QUANTUM GEOMETRY

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In pre-relativistic physics, geometry is the stage where the play takes place.

In modern physics, geometry is another actor in the play. It affects and is affected by other physical actors. That part of the play is described by general relativity.

However, general relativity describes gravity well only at large distances as compared to the *Planck length*

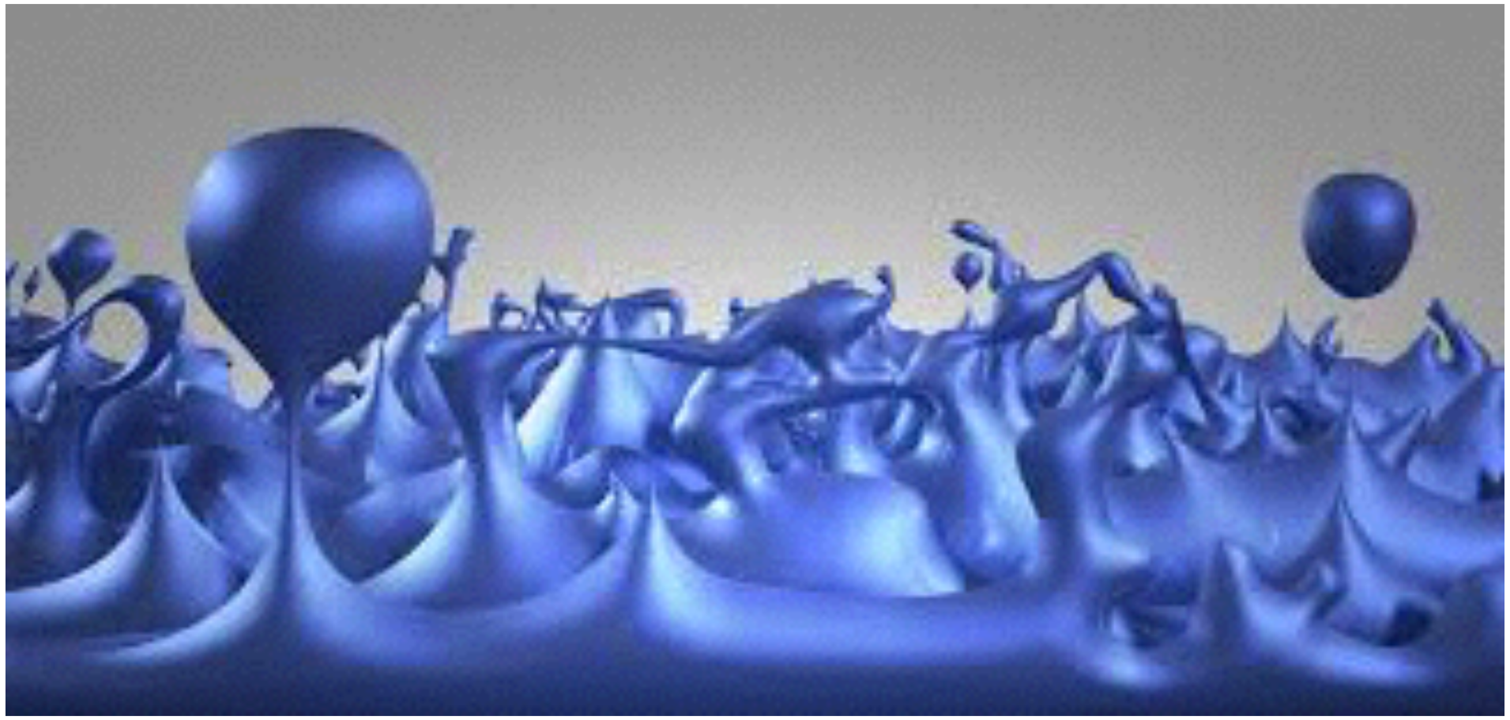
$$L \gg \ell_{\mathrm{P}} = \sqrt{\frac{\hbar G}{c^3}} = 1.616229 \times 10^{-35}\ \mathrm{m}$$

As we approach the Planck length, quantum effects should kick in. For example, the Newtonian potential is known to be corrected as [Donoghue et al.]

$$V(r) = -\frac{GMm}{r}\left[1 + \frac{41}{10\pi}\left(\frac{\ell_{\mathrm{P}}}{r}\right)^2 + \cdots\right]$$

More generally, at Planckian scales we expect to have a quantum theory of gravity. Since gravity is geometry, there might be a "quantum version" of Riemannian geometry, or *quantum geometry* for short.

