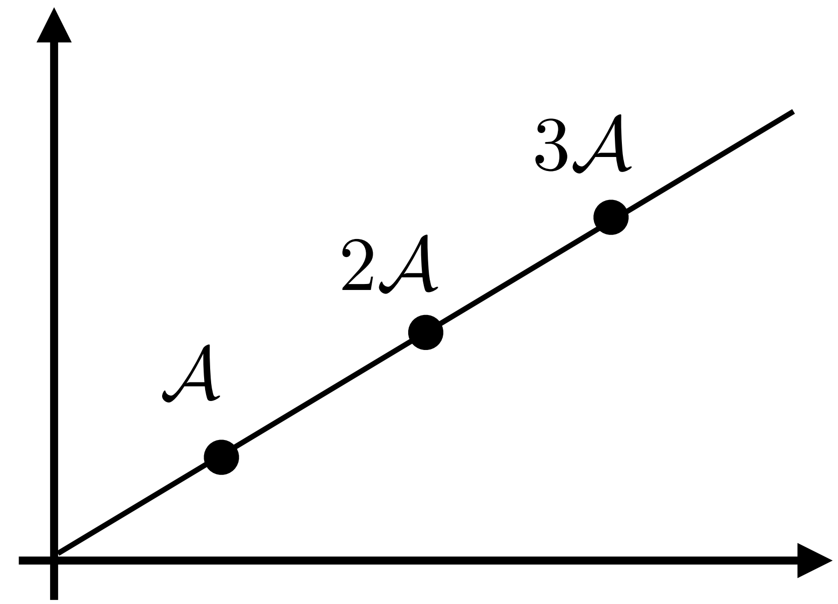
Multi-instantons and Stokes automorphisms

We have Borel singularities at positive integer multiples $\ell \mathcal{A}$ encoding **multi-instanton** amplitudes. They lead to a **Stokes discontinuity** which can be computed explicitly [Iwaki-M.M]. It is easier to write it in terms of the dual topological string partition function, or Zak transform

$$\tau(X, \lambda) = \sum_{k \in \mathbb{Z}} \lambda^k Z(X + k g_s)$$

$$\downarrow$$

$$Z = \mathrm{e}^F$$



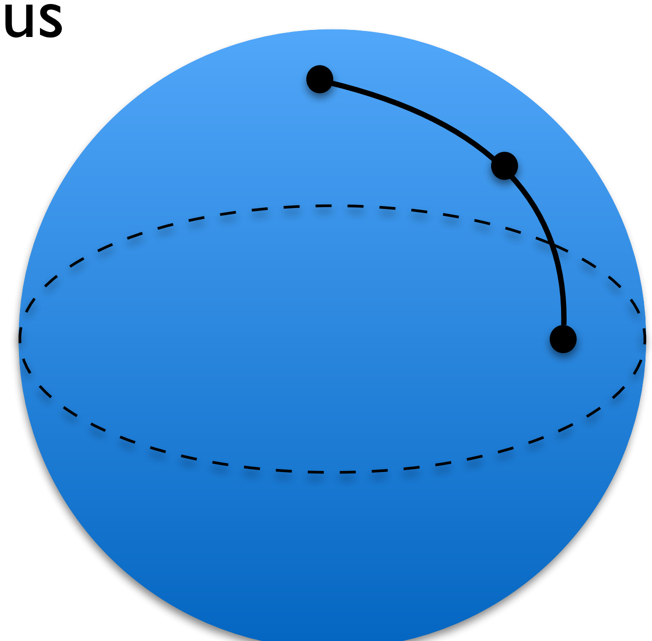
The Stokes automorphism across a ray of singularities acts as

$$\mathfrak{S}\tau(X,\lambda) = \mathrm{e}^{\mathsf{S}\,\mathrm{Li}_2(\lambda)}\tau(X - c g_s \mathsf{S}\log(1-\lambda),\lambda)$$

and induces a Delabare-Dillinger-Pham transformation on the CY periods! Similar results in different but related contexts have appeared before [Alexandrov-Pioline, Lisovyy-Roussillon, Coman-Longhi-Teschner, …]