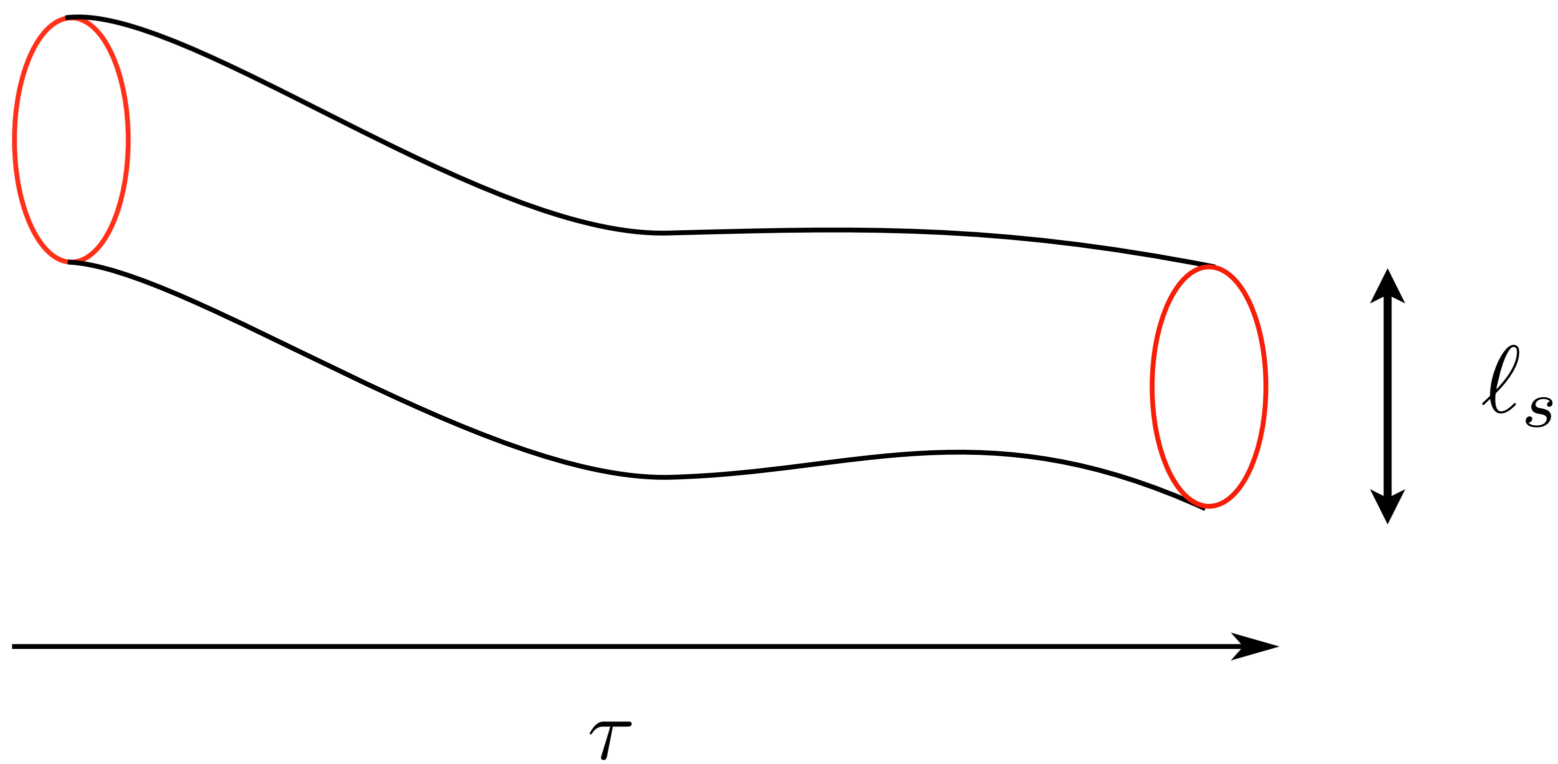
This is an unsolved, difficult and controversial problem, and nobody really knows what happens at Planckian scales.



String theory seems to be *a* consistent theory of quantum gravity. Moreover, some of the insights of string theory on this subject have been incorporated in well-defined mathematical theories.