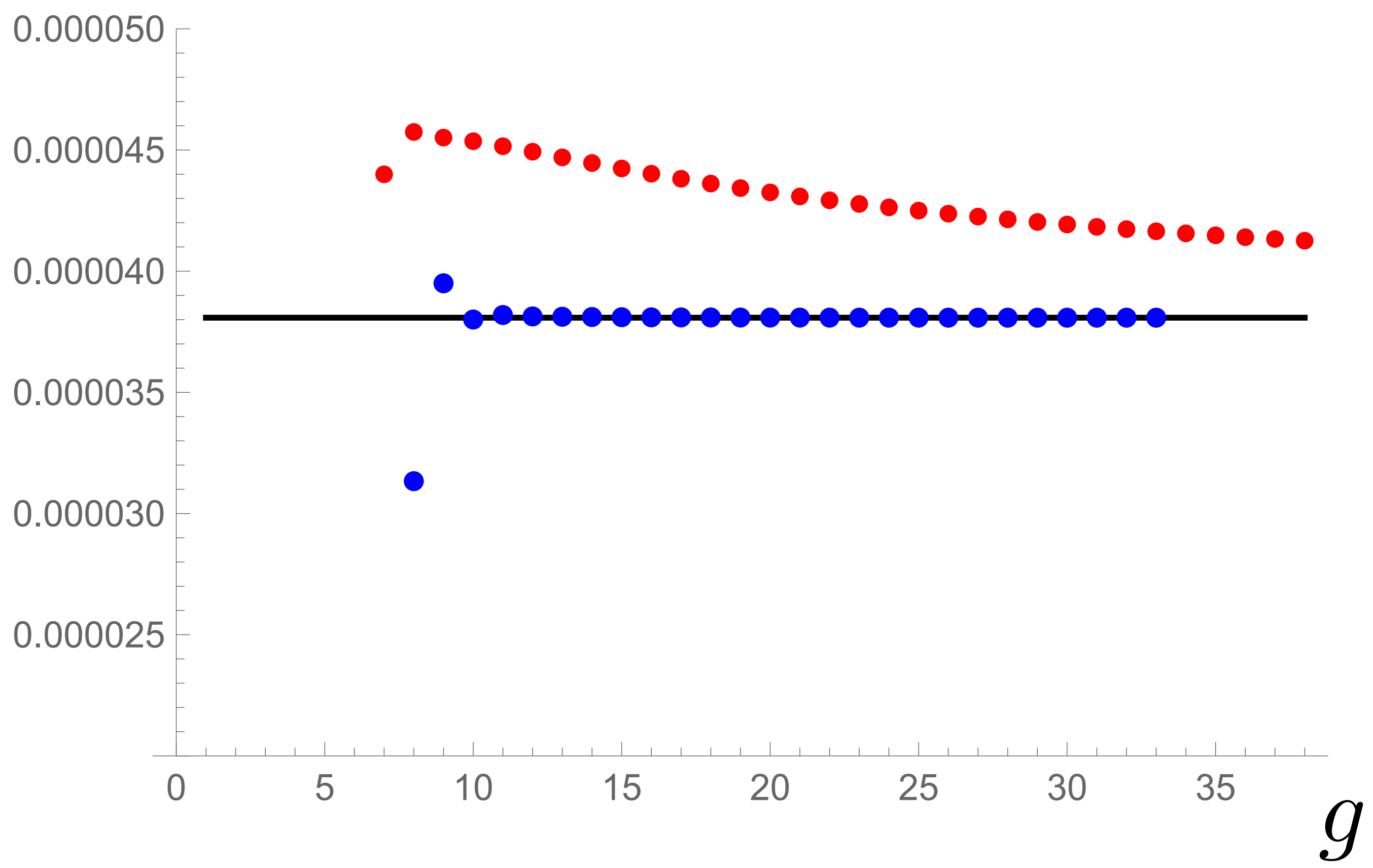
Experimental evidence: asymptotics of free energies in the quintic CY

The closest singularity of the Borel transform determines the asymptotics of perturbation theory, and we can test the corresponding one-instanton amplitude against perturbative data, in e.g. the famous quintic CY



$\longrightarrow \quad \mathcal{A}=\mathcal{F}_0$

vanishing period at the conifold point



red dots: sequence $$\frac{\mathcal{A}^{2g-1}}{\Gamma(2g-1)} F_g$$

blue dots: Richardson acceleration

black line: prediction from our one-instanton formula

Conclusions and outlook

The theory of resurgence gives a precise mathematical framework to understand non-perturbative sectors, which can be applied successfully to (topological) string theory.

To do this, we have developed an "instanton calculus" for the Kodaira-Spencer theory of BCOV. We have found **exact** solutions for multi-instanton amplitudes and conjectured the general form of the resurgent structure for topological strings.

Our results determine the instanton physics of other systems governed by the HAE, like large N matrix models

To know the resurgent structure we have to determine the precise location of Borel singularities and their Stokes constants. We expect a very rich mathematics and physics related to BPS invariants.

What is the physical meaning of the "instanton" amplitudes we obtained? They look like D-branes of the "wrong" type. Are they rather "renormalons" of the topological string?

What is the relation to proposals for non-perturbative definitions of the topological string, like the one in [Grassi-Hatsuda-M.M.]?