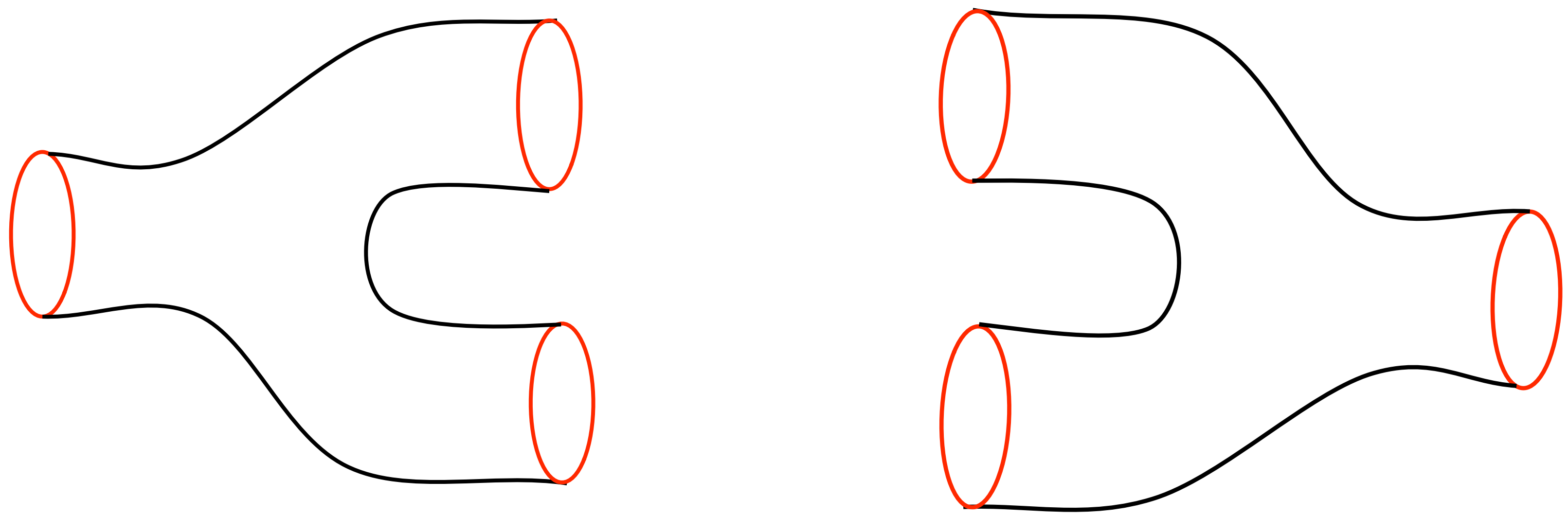
In this talk, I will try to give you an overview of some of these insights and specially of their applications in (enumerative) geometry

The basic idea of string theory is that elementary point-particles should be replaced by one-dimensional extended objects, i.e. elementary strings. This introduces a new fundamental parameter, the length of a string ℓ_s



ℓ_s can be regarded as a deformation parameter: when it goes to zero, we should recover a theory of point-particles

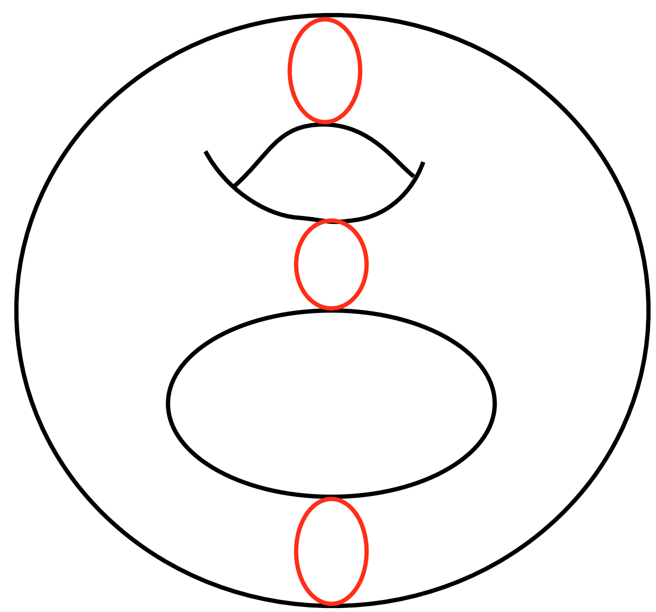
Strings can interact, but they do this in a purely geometric way, by joining or splitting. These events involve a "string vertex" or "pair of pants"

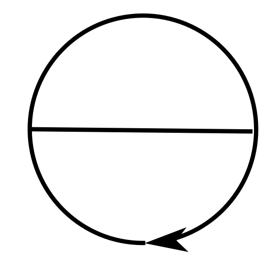


Each interaction vertex gives a factor of g_s

- the *string coupling constant*

The simplest quantum processes in string theory are "vacuum diagrams": the string follows periodic trajectories where it splits and rejoins. These diagrams are (closed) Riemann surfaces of genus *g*





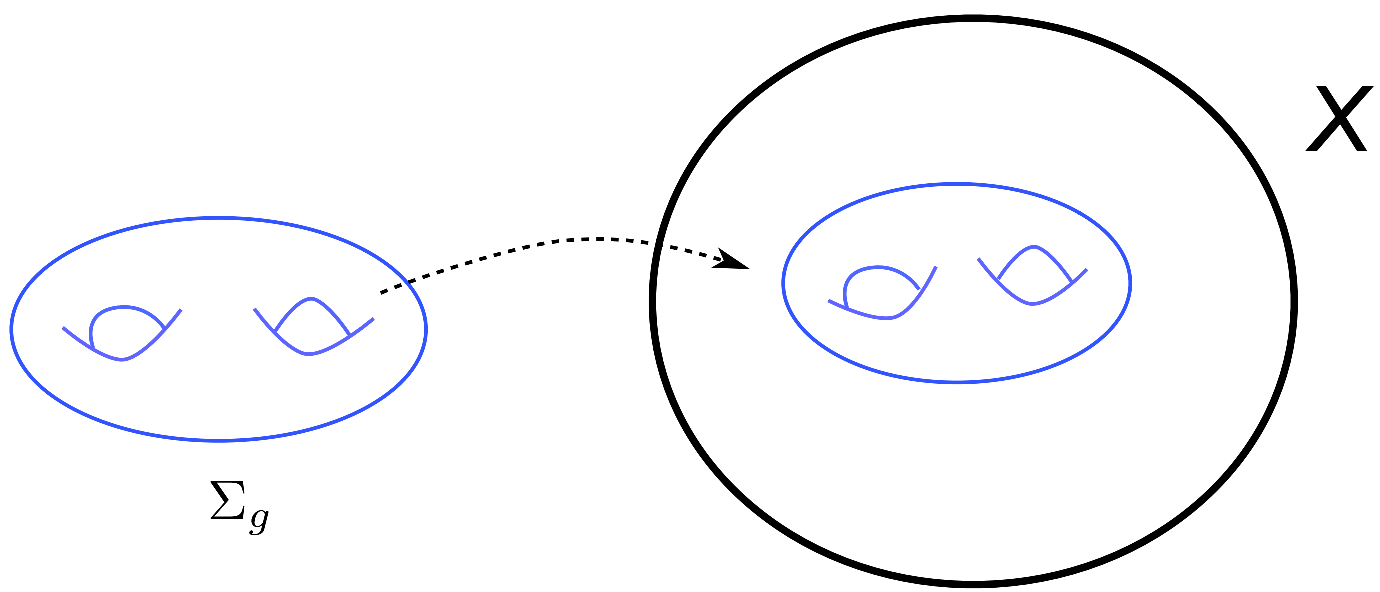
Note that a genus *g* surface is weighted by g_s^{2g-2}

Note that string theory has *two* quantum parameters, the string length and the string coupling constant.

Therefore, there are two types of quantum effects: one is due to the *extended* nature of the string, and the other to the *interacting* nature of the string.

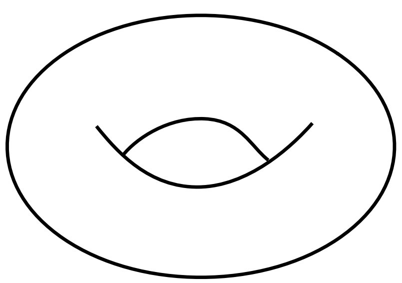
In classical geometry, manifolds are made out of points. When points are promoted to strings, we should expect a deformation of classical geometry. To implement this deformation, we enrich a manifold *X* by considering *embeddings of Riemann surfaces of arbitrary genus inside X*

$$x : \Sigma_g \to X$$



This is precisely the setting of modern enumerative geometry, in which one "counts" holomorphic maps from a Riemann surface to *X*!

In this talk, *X* will be a *Calabi-Yau (CY) threefold*, i.e. a complex, Kahler, Ricci-flat manifold of complex dimension 3. This is the case where the enumerative problem is simpler.



$\mathbb{T}^2$ CY one-fold

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0 \quad \text{in} \quad \mathbb{P}^4$$

quintic CY threefold

I will assume that my CY has $b_2(X) = 1$ so that the homology of embedded surfaces is labelled by a single integer *d*, their degree

There is a simplified version of string theory, called *topological string theory*, which captures precisely this enumerative information. It is a string theory version of topological quantum field theory.

The basic mathematical information in topological string theory is the appropriate counting of curves with a given degree d and a given genus g

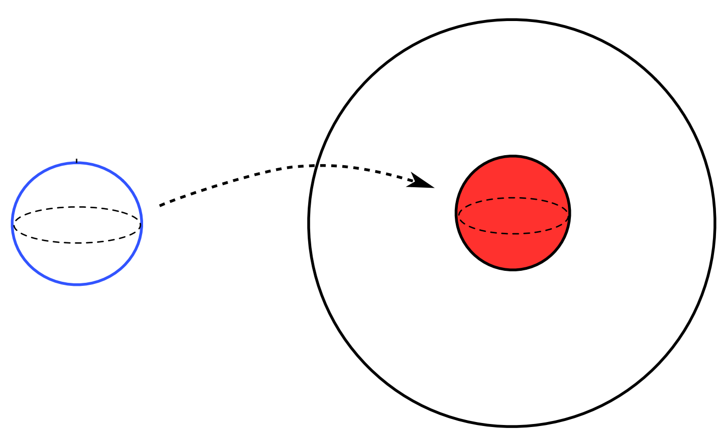
$$N_{g,d} \quad = \quad \text{Gromov-Witten (GW) invariant}$$

Calculating these invariants is a difficult problem in geometry.

In string theory, GW invariants do not come one by one, but appear in physical quantities which depend on the "size" of the manifold, or *Kähler parameter*

t = Area of the curve with degree one $/\ell_s^2$

The simplest case occurs when strings have *g=0* (i.e. they do not interact). The relevant quantity is the genus zero free energy:



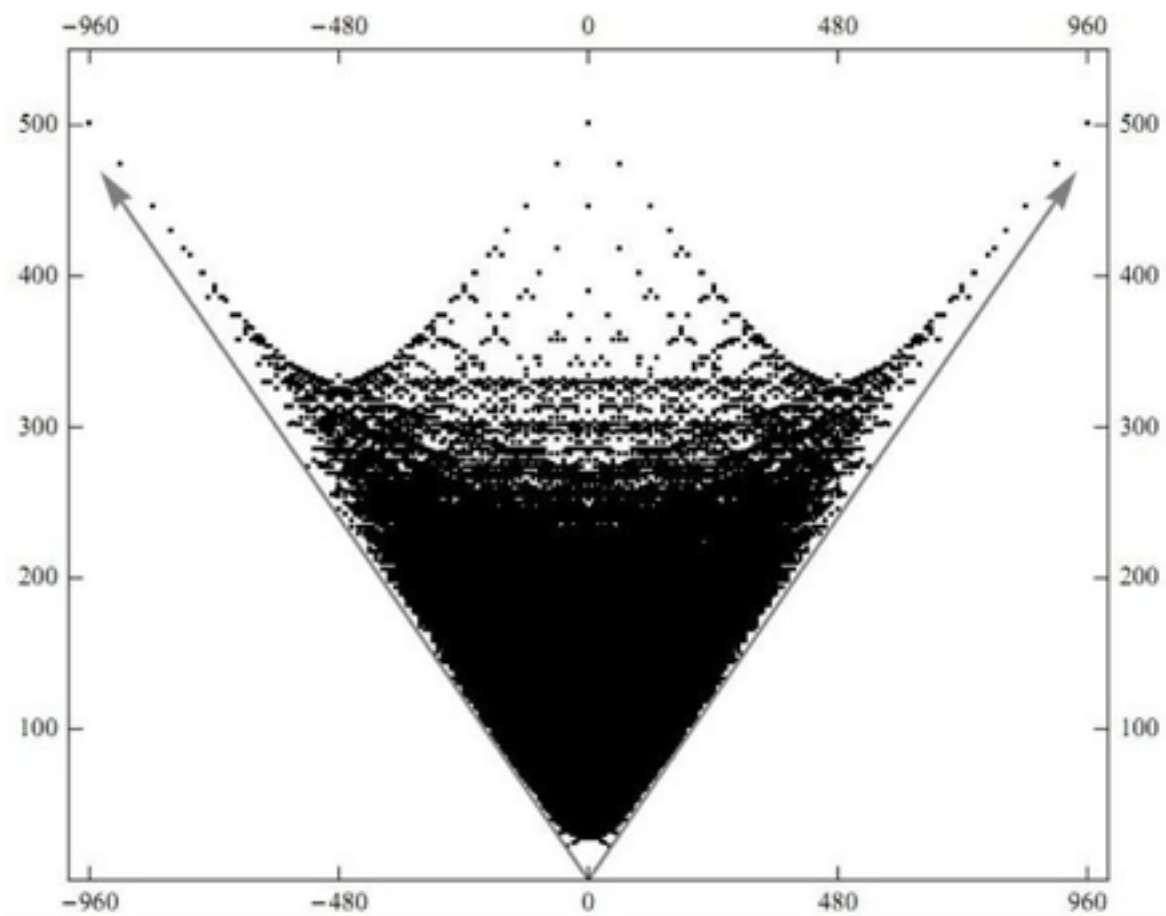
$$F_0(t) = ct^3 + \sum_{d\geq 1} N_{0,d} \mathrm{e}^{-dt}$$

triple intersection number: "classical" part

GW invariants: "stringy" part

Mirror symmetry

$b_2 + b_3/2 - 1$



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It turns out that CY threefolds come in *mirror pairs* (X, Y), related roughly by the exchange

$$H_2(X) \leftrightarrow H_3(Y)$$

An important consequence of mirror symmetry is that the calculation of the genus zero free energy of *X* (a difficult, "stringy" calculation) can be deduced from a simple, "classical" calculation in the mirror *Y*

This is an example of a *string theory duality*, in which a difficult calculation in one model gets mapped to an easy calculation in another model. For the quintic CY, Candelas *et al.* obtained in this way in 1991:

$$F_0(t) = \frac{5t^3}{6} + 2875\mathrm{e}^{-t} + \frac{4876875}{8}\mathrm{e}^{-2t} + \frac{8564575000}{27}\mathrm{e}^{-3t} + \cdots$$

Therefore, mirror symmetry determines the “stringy” deformation of the geometry at genus zero.

This “stringy” geometry is different from the standard one. For example, two topologically different CYs (related by a so-called “flop”) are equivalent.

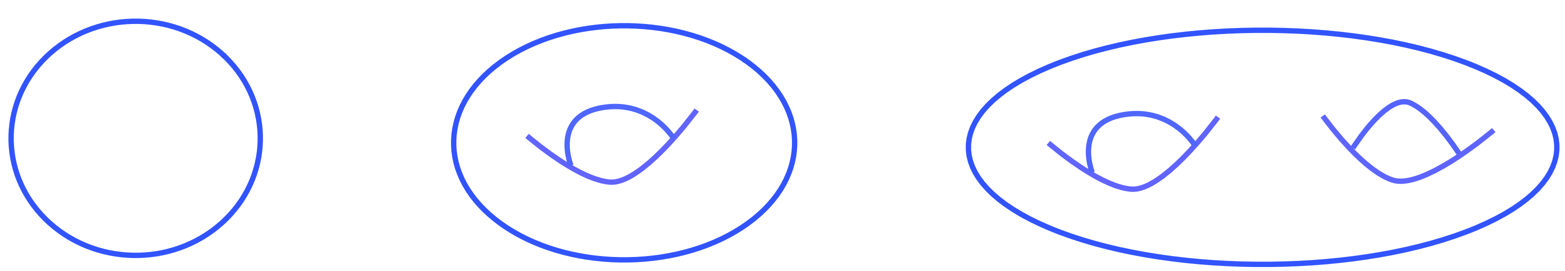
However, from the point of view of the second quantum parameter -the string coupling constant- genus zero is the “classical”, or non-interacting case. We have to consider interacting strings!

Interacting strings

To incorporate the string coupling constant, one has to construct a perturbative expansion. For example, in topological string theory, one first considers the genus g free energy

$$F_g(t) = \sum_{d \geq 1} N_{g,d}\, \mathrm{e}^{-dt}$$

and then we add the contributions of all genera in a formal power series



$$g_s^{-2} F_0(t) \quad + \quad F_1(t) \quad + \quad g_s^2 F_2(t) \quad + \quad \cdots$$

As in any quantum theory, going to higher loops becomes increasingly difficult. In fact, in topological string theory, there is no efficient, general algorithm to compute the higher genus free energies.

A much more serious problem is that the corrections due to interacting strings are "too large", and the formal perturbative series in genus is *asymptotic*, not convergent. This is a generic problem in string theory. More precisely, in topological string theory one has

$$F_g(t) \sim (2g)!, \qquad g \gg 1$$

Therefore, in standard perturbative string theory, we do not have a real understanding of the corrections due to higher genus strings. These could lead to deep modifications of the underlying geometry.

In the last 20 years, developments in string theory have indicated what these modifications might be. For example, they seem to involve *higher dimensional embedded objects*, like *membranes*.

But more radically, some developments in string theory indicate that the geometric picture is an *emergent phenomenon*. Classical geometry emerges in a "classical" or weak coupling limit, together with corrections due to strings of all genera, membranes, etc.

The most famous example of such a phenomenon is the *AdS/CFT correspondence*, in which string theory on certain gravitational backgrounds "emerges" from an underlying quantum gauge theory. This is somewhat similar to the emergence of a thermodynamic description from an underlying microscopic formulation.

Note however that the gauge theory description and the string theory description are very different, and they generally involve a different number of dimensions. In this sense, dimensionality is also an emergent property.

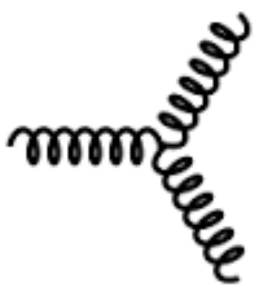
Mathematically, these correspondences suggest that *(enumerative) geometry can be reconstructed from an underlying algebraic object*

A gauge theory of the Yang-Mills type, with a gauge group *U(N)*, is characterized by two parameters:

$$g_{\mathrm{YM}} \quad \text{and} \quad N$$

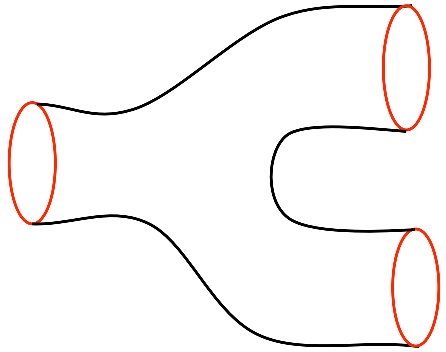
According to the gauge/string correspondence, they are related to the two parameters of string theory

g_{YM}



$\longleftrightarrow$

g_s



$\lambda = g_{\mathrm{YM}}^2 N$

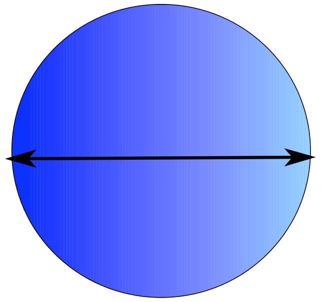
't Hooft coupling

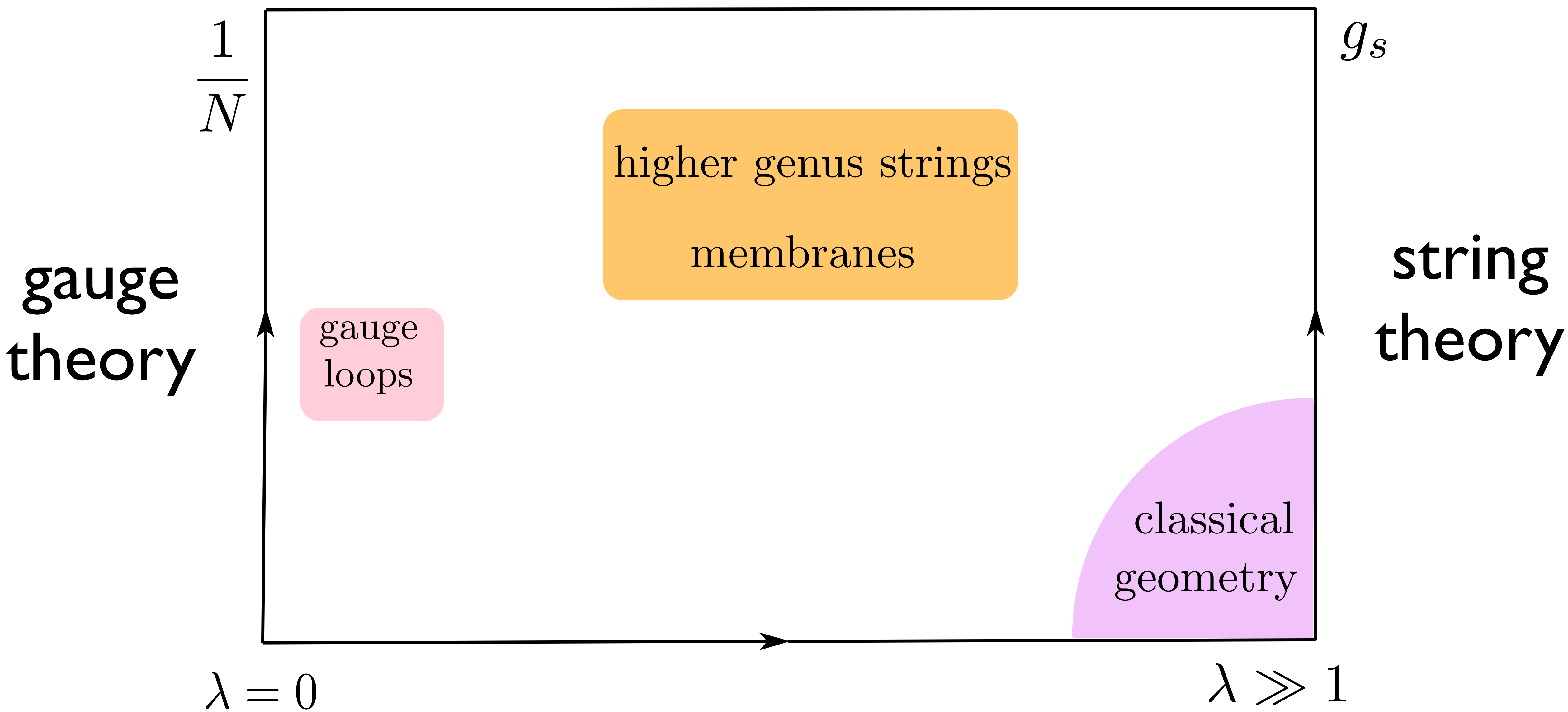
$\longleftrightarrow$

L/ℓ_s

or

t





In this picture, classical geometry "emerges" in the gauge theory regime with large *N* and large 't Hooft coupling. Note that, when the geometry becomes very quantum, one should better use the description in terms of gauge theory!

The dual description also solves the problem of the divergent string perturbative series. This series emerges now as the *asymptotic 1/N expansion* of a *well-defined function* in the gauge theory/quantum mechanical dual.

There are many examples where this correspondence holds, and they all involve strings whose background geometry includes an Anti de Sitter space. Their gauge theory duals are typically supersymmetric versions of Yang-Mills theory or Chern-Simons theory.

In these examples, a ten or eleven dimensional geometry emerges from the dynamics of a four or three dimensional gauge theory, respectively.

In the case of *topological* string theory, one could expect a similar dual description, but involving a simpler quantum theory. Such dual descriptions have been obtained in some cases.

Emergent enumerative geometry

Let us consider the toric CY given by

$$\mathcal{O}(-3) \to \mathbb{P}^2$$

This has a rich enumerative geometry accesible with various techniques (localization, BKMP conjecture, ...)

The dual theory is simply the spectral theory of the following operator on $L^2(\mathbb{R})$

$$\mathsf{O} = \mathrm{e}^{\mathsf{x}} + \mathrm{e}^{\mathsf{y}} + \mathrm{e}^{-\mathsf{x}-\mathsf{y}} \qquad [\mathsf{x}, \mathsf{y}] = \mathrm{i}\hbar$$

This is obtained by "quantizing" the mirror geometry

The Gromov-Witten invariants of this geometry are (conjecturally) encoded in the spectral traces of the inverse operator, at large $\hbar$ and large *N*

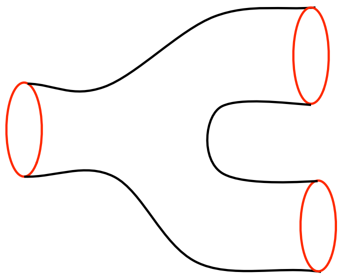
$$Z(N,\hbar) = \mathrm{tr}\,\left(\Lambda^N(\rho)\right) \qquad \rho = \mathsf{O}^{-1}$$

$$\log\, Z(N,\hbar) \sim \sum_{g\geq 0} \hbar^{2-2g} F_g(t) \qquad t = \frac{N}{\hbar}$$

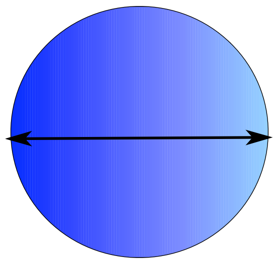
In this case, the duality dictionary is

modular dual
of Planck's
constant

$$\frac{1}{\hbar} \longleftrightarrow g_s$$



$$\frac{N}{\hbar} \longleftrightarrow t$$



The full enumerative geometry of this threefold can be reconstructed from a one-dimensional quantum system, in a strong coupling/thermodynamic limit!

Conclusions

String theory has led to new ideas in enumerative geometry, and it has provided concrete (conjectural) answers to many enumerative questions, particularly through mirror symmetry. We understand quite well the modifications of geometry due to the extended nature of the strings.

The next step is to understand the effects due to the string coupling constant. These might be addressed by using large N dualities with quantum mechanical systems. This leads to the notion of emergent (enumerative) geometry. Although some important progress has been made, much lies ahead!